A NEW FAMILY
OF ENNEPER TYPE MINIMAL SURFACES

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Abstract. An Enneper type surface is a complete immersed minimal surface in \( \mathbb{R}^3 \) with only one end and finite total curvature. In this paper we construct a family of Enneper type surfaces of genus 1, total curvature \(-8(2n + 1)\pi, n = 0, 1, 2, \ldots \). We use the Weierstrass \( \wp \) elliptic function as a tool and also prove some results about \( \wp \) on a square torus.

1. Introduction

An Enneper type minimal surface is a complete immersed minimal surface with finite total Gauss curvature and only one end; (i.e., conformally it is a closed genus \( k \) Riemannian surface with one puncture). The simplest example is Enneper’s surface. It has genus 0 and total curvature \(-4\pi \). There is also a family of genus 0 examples with total curvature \(-4n\pi, n = 1, 2, 3, \ldots \). In [2] Chen and Gackstätter constructed genus 1 and genus 2 examples with total curvature \(-8\pi \) and \(-12\pi \). In [6] Wohlgemuth constructed a family of genus 1 examples with total curvature \(-4\pi(2n + 1) \) for \( n \geq 1 \). In this paper we will construct a family of genus 1 examples with total curvature \(-8\pi(2n + 1) \) for \( n \geq 0 \). Our main tools are Weierstrass representations for minimal surfaces and the Weierstrass elliptic function \( \wp \) associated to a lattice \( L = [1, \tau] \) in the complex plane \( \mathbb{C} \). The by-products of this study are some properties of the Weierstrass \( \wp \) function. Having not seen these properties in publication, we list them as a theorem in this paper.

2. Weierstrass Representation

A very important tool used in the construction of minimal surfaces is the Weierstrass representation formula. Here we state one version of it; for details see [4] and [5].
Proposition 1. Let $\overline{M}$ be a compact Riemann surface and $M = \overline{M} - \{p_1, \ldots, p_n\}$. Suppose $\overline{g}: \overline{M} \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function and $\eta$ is a meromorphic 1-form such that whenever $g = \overline{g}|M$ has a pole of order $k$, then $\eta$ has a zero of order $2k$ and $\eta$ has no other zeros on $M$. Let

$$\omega_1 = \frac{1}{2}(1 - g^2)\eta, \quad \omega_2 = \frac{i}{2}(1 + g^2)\eta, \quad \omega_3 = g\eta.$$ 

If for any closed curve $\gamma$ in $M$,

$$(1) \quad \text{Re} \int_{\gamma} \omega_i = 0, \text{ for } i = 1, 2, 3,$$

then the surface $S$, defined by $X: M \to \mathbb{R}^3$, is a regular minimal surface, where

$$X(z) = \text{Re} \left( \int_{z_0}^{z} \omega_1, \int_{z_0}^{z} \omega_2, \int_{z_0}^{z} \omega_3 \right).$$

Here, $z_0$ is a fixed point of $M$. Moreover, if at any deleted point $p_i$, one of $\omega_1$, $\omega_2$, $\omega_3$ has a pole of order at least 2, then $S$ is also complete. The total curvature of $S$ is

$$C(S) = -4\pi m,$$

where $m$ is the degree of $\overline{g}$.

Proof. See, for example, [4, pp. 112–113] and [5, p. 82], Theorem 9.2. □

3. Weierstrass $\wp$ elliptic function

Let $L = [\omega_1, \omega_2]$ be a lattice $\mathbb{C}$. Associated to each $L$ there is a doubly periodic meromorphic function, the Weierstrass $\wp$ function. It is

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right),$$

where $\omega = m\omega_1 + n\omega_2$ for all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ and $(m, n) \neq (0, 0)$. It is easy to see $\wp$ is an even function. Moreover, we have

Lemma 1.

