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$\mathscr{C}_1$ IS UNIFORMLY KADEC-KLEE

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ABSTRACT. A dual Banach space X is Kadec-Klee in the weak * topology if weak * and norm convergence of sequences coincide in the unit sphere of X. We shall consider a stronger, uniform version of this property. A dual Banach space X is uniformly Kadec-Klee in the weak * topology (UKK^*) if for each $\varepsilon > 0$ we can find a δ in $(0, 1)$ such that every weak *-compact, convex subset C of the unit ball of X whose measure of norm compactness exceeds ε must meet the $(1-\delta)$-ball of X. We show in this paper that $C_1(\mathscr{H})$, the space of trace class operators on an arbitrary infinite-dimensional Hilbert space $\mathscr{H}$ is UKK^*. Consequently $C_1(\mathscr{H})$ has weak *-normal structure. This answers affirmatively a question of A. T. Lau and P. F. Mah. From this it follows that $C_1(\mathscr{H})$ has the weak *-fixed point property.

0. INTRODUCTION

We establish in this paper that the trace class $(\mathscr{C}_1, \|\cdot\|_1)$ has the weak * uniform Kadec-Klee property (UKK^*). This strengthens a result of Arazy [1] and of Simon [11] who show that $(\mathscr{C}_1, \|\cdot\|_1)$ has the weak * Kadec-Klee property (KK^*). They in fact show more—a unitary matrix space (minimal symmetrically normed ideal) $\mathscr{C}_E$ is KK^* if the symmetric sequence space E corresponding to $\mathscr{C}_E$ under the Calkin correspondence is KK^*. The analogous question for the UKK^* property remains open.

The fact that $(\mathscr{C}_1, \|\cdot\|_1)$ is UKK^* has a number of interesting consequences $(\mathscr{C}_1, \|\cdot\|_1)$ must have weak * normal structure which implies the weak * fixed point property, by van Dulst and Sims [2, Theorem 3]. This answers affirmatively a question of Lau and Mah [8], who show that $(\mathscr{C}_1, \|\cdot\|_1)$ has quasi-weak * normal structure (which is weaker than weak * normal structure).

We note that a Banach space $(X, \|\cdot\|)$ having ε-UKK^* for some $\varepsilon \in (0, 1)$ is sufficient to give us that X has weak * normal structure. That $(\mathscr{C}_1, \|\cdot\|_1)$ enjoys the stronger UKK^* property shows us, by van Dulst and Sims [2, Theorem 4], that the Chebychev centres of weak *-compact, convex sets in $\mathscr{C}_1$ are compact.

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For more information on the concepts discussed above, all of which can be related to fixed point properties for nonexpansive mappings, see van Dulst and Sims [2] or Sims [12, **13**]. Also we remark that van Dulst and de Valk [3] have recently investigated Kadec-Klee properties in the sequence spaces l_M and h_M, where M is an Orlicz function.

I wish to thank Brailey Sims for suggesting this problem and for many stimulating conversations concerning it.

1. Preliminaries

Recall that if $(X, \|\cdot\|)$ is a Banach space, a standard measure of compactness of a nonempty subset S of X is given by

$$\gamma(S) := \sup\{\operatorname{sep}(\{x_n\}_{n\in N}) \colon \{x_n\}_{n\in N} \subseteq S\},$$

where $\operatorname{sep}(\{x_n\})_{n\in N} := \inf_{n\neq m} \|x_n - x_m\|$. We will refer to γ as "the measure of compactness".

Define now, for a dual space X, and for $S \subseteq X$,

$$\beta(S) := \sup\Big\{ \operatorname{sep}(\{x_n\}_{n\in N}) \colon \{x_n\}_{n\in N} \subseteq S \text{ and weak}^* - \lim_{n\in N} x_n = x \text{ for some } x \in X \Big\}.$$

Clearly $\beta(S) \leq \gamma(S)$. In some cases more can be said. The proof of the following lemma is straightforward and is therefore omitted.

