THE SCHUR PRODUCT THEOREM
IN THE BLOCK CASE

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Abstract. Let $H$ be a positive semi-definite $mn$-by-$mn$ Hermitian matrix, partitioned into $m^2$ $n$-square blocks $H_{ij}$, $i,j = 1, \ldots, m$. We denote this by $H = [H_{ij}]$. Consider the function $f: M_n \to M_r$ given by $f(X) = X^k$ (ordinary matrix product) and denote $H_f = [f(H_{ij})]$. We shall show that if $H$ is positive semi-definite then under some restrictions on $H_{ij}$, $H_f$ is also positive semi-definite. This generalizes familiar results for Hadamard and ordinary products.

Introduction

Let $M_n$ denote the set of $n$-by-$n$ complex matrices. Let $H = [H_{ij}]$ be a positive semi-definite $mn$-by-$mn$ Hermitian matrix, partitioned into $m^2$ $n$-by-$n$ blocks $H_{ij}$, $i,j = 1, \ldots, m$. In this paper we are interested in functions $f: M_n \to M_r$ for which $H_f = [f(H_{ij})]$ is positive semi-definite whenever $H$ is positive semi-definite.

Functions of this type have been considered by many authors. The special case $n = r = 1$ leads to consideration of Hadamard products and functions, about which much is known [1, 2, 6]. John de Pillis [5] showed that if $H$ is positive semi-definite and $f(H_{ij})$ is the $q$th elementary symmetric function of $H_{ij}$ then $H_f \in M_m$ is also positive semi-definite for $1 \leq q \leq n$; this includes the special case $f(H_{ij}) = \det(H_{ij})$ that had been considered previously by R. C. Thompson [7]. Marvin Marcus and William Watkins [4] showed that if $f(H_{ij}) = \|H_{ij}\|_F^2$ (Frobenius norm of $H_{ij}$) and $H$ is positive semi-definite then $H_f$ is also positive semi-definite. They also showed that if $H = [H_{ij}]$, $i,j = 1,2$ is positive semi-definite and $f(H_{ij}) = \text{trace}(H_{ij}^2)$ then $H_f$ is also positive semi-definite. We observed that their second result is not true for 3-by-3 and larger partitions of $H$.

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Example. Consider the real Hermitian matrix

\[ H = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
H_{11} & H_{12} & H_{13} \\
H_{21} & H_{22} & H_{23} \\
H_{31} & H_{32} & H_{33} \\
\end{bmatrix} \]

whose eigenvalues are 0, 0, 0, 1, 2, 3. Thus, \( H \) is positive semi-definite. Notice that

\[ [H^2_{ij}] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
\end{bmatrix} \]

and the eigenvalues of \( [H^2_{ij}] \) are 1, 1, 2, 2.414, 0, - 0.414. Now

\[ H_f = \text{trace}(H^2_{ij}) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \]

and \( \det(H_f) = -2 \).

In view of the fact that \( H_f \) is positive semi-definite for \( f(H_{ij}) = |H_{ij}|^2 \) for \( n = 1 \), it is plausible to conjecture that if \( n \) and \( m \) are arbitrary and \( f(H_{ij}) = \|H_{ij}\|_{sp}^2 \) then \( H_f \) is also positive semi-definite. Unfortunately this is not the case.

Example. Consider the 3-by-3 real Hermitian matrix

\[ H = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix} = \begin{bmatrix}
H_{11} & H_{12} & H_{13} \\
H_{21} & H_{22} & H_{23} \\
H_{31} & H_{32} & H_{33} \\
\end{bmatrix} \]

We can also partition \( H \) as

\[ H = \begin{bmatrix} I & B \end{bmatrix} \]

where \( B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \).

Since \( \rho(BB^*) = 1 \) \( H \) is positive semi-definite. If we consider the function \( f(H_{ij}) = \|H_{ij}\|_{sp} \) (spectral norm) then

\[ H_f = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]

and \( \det(H_f) = -1 \).
In Marcus and William’s paper, there is an example to show that if \( f(H_{ij}) = H_{ij}^2 \) then \( H \) positive semi-definite does not imply \( H_f \) positive semi-definite even if all \( H_{ij} \)'s are Hermitian. However, this function does work for some special types of matrices. One simple preliminary result will be helpful. By \( \rho(A) \) we denote the spectral radius of the \( n \times n \) matrix \( A \).