1. $\wp^{(2k)}$ is an even function, $\wp^{(2k+1)}$ is an odd function, where $\wp^{(n)}$ denotes the nth derivative of $\wp$, $n \geq 0$.
2. $\wp^{(2k)} = P_k(\wp)$ and $\wp^{(2k+1)} = Q_k(\wp)\wp'$. Where $P_k$ and $Q_k$ are polynomials.
3. If $L = [1, \tau]$ and $\tau \in \mathbb{R}$ then $\wp(z + i) = \overline{\wp(z)}$ and $\wp(-z + 1) = \overline{\wp(z)}$. Also,

$$\int_{0}^{1} (\wp^{(n)})^2 \left( \frac{\tau}{2} + t \right) dt > 0,$$

for any $n \geq 0$. Furthermore,

$$\int_{0}^{1} (\wp^{(n)})^2 \left( \frac{\tau}{2} + t \right) \wp^m \left( \frac{\tau}{2} + t \right) dt$$
and
\[ \int_0^1 \varphi^m \left( \frac{\tau}{2} + t \right) dt \]
are nonzero real numbers, for any \( n, m \in \mathbb{Z} \) and \( n \geq 0 \).

4. If \( L = [1, i] \), \( \omega = \frac{1+i}{2} \), then \( \varphi(\omega + i\omega) = \overline{\varphi(\omega + \omega)} \), \( \varphi(\omega - i\omega) = -\overline{\varphi(\omega + \omega)} \), and \( \varphi(w + \omega) = \varphi(\omega + \omega) \). \( \varphi \) has a double pole at \( 0 \), a double zero at \( \omega \), and no other zeros or poles. Furthermore,

\[ \varphi^{(n)} \left( \frac{i}{2} + t \right) = -i^n \varphi^{(n)} \left( \frac{1}{2} + it \right). \]

\[ [\varphi^{(n)}]^2 \left( \frac{i}{2} + t \right) = (-1)^n [\varphi^{(n)}]^2 \left( \frac{1}{2} + it \right) \quad \text{for } 0 \leq t \leq 1. \]

Proof. See [1, pp. 658 and 631], and [6, pp. 17-18]. \( \square \)

**Minimal surfaces of Enneper type with genus 1**

**Theorem 1.** There is a family of Enneper type surfaces of genus 1 and total curvature \(-8\pi(2n+1), n = 0, 1, 2, \cdots \).

**Proof.** We consider the square lattice \( L = [1, i] \) in the complex plane \( \mathbb{C} \). By Lemma 1, we have \( \varphi^{(2k+1)}(z) = Q_k(\varphi(z))\varphi'(z) \), where \( Q_k(x) \) is a polynomial. Let \( Q_k(x) = x^n R_k(x) \) such that \( R_k(x) = a_0 + a_1x + \cdots + a_kx^k \) and \( a_0 \neq 0 \). Then the degree of \( Q_k \) is \( n + l = d \). We will denote \( n = n(k) \) to emphasize that \( n \) depends on \( k \). Since \( \varphi^{(2k+1)} \) has only one pole at \( 0 \) of order \( 2k+3 \), and \( Q_k(\varphi)\varphi' \) has only one pole at \( 0 \) of order \( 2d+3 \); hence \( d = k \) and \( n(k) \leq k \).

Let \( M = T - \{0\} = \mathbb{C}/L - \{0\} \). We choose

\[ g(z) = a \frac{\varphi^{(2k+1)}(z)}{\varphi^m(z)}, \]

where \( a \in \mathbb{C} \) is an nonzero constant to be determined and \( 0 \leq n(k) < m < k+1 \), \( k \geq 0 \). Then \( g \) has only poles at \( 0 \) of order \( 2(k-m)+3 \) and at \( \frac{1+i}{2} \) of order \( 2(m-n(k)) - 1 \). Hence we know that the degree of \( g \) is \( 2(k-m) + 3 + 2(m-n(k)) - 1 = 2(k-n(k)) + 2 \). We choose

\[ \eta = \varphi^{2(m-n(k))-1}(z) dz. \]

Then \( \eta \) is never zero on \( M \) except at \( \omega = \frac{1+i}{2} \), where \( g \) has a pole of order \( l = 2(m-n(k)) - 1 \), and \( \eta \) has an zero of order \( 4(m-n(k)) - 2 = 2l \). Let

\[ \Psi = \frac{[\varphi^{(2k+1)}]^2(z)}{\varphi^{2n(k)+1}(z)} dz = \frac{1}{a^2} g^2(z)\eta. \]