1.1. **Lemma.** *Let X be a dual Banach space with a weak*-sequentially compact unit ball. Then $\beta(S) = \gamma(S)$ for every nonempty set $S \subseteq X$.*

A dual Banach space $(X, \|\cdot\|)$ is said to be Kadec-Klee in the weak*-topology (KK^*) if weak* and norm convergence of sequences coincide in the unit sphere of X. We remark that the analogous Kadec-Klee property for the weak topology of a general Banach space (denoted by KK) is often called the Radon-Riesz property or property H.

It is easily seen that a Banach space (dual Banach space) $(X, \|\cdot\|)$ is KK (KK^*) if whenever $x_n \to x$ as $n \to \infty$ in the weak (weak*) topology on X and $\lim_{n\in N} \|x_n\| = \|x\|$, it follows that $\lim_{n\in N} \|x_n - x\| = 0$.

We introduce below the properties ε-UKK^* and UKK^* in a manner due to Sims [13].

1.2. **Definition.** Suppose that X is a dual Banach space and $\varepsilon > 0$. X is ε-uniformly Kadec-Klee in the weak* topology (ε-UKK^*) if there exists $\delta \in (0, 1)$ such that whenever C is a weak* compact, convex subset of B_X with $\gamma(C) \geq \varepsilon$ it follows that $C \cap (1-\delta)B_X \neq \varnothing$.

X is uniformly Kadec-Klee in the weak* topology (UKK^*) if it is ε-UKK^* for all $\varepsilon > 0$.

By substituting the weak topology on an arbitrary Banach space for the weak* topology on a dual Banach space in the above definitions we get the definitions

of the properties ε-UKK (also called $WUKK$) and UKK. UKK was introduced by Huff [5] and ε-UKK, ε-UKK^* and UKK^* are due to van Dulst and Sims [2].

It is not hard to see that $UKK \Rightarrow KK$ and, for dual spaces, $UKK^* \Rightarrow KK^*$. Moreover, both implications are strict. Indeed set $X = (l_2 \oplus l_3 \oplus \cdots \oplus l_n \oplus \cdots)_2$. Then, as noted by Huff [5], X is reflexive, X is KK but X is not UKK. Further, X does not have an equivalent norm for which $(X, \|\cdot\|)$ is UKK. In contrast to this, from Kadec [6] and Klee [7], we have that every separable Banach space (dual Banach space) admits an equivalent norm (dual norm) such that X becomes KK (KK^*). This result may also be found in Lindenstrauss and Tzafriri [9].

If a Banach space $(X, \|\cdot\|)$ is uniformly convex (which forces X to be reflexive) then X must be UKK^*. The converse fails. $(l_1, \|\cdot\|_1)$ and $(\mathscr{C}_1, \|\cdot\|_1)$ are both UKK^* but fail to be uniformly convex (or even uniformly convexifiable).

We present below a simple characterization of dual Banach spaces with ε-UKK^* (and consequently also those with UKK^*) due to Sims [13].

1.3. **Proposition.** *Let $\varepsilon > 0$. A dual Banach space $(X, \|\cdot\|)$ over $\mathbf{K}$ ($= \mathbf{R}$ or $\mathbf{C}$) is ε-UKK^* if and only if there exists $k \in (0, 1)$ such that for all weak* continuous linear functionals f with $\|f\| = 1$ we have $\gamma(S[f, k]) \leq \varepsilon$, where*

$$S[f, k] := \{x \in B_X : \operatorname{Re} f(x) \geq k\}.$$

Throughout this paper $(\mathscr{H}, (\cdot, \cdot))$ will denote an arbitrary infinite-dimensional real or complex Hilbert space. We will denote the field by $\mathbf{K}$ i.e. $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. $(\mathscr{B}(\mathscr{H}), \|\cdot\|_\infty)$ is the Banach algebra of all bounded linear operators on $\mathscr{H}$, with the usual supremum norm. Any $B \in \mathscr{B}(\mathscr{H})$ has a polar decomposition B=U[B], where $[B] := (B^*B)^{1/2}$ is the absolute value of B and U is that partial isometry with initial set the closure of the range of $[B]$, and final set the closure of the range of B.

