**C** IS UNIFORMLY KADEC-KLEE

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Abstract. A dual Banach space $X$ is Kadec-Klee in the weak* topology if weak* and norm convergence of sequences coincide in the unit sphere of $X$. We shall consider a stronger, uniform version of this property. A dual Banach space $X$ is uniformly Kadec-Klee in the weak* topology (UKK*) if for each $\varepsilon > 0$ we can find a $\delta$ in $(0, 1)$ such that every weak*-compact, convex subset $C$ of the unit ball of $X$ whose measure of norm compactness exceeds $\varepsilon$ must meet the $(1 - \delta)$-ball of $X$. We show in this paper that $C_1(\mathcal{H})$, the space of trace class operators on an arbitrary infinite-dimensional Hilbert space $\mathcal{H}$ is UKK*. Consequently $C_1(\mathcal{H})$ has weak*-normal structure. This answers affirmatively a question of A. T. Lau and P. F. Mah. From this it follows that $C_1(\mathcal{H})$ has the weak*-fixed point property.

0. Introduction

We establish in this paper that the trace class $(\mathcal{C}_1, \| \cdot \|_1)$ has the weak* uniform Kadec-Klee property (UKK*). This strengthens a result of Arazy [1] and of Simon [11] who show that $(\mathcal{C}_1, \| \cdot \|_1)$ has the weak* Kadec-Klee property (KK*). They in fact show more—a unitary matrix space (minimal symmetrically normed ideal) $\mathcal{E}_E$ is KK* if the symmetric sequence space $E$ corresponding to $\mathcal{E}_E$ under the Calkin correspondence is KK*. The analogous question for the UKK* property remains open.

The fact that $(\mathcal{C}_1, \| \cdot \|_1)$ is UKK* has a number of interesting consequences $(\mathcal{C}_1, \| \cdot \|_1)$ must have weak* normal structure which implies the weak* fixed point property, by van Dulst and Sims [2, Theorem 3]. This answers affirmatively a question of Lau and Mah [8], who show that $(\mathcal{C}_1, \| \cdot \|_1)$ has quasi-weak* normal structure (which is weaker than weak* normal structure).

We note that a Banach space $(X, \| \cdot \|)$ having $\varepsilon$-UKK* for some $\varepsilon \in (0, 1)$ is sufficient to give us that $X$ has weak* normal structure. That $(\mathcal{C}_1, \| \cdot \|_1)$ enjoys the stronger UKK* property shows us, by van Dulst and Sims [2, Theorem 4], that the Chebychev centres of weak*-compact, convex sets in $\mathcal{C}_1$ are compact.
For more information on the concepts discussed above, all of which can be related to fixed point properties for nonexpansive mappings, see van Dulst and Sims [2] or Sims [12, 13]. Also we remark that van Dulst and de Valk [3] have recently investigated Kadec-Klee properties in the sequence spaces $l_M$ and $h_M$, where $M$ is an Orlicz function.

I wish to thank Brailey Sims for suggesting this problem and for many stimulating conversations concerning it.

1. Preliminaries

Recall that if $(X, \| \cdot \|)$ is a Banach space, a standard measure of compactness of a nonempty subset $S$ of $X$ is given by

$$\gamma(S) := \sup \{ \text{sep}(\{x_n\}_n) : \{x_n\}_n \subseteq S \},$$

where $\text{sep}(\{x_n\}_n) := \inf_{n \neq m} \|x_n - x_m\|$. We will refer to $\gamma$ as “the measure of compactness”.

Define now, for a dual space $X$, and for $S \subseteq X$,

$$\beta(S) := \sup \left\{ \text{sep}(\{x_n\}_n) : \{x_n\}_n \subseteq S \right\},$$

and $\text{weak}^* - \lim \limits_{n \to \infty} x_n = x$ for some $x \in X$.

Clearly $\beta(S) \leq \gamma(S)$. In some cases more can be said. The proof of the following lemma is straightforward and is therefore omitted.

1.1. Lemma. Let $X$ be a dual Banach space with a weak*-sequentially compact unit ball. Then $\beta(S) = \gamma(S)$ for every nonempty set $S \subseteq X$.

A dual Banach space $(X, \| \cdot \|)$ is said to be Kadec-Klee in the weak*-topology ($KK^*$) if weak* and norm convergence of sequences coincide in the unit sphere of $X$. We remark that the analogous Kadec-Klee property for the weak topology of a general Banach space (denoted by $KK$) is often called the Radon-Riesz property or property $H$.