Since \( \varphi^{2(m-n(k))-1}(z) \) and \( [\varphi^{(2k+1)}]^2(z)/\varphi^{2n(k)+1}(z) \) are both even functions, they have no residues at \( 0 \). Now in the Weierstrass representation formula with
these \( g \) and \( \eta \), we have
\[
\omega_1 = \frac{1}{2}(\eta - a^2 \Psi),
\]
\[
\omega_2 = \frac{i}{2}(\eta + a^2 \Psi),
\]
\[
\omega_3 = g \eta = a \wp^{(2k+1)}(z) \wp^{m-2n(k)-1}(z) \, dz
\]
\[
= a \wp^{(2k+1)}(z) \wp^{m-2n(k)-1}(z) \, dz = a \wp^{m-n(k)-1}(z) R_\lambda(\wp(z)) \wp'(z) \, dz.
\]
Notice that \( \omega_3 \) has a pole at \( 0 \) of order greater or equal to 3. By Proposition 1, this Weierstrass representation will generate an Enneper type minimal surface if equation (1) of Proposition 1 is satisfied. Since \( \omega_3 \) is exact, we do not need to worry about its periods. Now let \( \gamma_1(t) = i/2 + t \), \( \gamma_2(t) = 1/2 + it \), \( 0 \leq t \leq 1 \). Then \( \gamma_1 \) and \( \gamma_2 \) are generators of the fundamental group of \( T = C/L \). Hence it is enough to prove that we can choose an \( a \) such that
\[
\text{Re} \int_{\gamma_i} \omega_j = 0,
\]
for \( i, j = 1, 2 \). Notice that by Lemma 1 we have
\[
\left[ \wp^{(2k+1)} \right]^2 \left( \frac{i}{2} + t \right) = -\left[ \wp^{(2k+1)} \right]^2 \left( \frac{1}{2} + it \right),
\]
\[
\wp' \left( \frac{i}{2} + t \right) = (-1)^l \wp' \left( \frac{1}{2} + it \right).
\]
Hence
\[
\int_{\gamma_2} \eta = \int_0^1 \wp^{2(m-n(k))-1} \left( \frac{1}{2} + it \right) \, dt
\]
\[
= -i \int_0^1 \wp^{2(m-n(k))-1} \left( \frac{i}{2} + t \right) \, dt = -i \int_{\gamma_1} \eta,
\]
that is,
\[
\int_{\gamma_2} \eta = -i \int_{\gamma_1} \eta.
\]
Also
\[
\int_{\gamma_2} \Psi = \int_0^1 \frac{\left[ \wp^{(2k+1)} \right]^2 \left( \frac{1}{2} + it \right)}{\wp^{2n(k)+1} \left( \frac{1}{2} + it \right)} \, dt
\]
\[
= i \int_0^1 \frac{\left[ \wp^{(2k+1)} \right]^2 \left( \frac{i}{2} + t \right)}{\wp^{2n(k)+1} \left( \frac{i}{2} + t \right)} \, dt = i \int_{\gamma_1} \Psi,
\]
that is,
\[
\int_{\gamma_2} \Psi = i \int_{\gamma_1} \Psi.
\]
By Lemma 1, we know that $\int_{\gamma_1} \eta$ and $\int_{\gamma_1} \Psi$ are nonzero real numbers, so let $a^2 = \int_{\gamma_1} \eta / \int_{\gamma_1} \Psi$; then $a^2 \in \mathbb{R}$ and $a^2 \neq 0$. We have

$$2 \int_{\gamma_1} \omega_1 = \int_{\gamma_1} \eta - a^2 \int_{\gamma_1} \Psi = 0,$$

$$2 \int_{\gamma_2} \omega_1 = -i \int_{\gamma_1} \eta - i a^2 \int_{\gamma_1} \Psi \in i\mathbb{R},$$

$$2 \int_{\gamma_1} \omega_2 = i \int_{\gamma_1} \eta + i a^2 \int_{\gamma_1} \Psi \in i\mathbb{R},$$

$$2 \int_{\gamma_2} \omega_2 = i \int_{\gamma_2} \eta + i a^2 \int_{\gamma_2} \Psi = \int_{\gamma_1} \eta - a^2 \int_{\gamma_1} \Psi = 0.$$