$\mathscr{C}_\infty(\mathscr{H})$ is the ideal of compact operators in $\mathscr{B}(\mathscr{H})$. Any $C \in \mathscr{C}_\infty$ has a Schmidt decomposition

$$C = \sum_{j=1}^{\infty} s_j(C) \varphi_j \otimes \overline{\psi_j},$$

where $s(C) := \{s_j(C)\}_j \subseteq c_0$ is the sequence of singular values of C, $\{\varphi_j\}_{j \in \mathbf{N}}$ and $\{\psi_j\}_{j \in \mathbf{N}}$ are orthonormal sequences in $\mathscr{H}$ and $f \otimes \bar{g} := (\cdot, g) f$ for all $f, g \in \mathscr{H}$.

$\mathscr{C}_1(\mathscr{H})$ is the ideal of all trace class operators in $\mathscr{B}(\mathscr{H})$ and $\|\cdot\|_1$ is given by

$$\|A\|_1 = \operatorname{tr}[A],$$

for all $A \in \mathscr{C}_1$. $(\mathscr{C}_1, \|\cdot\|_1)$ is isometrically isomorphic to the dual of the Banach space $(\mathscr{C}_\infty, \|\cdot\|_\infty)$, and the duality is given by the trace functional i.e. each $\varphi \in \mathscr{C}_\infty^*$ is of the form

$$\varphi : C \mapsto \operatorname{tr}(AC) : \mathscr{C}_\infty \to \mathbf{K}.$$

For more information on $\mathscr{B}(\mathscr{H})$, the trace class and trace duality see Schatten [10] or Gohberg and Kreĭn [4].

$\mathscr{B}_{\text{s.a.}}(\mathscr{H})$ is the class of all selfadjoint operators in $\mathscr{B}(\mathscr{H})$, and it is partially ordered by the quadratic form ordering $\leq$ i.e. for B, $B' \in \mathscr{B}_{\text{s.a.}}$, $B \leq B'$ if for all $f \in \mathscr{H}$

$$(Bf, f) \leq (B'f, f).$$

For a family $\{B_\tau\}_\tau$ in $\mathscr{B}_{\text{s.a.}}$ and $B \in \mathscr{B}_{\text{s.a.}}$, we denote by $B_\tau \uparrow_\tau B (B_\tau \downarrow_\tau B)$ the fact that $\{B_\tau\}_\tau$ is upwards directed (downwards directed) to B.

2. $\mathscr{C}_1$ IS UKK^*

We will need the following lemma. The proof, using the Schmidt decomposition mentioned above, is straightforward and is therefore omitted.

2.1. **Lemma.** *Let $\{Q_\tau\}_\tau$ be a family of orthogonal projections in $\mathscr{B}(\mathscr{H})$ with $Q_\tau \downarrow_\tau 0$ and $C \in \mathscr{C}_\infty$. Then*

$$\lim_\tau \|CQ_\tau\|_\infty = 0.$$

The proposition below, upon which our main result depends, is due to Arazy [1]. We include a proof for the sake of completeness.

2.2. **Proposition.** *Let $A \in \mathscr{C}_1$, let P be an orthogonal projection in $\mathscr{B}(\mathscr{H})$ and $Q = I - P$. Then*

$$\text{(a)} \qquad \|A\|_1^2 \geq \|PAP\|_1^2 + \|PAQ\|_1^2 + \|QAP\|_1^2 + \|QAQ\|_1^2$$

and

$$\text{(b)} \qquad \|PAQ\|_1 + \|QAP\|_1 + \|QAQ\|_1 \leq \sqrt{3}(\|A\|_1^2 - \|PAP\|_1^2)^{1/2}.$$

Proof. Clearly (b) follows from (a). To prove (a), we begin by letting $A = U[A]$ be the polar decomposition of A. Note that $[A^*] = U[A]U^*$.