It is easily seen that a Banach space (dual Banach space) $(X, \| \cdot \|)$ is $KK$ ($KK^*$) if whenever $x_n \to x$ as $n \to \infty$ in the weak (weak*) topology on $X$ and $\lim_{n \to \infty} \|x_n\| = \|x\|$, it follows that $\lim_{n \to \infty} \|x_n - x\| = 0$.

We introduce below the properties $\varepsilon$-$UKK^*$ and $UKK^*$ in a manner due to Sims [13].

1.2. Definition. Suppose that $X$ is a dual Banach space and $\varepsilon > 0$. $X$ is $\varepsilon$-uniformly Kadec-Klee in the weak* topology ($\varepsilon$-$UKK^*$) if there exists $\delta \in (0, 1)$ such that whenever $C$ is a weak* compact, convex subset of $B_X$ with $\gamma(C) \geq \varepsilon$ it follows that $C \cap (1 - \delta)B_X \neq \emptyset$.

$X$ is uniformly Kadec-Klee in the weak* topology ($UKK^*$) if it is $\varepsilon$-$UKK^*$ for all $\varepsilon > 0$.

By substituting the weak topology on an arbitrary Banach space for the weak* topology on a dual Banach space in the above definitions we get the definitions
of the properties $\varepsilon$-UKK (also called $WUKK$) and UKK. UKK was introduced by Huff [5] and $\varepsilon$-UKK, $\varepsilon$-UKK* and UKK* are due to van Dulst and Sims [2].

It is not hard to see that $UKK \Rightarrow KK$ and, for dual spaces, $UKK^* \Rightarrow KK^*$. Moreover, both implications are strict. Indeed set $X = (l_2 \oplus l_3 \oplus \cdots \oplus l_n \oplus \cdots)_2$. Then, as noted by Huff [5], $X$ is reflexive, $X$ is KK but $X$ is not UKK. Further, $X$ does not have an equivalent norm for which $(X, \| \cdot \|)$ is UKK. In contrast to this, from Kadec [6] and Klee [7], we have that every separable Banach space (dual Banach space) admits an equivalent norm (dual norm) such that $X$ becomes $KK$ ($KK^*$). This result may also be found in Lindenstrauss and Tzafriri [9].

If a Banach space $(X, \| \cdot \|)$ is uniformly convex (which forces $X$ to be reflexive) then $X$ must be $UKK^*$. The converse fails. $(l_1, \| \cdot \|)$ and $(C_1, \| \cdot \|)$ are both $UKK^*$ but fail to be uniformly convex (or even uniformly convexifiable).

We present below a simple characterization of dual Banach spaces with $\varepsilon$-UKK* (and consequently also those with $UKK^*$) due to Sims [13].

1.3. Proposition. Let $\varepsilon > 0$. A dual Banach space $(X, \| \cdot \|)$ over $K$ ( = $\mathbb{R}$ or $\mathbb{C}$) is $\varepsilon$-UKK* if and only if there exists $k \in (0, 1)$ such that for all weak* continuous linear functionals $f$ with $\|f\| = 1$ we have $\gamma(S[f, k]) \leq \varepsilon$, where

$$S[f, k] := \{x \in B_X : \text{Re } f(x) \geq k\}.$$

Throughout this paper $(\mathcal{H}, (\cdot, \cdot))$ will denote an arbitrary infinite-dimensional real or complex Hilbert space. We will denote the field by $K$ i.e. $K = \mathbb{R}$ or $\mathbb{C}$. $(B(\mathcal{H}), \| \cdot \|_\infty)$ is the Banach algebra of all bounded linear operators on $\mathcal{H}$, with the usual supremum norm. Any $B \in B(\mathcal{H})$ has a polar decomposition $B = U[B]$, where $[B] := (B^*B)^{1/2}$ is the absolute value of $B$ and $U$ is that partial isometry with initial set the closure of the range of $[B]$, and final set the closure of the range of $B$.

$C_\infty(\mathcal{H})$ is the ideal of compact operators in $B(\mathcal{H})$. Any $C \in C_\infty$ has a Schmidt decomposition

$$C = \sum_{j=1}^{\infty} s_j(C) \varphi_j \otimes \psi_j,$$

where $s(C) := \{s_j(C)\}_{j} \subseteq c_0$ is the sequence of singular values of $C$, $\{\varphi_j\}_{j \in \mathbb{N}}$ and $\{\psi_j\}_{j \in \mathbb{N}}$ are orthonormal sequences in $\mathcal{H}$ and $f \otimes g := (\cdot, g)f$ for all $f, g \in \mathcal{H}$.