Hence $\text{Re} \int_{\gamma_1} \omega_j = 0$. By Proposition 1 we get a complete minimal surface with genus 1 and one end. Since the degree of $g$ is $2(k - n(k)) + 2$, the total curvature is $C(S) = -4\pi(2(k - n(k)) + 2)$. Let $d = k - n(k) + 1 \geq 1$. The only thing that remains to be proved is that $d$ can be any odd positive integer. The next proposition will complete the proof of this theorem. □

**Proposition 2.** The $n(k)$ defined in Theorem 2 satisfies

$$n(k) = \begin{cases} 
0 & \text{if } k \text{ is even}, \\
1 & \text{if } k \text{ is odd.}
\end{cases}$$

**Proof.** First we look at the formula in Lemma 1 stating that

$$\varphi^{(2k)} = P_k(\varphi), \quad \varphi^{(2k+1)} = Q_k(\varphi)\varphi'.$$

$P_k$ and $Q_k$ are polynomials. Let $Q_k(x) = a_0 + a_2 x + \cdots + a_{2k} x^{2k}$. We claim that

1. $a_j = 0$ if $k \not\equiv j \mod 2$, and $a_j \in \mathbb{R}$,

2. $a_0^{2k} \neq 0$, $a_1^{2k+1} \neq 0$.

Clearly claim 1 and claim 2 imply the proposition.

Now $\varphi'^2 = 4\varphi^3 - g_2 \varphi - g_3$, where $g_2$ and $g_3$ depend on the lattice $L = [1, \tau]$. When $\tau = i$, we have $g_3 = 0$, and $g_2 \in \mathbb{R}$, $g_3 \neq 0$ (see [3], Corollary 3, p. 40). We prove claim 1 by induction. Since $\varphi' = 1 \cdot \varphi'$, so $Q_0 = 1$, and $\varphi'' = 6\varphi^2 - g_2/2$, $\varphi''' = 12\varphi \varphi'$, so that $Q_1(x) = 12x$. Thus claim 1 is true for $k = 0$ and $k = 1$. Suppose claim 1 is true for $k = 2n$ and $2n + 1$, $n \geq 0$. Then

$$\varphi^{(4n+1)} = \varphi' \sum_{j=0}^{n} a_{2j} \varphi^{2j}, \quad \varphi^{(4n+3)} = \varphi' \sum_{j=0}^{n} a_{2j+1} \varphi^{2j+1}.$$
We have that
\[
\varphi^{(4n+4)} = \varphi' \sum_{j=0}^{n} (2j+1) a_{2j+1}^{2n+1} \varphi^{2j} + \varphi'' \sum_{j=0}^{n} a_{2j+1}^{2n+1} \varphi^{2j+1}
\]
\[
= (4\varphi^3 - g_2 \varphi) \sum_{j=0}^{n} (2j+1) a_{2j+1}^{2n+1} \varphi^{2j} + (6\varphi^2 - g_2/2) \sum_{j=0}^{n} a_{2j+1}^{2n+1} \varphi^{2j+1}
\]
\[
= \sum_{j=0}^{n} (4(2j+1) + 6) a_{2j+1}^{2n+1} \varphi^{2j+3} - \frac{g_2}{2} \sum_{j=0}^{n} (2(2j+1) + 1) a_{2j+1}^{2n+1} \varphi^{2j+1}.
\]
Hence,
\[
\varphi^{(4n+5)} = \varphi' \left\{ \sum_{j=0}^{n} (2j+3)(8j+10) a_{2j+1}^{2n+1} \varphi^{2j+2} - \frac{g_2}{2} \sum_{j=0}^{n} (2j+1)(4j+3) a_{2j+1}^{2n+1} \varphi^{2j} \right\}
\]
\[
= \varphi' \left\{ -\frac{3g_2}{2} a_1^{2n+1} + \sum_{j=0}^{n} (2j+1) \left[ (8j+2) a_{2j-1}^{2n+1} - \frac{g_2}{2} (4j+3) a_{2j+1}^{2n+1} \right] \varphi^{2j} + (2n+3)(8n+10) a_{2n+2}^{2n+1} \varphi^{2n+2} \right\}
\]
\[
= \varphi' \left( \sum_{j=0}^{n+1} a_{2j}^{2n+2} \varphi^{2j} \right).
\]