**Lemma 1.** Let \( A \) be a Hermitian matrix. Then \( \rho(A) = \|A\|_{sp} \).

**Proof.** By definition

\[
\|A\|_{sp} = \max\{\sqrt{\lambda^2}: \lambda^2 \text{ is an eigenvalue of } A^*A\}
\]

\[
= \max\{\sqrt{\lambda^2}: \lambda^2 \text{ is an eigenvalue of } A^2\}
\]

\[
= \max\{|\lambda|: \lambda \text{ is an eigenvalue of } A\}
\]

\[
= \rho(A).
\]

**Theorem 2.** If

\[
H = \begin{bmatrix}
I & 0 & 0 & \cdots & 0 & B_1 \\
0 & I & 0 & \cdots & 0 & B_2 \\
0 & 0 & I & \cdots & 0 & B_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I & B_{m-1} \\
B_1^* & B_2^* & B_3^* & \cdots & B_{m-1}^* & I
\end{bmatrix}
\]

and

\[
G = \begin{bmatrix}
I & 0 & 0 & \cdots & 0 & A_1 \\
0 & I & 0 & \cdots & 0 & A_2 \\
0 & 0 & I & \cdots & 0 & A_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I & A_{m-1} \\
A_1^* & A_2^* & A_3^* & \cdots & A_{m-1}^* & I
\end{bmatrix}
\]

and both are positive semi-definite matrices, then

\[
H \square G = \begin{bmatrix}
I & 0 & 0 & \cdots & 0 & B_1A_1 \\
0 & I & 0 & \cdots & 0 & B_2A_2 \\
0 & 0 & I & \cdots & 0 & B_3A_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I & B_{m-1}A_{m-1} \\
B_1A_1^* & B_2A_2^* & B_3A_3^* & \cdots & B_{m-1}A_{m-1}^* & I
\end{bmatrix}
\]

is also positive semi-definite if \( A_iB_i = B_iA_i \) for \( i = 1, \ldots, m \).

**Proof.** Since \( H \) and \( G \) are positive semi-definite, we have

\[
\|B_i\|_{sp} \leq 1, \|A_i\|_{sp} \leq 1 \quad \text{for } i = 1, \ldots, m
\]

(3)
and also

\[
\begin{bmatrix}
B_1 B_1^* & B_1 B_2^* & \cdots & B_1 B_{m-1}^* \\
B_2 B_1^* & B_2 B_2^* & \cdots & B_2 B_{m-1}^* \\
\vdots & \vdots & \ddots & \vdots \\
B_{m-1} B_1^* & B_{m-1} B_2^* & \cdots & B_{m-1} B_{m-1}^*
\end{bmatrix} \leq 1.
\]

We know that \( H H G \) will be positive semi-definite if

\[
\rho\left(\begin{bmatrix}
B_1 A_1 \\
B_2 A_2 \\
\vdots \\
B_{m-1} A_{m-1}
\end{bmatrix} \begin{bmatrix}
B_1^* A_1^* & B_2^* A_2^* & \cdots & B_{m-1}^* A_{m-1}^*
\end{bmatrix}\right) \leq 1.
\]

Now the spectral radius of the matrix within the brackets is

\[
\rho\left(\begin{bmatrix}
B_1 A_1 B_1^* A_1^* & B_1 A_1 B_2^* A_2^* & \cdots & B_1 A_1 B_{m-1}^* A_{m-1}^* \\
B_2 A_2 B_1^* A_1^* & B_2 A_2 B_2^* A_2^* & \cdots & B_2 A_2 B_{m-1}^* A_{m-1}^* \\
\vdots & \vdots & \ddots & \vdots \\
B_{m-1} A_{m-1} B_1^* A_1^* & B_{m-1} A_{m-1} B_2^* A_2^* & \cdots & B_{m-1} A_{m-1} B_{m-1}^* A_{m-1}^*
\end{bmatrix}\right)
\]

\[
= \rho\left(\begin{bmatrix}
A_1 & 0 & 0 & \cdots & 0 \\
0 & A_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{m-1}
\end{bmatrix} \begin{bmatrix}
B_1 B_1^* & B_1 B_2^* & \cdots & B_1 B_{m-1}^* \\
B_2 B_1^* & B_2 B_2^* & \cdots & B_2 B_{m-1}^* \\
\vdots & \vdots & \ddots & \vdots \\
B_{m-1} B_1^* & B_{m-1} B_2^* & \cdots & B_{m-1} B_{m-1}^*
\end{bmatrix}\right)
\]