$C_1(\mathcal{H})$ is the ideal of all trace class operators in $B(\mathcal{H})$ and $\| \cdot \|_1$ is given by

$$\|A\|_1 = \text{tr}[A],$$

for all $A \in C_1$. $(C_1, \| \cdot \|_1)$ is isometrically isomorphic to the dual of the Banach space $(C_\infty, \| \cdot \|_\infty)$, and the duality is given by the trace functional i.e. each $\varphi \in C_\infty^*$ is of the form

$$\varphi : C \mapsto \text{tr}(AC) : C_\infty \to K.$$
For more information on $\mathcal{B}(\mathcal{H})$, the trace class and trace duality see Schatten [10] or Gohberg and Krein [4].

$\mathcal{B}_{\text{s.a.}}(\mathcal{H})$ is the class of all selfadjoint operators in $\mathcal{B}(\mathcal{H})$, and it is partially ordered by the quadratic form ordering $\leq$ i.e. for $B, B' \in \mathcal{B}_{\text{s.a.}}$, $B \leq B'$ if for all $f \in \mathcal{H}$

$$(Bf, f) \leq (B'f, f).$$

For a family $\{B_t\}_t$ in $\mathcal{B}_{\text{s.a.}}$ and $B \in \mathcal{B}_{\text{s.a.}}$, we denote by $B_t \uparrow_t B(B_t \downarrow_t B)$ the fact that $\{B_t\}_t$ is upwards directed (downwards directed) to $B$.

2. $\mathcal{C}$ is UKK$^*$

We will need the following lemma. The proof, using the Schmidt decomposition mentioned above, is straightforward and is therefore omitted.

2.1. Lemma. Let $\{Q_t\}_t$ be a family of orthogonal projections in $\mathcal{B}(\mathcal{H})$ with $Q_t \downarrow_t 0$ and $C \in \mathcal{C}$. Then

$$\lim_{t} \|CQ_t\|_{\infty} = 0.$$

The proposition below, upon which our main result depends, is due to Arazy [1]. We include a proof for the sake of completeness.

2.2. Proposition. Let $A \in \mathcal{C}$, let $P$ be an orthogonal projection in $\mathcal{B}(\mathcal{H})$ and $Q = I - P$. Then

(a) $\|A\|_1^2 \geq \|PAP\|_1^2 + \|PAQ\|_1^2 + \|QAP\|_1^2 + \|QAQ\|_1^2$

and

(b) $\|PAP\|_1 + \|PAQ\|_1 + \|QAP\|_1 \leq \sqrt{3}(\|A\|_1^2 - \|PAP\|_1^2)^{1/2}$.

Proof. Clearly (b) follows from (a). To prove (a), we begin by letting $A = U[A]$ be the polar decomposition of $A$. Note that $[A^*] = U[A]U^*$.

We remark that

$$\text{(1)} \quad \text{if } 0 \leq X \in \mathcal{C} \text{ and } 0 \leq Y \leq Z \in \mathcal{B}(\mathcal{H}), \text{ then } \text{tr}(XY) \leq \text{tr}(XZ).$$

Now consider $PAQ$. Let $PAQ = V[PAQ]$ be its polar decomposition. We have that

$$\|PAQ\|_1^2 = (\text{tr}[PAQ])^2 = (\text{tr}(V^*PAQ))^2,$$

where $S = [A]^{1/2}U^*PV$ and $T = [A]^{1/2}Q$, $\leq \text{tr}(S^*S)\text{tr}(T^*T), \text{ by the Cauchy-Schwarz inequality,}$

$$\leq \text{tr}(V^*PU[A]U^*PV) \text{tr}(Q[A]Q)$$

$$= \text{tr}(P[A^*]PVV^*) \text{tr}(Q[A]Q)$$

$$\leq \text{tr}(P[A^*]) \text{tr}(Q[A]Q),$$
by (1) above, as $VV^*$ is an orthogonal projection and so $VV^* \leq I$. So we have
\[
\|PAQ\|^2 \leq \text{tr}(PA^*P) \text{tr}(QAQ),
\]
and of course we also have three analogous upper estimates for each of $\|QAP\|^2$, $\|PAP\|^2$ and $\|QAQ\|^2$.