Since there are only even terms and all the \(a_{2j+1}^{2n+1}\) and \(j\) are real, so \(a_{2n+2}^{2n+2}\) is real and claim 1 is true for \(k = 2n + 2\). Also we can see by the computation that \(a_0^{2n+2} = -\frac{3}{2} g_2 a_1^{2n+1}\). Similarly,
\[
\varphi^{(4n+6)} = \varphi' \sum_{j=0}^{n+1} 2j a_{2j}^{2n+2} \varphi^{2j-1} + \varphi'' \sum_{j=0}^{n+1} a_{2j}^{2n+2} \varphi^{2j}
\]
\[
= \sum_{j=0}^{n+1} (8j+6) a_{2j}^{2n+2} \varphi^{2j+2} - \frac{g_2}{2} \sum_{j=0}^{n+1} (8j+1) a_{2j}^{2n+2} \varphi^{2j} = \sum_{j=0}^{n+1} b_{2j+2} \varphi^{2j+2},
\]
so
\[
\varphi^{(4n+7)} = \varphi' \sum_{j=0}^{n+1} (2j+2) b_{2j+2} \varphi^{2j+1} = \varphi' \sum_{j=0}^{n+1} a_{2j+1}^{2n+3} \varphi^{2j+1}.
\]
Hence \(k = 2n + 3\) is true for claim 1. This completes the proof of claim 1.

Notice that \(a_0^0 = 1\), \(a_1^1 = 12\), and \(a_0^{2n} = -\frac{3}{2} g_2 a_1^{2n-1}\) for \(n \geq 1\). So we need only to prove that \(a_0^{2n} \neq 0\). Let \(\omega = \frac{1+\sqrt{13}}{2}\). Since \(\varphi(\omega) = \varphi'(\omega) = 0\),
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In (2) of Lemma 1 set \( t = 1/2 \) and \( n = 4k \). We get \( \varphi^{(4k)}(\omega) = -\varphi^{(4k)}(\omega) \), so \( \varphi^{(4k)}(\omega) = 0 \). Also by \( \varphi^{(2k+1)} = \varphi' Q_k(\varphi) \) we know that \( \varphi^{(2k+1)}(\omega) = \varphi^{(2k+1)}(\omega + 1/2) = 0 \). If \( a_0^{2n} = 0 \), then \( \omega \) will be a zero of \( \varphi^{(4n+1)} \) of order at least 3. So if we prove that \( \varphi^{(4n+2)}(\omega) \neq 0 \), then \( \varphi^{(4n+1)} \) will have only a single zero at \( \omega \) and thus \( a_0^{2n} \neq 0 \). We will count the number of zeros for \( \varphi^{(4n+2)} \). Let \( I \) be the open interval \((0, 1/2)\). Then on \( \omega + I \) (for \( \omega + I \) we mean the interval \( 1/2 + i/2 + t, 0 < t < 1/2 \)), similarly \( \omega + iI \) means the interval \( 1/2 + i/2 + it, 0 < t < 1/2 \), \( \varphi \) and \( \varphi' \) are real, and \( \varphi(\omega + z) = \varphi(\omega + z), \varphi'(\omega + z) = \varphi'(\omega + z) \). By claim 1, any \( \varphi^{(k)}, k \geq 0 \), has these properties. If \( z = x + iy \), then

\[
\frac{\partial \text{Re} \varphi^{(k)}(\omega + z)}{\partial y} \bigg|_{y=0} = 0
\]

on \( \omega + I \). For a holomorphic function \( f \),

\[
f' = 2(\text{Re } f)_x = \frac{\partial \text{Re } f}{\partial x} - i \frac{\partial \text{Re } f}{\partial y},
\]

on \( \omega + I \) we have

\[
\varphi^{(k+1)}(\omega + x) = \frac{\partial \text{Re} \varphi^{(k)}(\omega + x)}{\partial x} - i \frac{\partial \text{Re } \varphi(\omega + x)}{\partial y} = \frac{\partial \varphi^{(k)}(\omega + x)}{\partial x}
\]

for all \( x \in I \). We claim that for any \( n \geq 0 \), \( \varphi^{(4n+2)} \) has at least \( n + 1 \) different zeros on \( \omega + I \). Since \( \varphi'(\omega + 1/2) = \varphi'(\omega) = 0 \), by Rolle’s theorem there is at least one \( x_0 \in I \) such that \( \varphi''(\omega + x_0) = 0 \). Hence for \( n = 0 \) the claim is true.