\[
\times \begin{bmatrix}
A_1^* & 0 & 0 & \cdots & 0 \\
0 & A_2^* & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{m-1}^*
\end{bmatrix}.
\]
By Lemma 1, this is
\[
\begin{bmatrix}
A_1 & 0 & 0 & \cdots & 0 \\
0 & A_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{m-1}
\end{bmatrix}
\begin{bmatrix}
B_1 B_1^* & B_1 B_2^* & \cdots & B_1 B_{m-1}^*
B_2 B_1^* & B_2 B_2^* & \cdots & B_2 B_{m-1}^*
\vdots & \vdots & \ddots & \vdots \\
B_{m-1} B_1^* & B_{m-1} B_2^* & \cdots & B_{m-1} B_{m-1}^*
\end{bmatrix}
\]
and by the submultiplicative property of the matrix norm it is
\[
\left\| \begin{bmatrix}
A_1^* & 0 & 0 & \cdots & 0 \\
0 & A_2^* & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{m-1}^*
\end{bmatrix} \right\|_{sp}
\]
\[
\times \left\| \begin{bmatrix}
B_1 B_1^* & B_1 B_2^* & \cdots & B_1 B_{m-1}^*
B_2 B_1^* & B_2 B_2^* & \cdots & B_2 B_{m-1}^*
\vdots & \vdots & \ddots & \vdots \\
B_{m-1} B_1^* & B_{m-1} B_2^* & \cdots & B_{m-1} B_{m-1}^*
\end{bmatrix} \right\|_{sp}
\]
\[
\leq 1 \text{ by the relationships (3) and (4). Hence } H \Box G \text{ is positive semi-definite. \qed}
\]

Unfortunately, the \( \Box \) product of positive definite matrices is not always positive definite if the blocks are greater than 1-by-1. See [4, p. 238] for an example of a positive definite \( H \) for which \( \Box \) product is not positive definite for \( n = 2 \). The following theorem gives a generalization of the Schur product theorem for special kinds of matrices.

**Theorem 5.** Let \( H = [H_{ij}] \) be a given positive semi-definite \( mn \)-by-\( mn \) matrix, assume that each \( H_{ij} \) is a normal \( n \)-by-\( n \) matrix for \( i, j = 1, \ldots, m \) and assume that the \( m^2 \) matrices \{\( H_{ij}; 1 \leq i, j \leq m \)\} are a commuting family. If \( H \) is positive semi-definite, then so is \( [H_{ij}^\alpha] \) for all \( \alpha = 1, 2, \ldots \). If, in addition each \( H_{ij} \) is positive semi-definite, then \( [H_{ij}^\alpha] \) is positive semi-definite for all real \( \alpha \geq mn - 2 \).

**Proof.** Since the \( m^2 \) normal matrices \{\( H_{ij}; 1 \leq i, j \leq m \)\} are a commuting family, they are simultaneously diagonalizable by a unitary matrix \( U \), i.e.,
\[
H_{ij} = U \Lambda_{ij} U^* \quad \text{for } i, j = 1, \ldots, m
\]
where each $\Lambda_{ij}$ is an $n$-by-$n$ diagonal matrix. Thus we can write

$$H = \begin{bmatrix}
U\Lambda_{11}U^* & U\Lambda_{12}U^* & \cdots & U\Lambda_{1m}U^* \\
U\Lambda_{21}U^* & U\Lambda_{22}U^* & \cdots & U\Lambda_{2m}U^* \\
\vdots & \vdots & \ddots & \vdots \\
U\Lambda_{m1}U^* & U\Lambda_{m2}U^* & \cdots & U\Lambda_{mm}U^*
\end{bmatrix} = T\Lambda T^*,$$

where

$$T = \begin{bmatrix}
U & 0 & 0 & \cdots & 0 \\
0 & U & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & U
\end{bmatrix}, \quad \Lambda = \begin{bmatrix}
\Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1m} \\
\Lambda_{21} & \Lambda_{22} & \cdots & \Lambda_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\Lambda_{m1} & \Lambda_{m2} & \cdots & \Lambda_{mm}
\end{bmatrix}.$$

Since $H$ is positive semi-definite and $T$ is unitary, $\Lambda$ is positive semi-definite. Again since $\Lambda$ is positive semi-definite the matrix $[\Lambda^\alpha_{ij}]$ is also positive semi-definite for $\alpha = 1, 2, \ldots$ by the Schur product theorem. Therefore, any matrix that is unitarily similar to this matrix is also positive semi-definite. Thus for $\alpha = 1, 2, \ldots$