We remark that

$$(1) \qquad \text{if } 0 \leq X \in \mathscr{C}_1 \text{ and } 0 \leq Y \leq Z \in \mathscr{B}(\mathscr{H}), \text{ then } \operatorname{tr}(XY) \leq \operatorname{tr}(XZ).$$

Now consider PAQ. Let $PAQ = V[PAQ]$ be its polar decomposition. We have that

$$\begin{aligned}
\|PAQ\|_1^2 = (\operatorname{tr}[PAQ])^2 &= (\operatorname{tr}(V^*PAQ))^2, \\
&= (\operatorname{tr}(V^*PU[A]Q))^2 = (\operatorname{tr}(S^*T))^2, \\
&\qquad \text{where } S = [A]^{1/2}U^*PV \text{ and } T = [A]^{1/2}Q, \\
&\leq \operatorname{tr}(S^*S)\operatorname{tr}(T^*T), \quad \text{by the Cauchy-Schwarz inequality}, \\
&= \operatorname{tr}(V^*PU[A]U^*PV)\operatorname{tr}(Q[A]Q) \\
&= \operatorname{tr}(P[A^*]PVV^*)\operatorname{tr}(Q[A]Q) \\
&\leq \operatorname{tr}(P[A^*]P)\operatorname{tr}(Q[A]Q),
\end{aligned}$$

by (1) above, as VV^* is an orthogonal projection and so $VV^* \leq I$. So we have

$$\|PAQ\|_1^2 \leq \operatorname{tr}(P[A^*]P)\operatorname{tr}(Q[A]Q),$$

and of course we also have three analogous upper estimates for each of $\|QAP\|_1^2$, $\|PAP\|_1^2$ and $\|QAQ\|_1^2$.

Consequently,

$$\begin{aligned}\|PAP\|_1^2 + \|PAQ\|_1^2 + \|QAP\|_1^2 + \|QAQ\|_1^2 \\ &\leq (\operatorname{tr}(P[A]P) + \operatorname{tr}(Q[A]Q))(\operatorname{tr}(P[A^*]P) + \operatorname{tr}(Q[A^*]Q)) \\ &= (\operatorname{tr}[A])(\operatorname{tr}[A^*]) = \|A\|_1\|A^*\|_1 \\ &= \|A\|_1^2.\end{aligned}$$

2.3. **Lemma.** *$\mathscr{C}_1$ has a weak* sequentially compact unit ball.*

Proof. Any sequence $\{A_n\}_{n\in\mathbf{N}}$ in $\mathscr{C}_1 = \mathscr{C}_1(\mathscr{H})$ lives essentially in $\mathscr{C}_1(\mathscr{H}_0)$, where $\mathscr{H}_0$ is the separable Hilbert space

$$\operatorname{closure}\left(\operatorname{span}\left(\bigcup_{n=1}^{\infty}(\operatorname{range}(A_n) \cup \operatorname{range}(A_n^*))\right)\right);$$

while the unit ball of $\mathscr{C}_1(\mathscr{H}_0)$ is weak* sequentially compact since $\mathscr{C}_1(\mathscr{H}_0) = \mathscr{C}_\infty^*(\mathscr{H}_0)$ and $\mathscr{C}_\infty(\mathscr{H}_0)$ is separable. From this, using the trace functional, which (as noted above) gives the duality between $\mathscr{C}_1(\mathscr{H})$ and $\mathscr{C}_\infty(\mathscr{H})$, the result follows. □

We now present the main result. The argument is similar to that of Brailey Sims in [13] for the case of $(l_1, \|\cdot\|_1)$. We use the characterization of UKK^* that comes from Proposition 1.3.

2.4. **Theorem.** *$(\mathscr{C}_1, \|\cdot\|_1)$ is UKK^*.*

Proof. Fix $k \in (0, 1)$ and then fix $C \in \mathscr{C}_\infty$ with $\|C\|_\infty = 1$. Choose a family $\{P_\tau\}_\tau$ of finite-dimensional orthogonal projections with $P_\tau \uparrow_\tau I$. Let $Q_\tau = I - P_\tau$ for each τ and note that $Q_\tau \downarrow_\tau 0$.