We now apply induction. Suppose for \( n = k \geq 0 \), \( \varphi^{(4k+2)} \) has at least \( k + 1 \) different zero points on \( \omega + I \). Then by Rolle’s theorem, \( \varphi^{(4k+3)} \) has at least \( k \) different zero points on \( \omega + I \). Note \( \varphi^{(4k+3)}(\omega) = \varphi^{(4k+3)}(\omega + 1/2) = 0 \), so on the closure of \( \omega + I \), \( \varphi^{(4k+3)} \) has at least \( k + 2 \) different zeros. Again by Rolle’s theorem \( \varphi^{(4k+4)} \) has at least \( k + 1 \) different zeros on \( \omega + I \). Because \( \varphi^{(4k+4)}(\omega) = 0 \), on the closure of \( \omega + I \), \( \varphi^{(4k+4)} \) has at least \( k + 2 \) different zeros. Hence \( \varphi^{(4k+5)} \) has at least \( k + 1 \) different zeros on \( \omega + I \). Again, \( \varphi^{(4k+5)}(\omega) = \varphi^{(4k+5)}(\omega + 1/2) = 0 \), so that \( \varphi^{(4k+6)} \) has at least \( k + 2 \) different zeros on \( \omega + I \). Hence we have thus proved this claim. For \( x \in I \), \( \varphi(\omega + x) = \varphi(\omega - x), \varphi'(\omega + x) = \varphi'(\omega - x) \). Hence \( \varphi^{(4n+2)} \) has at least \( 4n + 4 \) different zeros. But \( \deg \varphi^{(4n+2)} = 4n + 4 \), so \( \varphi^{(4n+2)} \) can have only \( 4n + 4 \) zeros. Thus we have found all of the zeros of \( \varphi^{(4n+2)} \). Because these zeros are in \( \omega + I, \omega - I, \omega + iI, \omega - iI \), we conclude that \( \varphi^{(4n+2)}(\omega) \neq 0 \). Thus the proof of this proposition is complete. \( \square \)

Remark 1. When setting \( k = 0 \), we get Chen and Gackstätter’s genus 1 example.

Remark 2. By the proof of Proposition 2, we get some properties of the Weierstrass \( \varphi \) function associated to \( L = [1, i] \). We list these properties as a separate theorem.
Theorem 2. The Weierstrass \( \wp \) function associated with \([1, i]\) has the following properties:

1. For \( n \geq 0 \), all the zeros of \( \wp^{(n)} \) are in the two lines \( \gamma_1(t) = i/2 + t \), \( \gamma_2(t) = 1/2 + it \), \( 0 \leq t \leq 1 \). The zeros are symmetric about \( \omega = \frac{1+i}{2} \).
2. For \( n \geq 0 \), \( \wp^{(4n)} \) has a double zero at \( \omega \), \( \wp^{(4n+3)} \) has a triple zero at \( \omega \). Any other zeros of \( \wp^{(n)} \) are simple.
3. \( \wp^{(4n)}(\omega) = \wp^{(4n+1)}(\omega) = \wp^{(4n+3)}(\omega) = 0 \) and \( \wp^{(4n+2)}(\omega) \neq 0 \) for \( n \geq 0 \).

Proof. Just as in the proof of Proposition 1 count the number of zeros on \( \omega + 1 \), using the three symmetries and \( \deg \wp^{(k)} = k + 2 \). Note that because \( \wp^{(4n-1)}(\omega) = \wp^{(4n)}(\omega) = \wp^{(4n+1)}(\omega) = 0 \), \( \wp^{(4n+2)}(\omega) \neq 0 \), we get claim 2. \( \square \)

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References