$$T \begin{bmatrix}
\Lambda_{11}^\alpha & \Lambda_{12}^\alpha & \cdots & \Lambda_{1m}^\alpha \\
\Lambda_{21}^\alpha & \Lambda_{22}^\alpha & \cdots & \Lambda_{2m}^\alpha \\
\vdots & \vdots & \ddots & \vdots \\
\Lambda_{m1}^\alpha & \Lambda_{m2}^\alpha & \cdots & \Lambda_{mm}^\alpha
\end{bmatrix} T^{-1} = \begin{bmatrix}
U & 0 & 0 & \cdots & 0 \\
0 & U & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & U
\end{bmatrix} \begin{bmatrix}
\Lambda_{11}^\alpha & \Lambda_{12}^\alpha & \cdots & \Lambda_{1m}^\alpha \\
\Lambda_{21}^\alpha & \Lambda_{22}^\alpha & \cdots & \Lambda_{2m}^\alpha \\
\vdots & \vdots & \ddots & \vdots \\
\Lambda_{m1}^\alpha & \Lambda_{m2}^\alpha & \cdots & \Lambda_{mm}^\alpha
\end{bmatrix}^{-1} \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
U^* & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & U^*
\end{bmatrix}$$

is positive semi-definite.

If all $H_{ij}$'s are positive semi-definite then $\Lambda$ is a real symmetric positive semi-definite matrix with nonnegative entries. We know [1] $\Lambda^\alpha = [\Lambda^\alpha_{ij}]$ is also positive definite for $\alpha \geq mn - 2$. By the same argument as before it is easy to show that $[H^\alpha_{ij}]$ is positive definite for $\alpha \geq mn - 2$. \(\square\)

The last part of Theorem 5 can be viewed as a generalization of the result on fractional Hadamard powers on positive definite matrices by FitzGerald and Horn [1].

For the function $f(H_{ij}) = \|H_{ij}\|_{sp}$, there is not hope of proving that $H_f$ is always positive semi-definite whenever $H$ is positive semi-definite, because
when \( n = 1 \), \( H_f = |H| \) (entry-wise absolute values). See [3, Chapter 7.5, Problem 6] for an example of a positive definite \( H \) for which \( |H| \) is not positive definite. However, since the absolute value function does preserve positive definiteness for 2-by-2 matrices we can show that if \( f(H_{ij}) = \|H_{ij}\|_{sp} \), then \( H_f \) is positive semi-definite when \( H \) is positive semi-definite for \( m = 2 \), \( n \) arbitrary.

**Theorem 6.** If \( H = [H_{ij}] \), \( i, j = 1, 2 \), and \( f(H_{ij}) = \|H_{ij}\|_{sp} \), then \( H_f \) is positive semi-definite whenever \( H \) is positive semi-definite.

**Proof.** Since \( H \) is positive semi-definite we know that

\[
|x^* H_{11} x (y^* H_{22} y)| \geq |x^* H_{12} y|^2 \quad \text{for all} \ x, y \in \mathbb{C}^n.
\]

By definition

\[
\|H_{12}\|_{sp}^2 = \max_{\|v\| = 1} v^* H_{12}^* H_{12} v = z^* H_{12}^* H_{12} z
\]

for some \( z \in \mathbb{C}^n \). Let \( x = H_{12} z \) and \( y = z \) then from (7) we have

\[
(H_{12} z)^* H_{11} (H_{12} z) (z^* H_{22} z) \geq |(H_{12} z)^* H_{12} z|^2.
\]

If \( H_{12} \) is the \( n \)-by-\( n \) zero matrix, then \( H_f \) is positive semi-definite. Assume \( H_{12} \) is not a zero matrix. Therefore, \( (H_{12} z)^* (H_{12} z) \neq 0 \). If we divide both sides of the inequality (8) by \( (H_{12} z)^* (H_{12} z) \), we have

\[
\frac{(H_{12} z)^* H_{11} (H_{12} z)}{(H_{12} z)^* (H_{12} z)} (z^* H_{22} z) \geq \frac{(\|H_{12}\|_{sp}^2)^2}{(H_{12} z)^* (H_{12} z)}.
\]

Now

\[
\|H_{11}\|_{sp} \|H_{22}\|_{sp} \geq \frac{(H_{12} z)^* H_{11} (H_{12} z)}{(H_{12} z)^* (H_{12} z)} (z^* H_{22} z) \geq \|H_{12}\|_{sp}^2.
\]

This shows that \( H_f \) is also positive semi-definite.

The relationship (9) can also be written as

\[
\|H_{11}\|_{sp} \|H_{22}\|_{sp} \geq \rho(H_{11}) \rho(H_{22}) \geq \frac{(H_{12} z)^* H_{11} (H_{12} z)}{(H_{12} z)^* (H_{12} z)} (z^* H_{22} z) \geq \|H_{12}\|_{sp}^2 \geq \{\rho(H_{12})\}^2
\]

which leads us to the following theorem.

**Theorem 10.** If \( H = [H_{ij}], i, j = 1, 2 \), and \( f(H_{ij}) = \rho(H_{ij}) \), then \( H_f \) is positive semi-definite whenever \( H \) is positive semi-definite.

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