Consider $E \in S[C, k]$ and define for each τ,

$$\alpha_\tau(E) = \|P_\tau E Q_\tau\|_1 + \|Q_\tau E P_\tau\|_1 + \|Q_\tau E Q_\tau\|_1.$$

By Proposition 2.2, for each τ,

$$\alpha_\tau(E) \leq \sqrt{3}(1 - \|P_\tau E P_\tau\|_1^2)^{1/2}. \tag{1}$$

Now,

$$\begin{aligned}\|P_\tau E P_\tau\|_1 &\geq |\operatorname{tr}(P_\tau E P_\tau C)| \geq \operatorname{Retr}(E P_\tau C P_\tau) \\ &= \operatorname{Retr}(EC) - \operatorname{Retr}(E P_\tau C Q_\tau) - \operatorname{Retr}(E Q_\tau C P_\tau) - \operatorname{Retr}(E Q_\tau C Q_\tau) \\ &\geq k - \|E\|_1\|P_\tau C Q_\tau\|_\infty - \|E\|_1\|Q_\tau C P_\tau\|_\infty - \|E\|_1\|Q_\tau C Q_\tau\|_\infty \\ &\geq k - \|C Q_\tau\|_\infty - \|Q_\tau C\|_\infty - \|C Q_\tau\|_\infty \\ &\geq k - 3\max\{\|C Q_\tau\|_\infty, \|C^* Q_\tau\|_\infty\}.\end{aligned}$$

Let us fix $\eta \in (0, k)$. As C and C^* are compact operators, by Lemma 2.1 we have that there exists τ_0 such that for all τ with $Q_\tau \leq Q_{\tau_0}$,

$$3\max\{\|CQ_\tau\|_\infty, \|C^*Q_\tau\|_\infty\} < \eta.$$

From inequality (1), for each $E \in S[C, k]$ and for every τ with $Q_\tau \leq Q_{\tau_0}$,

$$\alpha_\tau(E) \leq \sqrt{3}(1-(k-\eta)^2)^{1/2}. \tag{2}$$

Consider an arbitrary sequence $\{A_n\}_{n\in\mathbf{N}} \subseteq S[C, k]$. From Lemma 2.3 we know that $\mathscr{B}_{\mathscr{C}_1}$ is weak* sequentially compact. We wish to estimate $\gamma(S[C, k])$ and so by Lemma 1.1 there is no loss of generality in assuming that $\{A_n\}_{n\in\mathbf{N}}$ is weak* convergent.

Note that since weak* and norm convergence coincide on finite-dimensional subspaces of a dual Banach space, we have

$$\|P_{\tau_0}(A_n - A_m)P_{\tau_0}\|_1 \to 0 \quad \text{as } n, m \to \infty. \tag{3}$$

Consequently, for each $n, m \in \mathbf{N}$,

$$\begin{aligned}\|A_n - A_m\|_1 &\leq \|P_{\tau_0}(A_n - A_m)P_{\tau_0}\|_1 + \alpha_{\tau_0}(A_n) + \alpha_{\tau_0}(A_m)\\ &\leq \|P_{\tau_0}(A_n - A_m)P_{\tau_0}\|_1 + 2\sqrt{3}(1-(k-\eta)^2)^{1/2},\end{aligned}$$

from inequality (2); and hence using (3) we see that

$$\inf_{n\neq m}\|A_n - A_m\|_1 \leq 2\sqrt{3}(1-(k-\eta)^2)^{1/2}.$$

It is clear now that

$$\gamma(S[C, k]) \leq 2\sqrt{3}(1-(k-\eta)^2)^{1/2},$$

and since $\eta \in (0, k)$ is arbitrary, we have

$$\gamma(S[C, k]) \leq 2\sqrt{3}(1-k^2)^{1/2}. \tag{4}$$

Finally, if $\varepsilon \in (0, 2\sqrt{3})$ is given, choose

$$k = (1-(\varepsilon/2\sqrt{3})^2)^{1/2}.$$

$k \in (0, 1)$ and inequality (4) gives us that for each $C \in S_{\mathscr{C}_\infty}$,

$$\gamma(S[C, k]) \leq \varepsilon. \quad \square$$

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