Consequently,
\[
\|PAP\|^2 + \|PAQ\|^2 + \|QAP\|^2 + \|QAQ\|^2 \\
\leq (\text{tr}(PA^*P) + \text{tr}(QAQ))(\text{tr}(PA^*P) + \text{tr}(QA^*Q)) \\
= (\text{tr}[A])(\text{tr}[A^*]) = \|A\|_1 \|A^*\|_1 \\
= \|A\|^2.
\]

2.3. Lemma. $C_1$ has a weak* sequentially compact unit ball.

Proof. Any sequence $\{A_n\}_{n \in \mathbb{N}}$ in $C_1 = C_1(\mathcal{H})$ lives essentially in $C_1(\mathcal{H}_0)$, where $\mathcal{H}_0$ is the separable Hilbert space
\[
\text{closure} \left( \text{span} \left( \bigcup_{n=1}^{\infty} (\text{range}(A_n) \cup \text{range}(A_n^*)) \right) \right);
\]
while the unit ball of $C_1(\mathcal{H}_0)$ is weak* sequentially compact since $C_1(\mathcal{H}_0) = C_\infty(\mathcal{H}_0)$ and $C_\infty(\mathcal{H}_0)$ is separable. From this, using the trace functional, which (as noted above) gives the duality between $C_1(\mathcal{H})$ and $C_\infty(\mathcal{H})$, the result follows. $\Box$

We now present the main result. The argument is similar to that of Brailey Sims in [13] for the case of $(l_1, \| \cdot \|_1)$. We use the characterization of UKK* that comes from Proposition 1.3.

2.4. Theorem. $(C_1, \| \cdot \|_1)$ is UKK*.

Proof. Fix $k \in (0, 1)$ and then fix $C \in C_\infty$ with $\|C\|_\infty = 1$. Choose a family $\{P_\tau\}_\tau$ of finite-dimensional orthogonal projections with $P_\tau \uparrow 1$. Let
\[
Q_\tau = I - P_\tau \text{ for each } \tau \text{ and note that } Q_\tau \downarrow 0.
\]

Consider $E \in S[C, k]$ and define for each $\tau$,
\[
\alpha_\tau(E) = \|P_\tau EQ_\tau\|_1 + \|Q_\tau EP_\tau\|_1 + \|Q_\tau EQ_\tau\|_1.
\]

By Proposition 2.2, for each $\tau$,
\[
(1) \quad \alpha_\tau(E) \leq \sqrt{3}(1 - \|P_\tau EP_\tau\|_1^2)^{1/2}.
\]

Now,
\[
\|P_\tau EP_\tau\|_1 \geq |\text{tr}(P_\tau EP_\tau C)| \geq \text{Retr}(EP_\tau CP_\tau) \\
= \text{Retr}(EC) - \text{Retr}(EP_\tau CQ_\tau) - \text{Retr}(EQ_\tau CP_\tau) - \text{Retr}(EQ_\tau CQ_\tau) \\
\geq k - \|E\|_1 \|P_\tau CQ_\tau\|_\infty - \|E\|_1 \|Q_\tau CP_\tau\|_\infty - \|E\|_1 \|Q_\tau CQ_\tau\|_\infty \\
\geq k - \|CQ_\tau\|_\infty - \|Q_\tau C\|_\infty - \|CQ_\tau\|_\infty \\
\geq k - 3 \max\{\|CQ_\tau\|_\infty, \|C^* Q_\tau\|_\infty\}.
\]
Let us fix $\eta \in (0, k)$. As $C$ and $C^*$ are compact operators, by Lemma 2.1 we have that there exists $\tau_0$ such that for all $\tau$ with $Q_\tau \leq Q_{\tau_0}$,

$$3 \max \{\|CQ_\tau\|_\infty, \|C^*Q_\tau\|_\infty\} < \eta.$$ 

From inequality (1), for each $E \in S[C, k]$ and for every $\tau$ with $Q_\tau \leq Q_{\tau_0}$,

$$\alpha_\tau(E) \leq \sqrt{3}(1 - (k - \eta)^2)^{1/2}. \tag{2}$$

Consider an arbitrary sequence $\{A_n\}_{n \in \mathbb{N}} \subseteq S[C, k]$. From Lemma 2.3 we know that $\mathcal{B}_{\mathfrak{C}}$ is weak* sequentially compact. We wish to estimate $\gamma(S[C, k])$ and so by Lemma 1.1 there is no loss of generality in assuming that $\{A_n\}_{n \in \mathbb{N}}$ is weak* convergent.

Note that since weak* and norm convergence coincide on finite-dimensional subspaces of a dual Banach space, we have

$$\|P_{\tau_0}(A_n - A_m)P_{\tau_0}\|_1 \to 0 \quad \text{as } n, m \to \infty. \tag{3}$$

Consequently, for each $n, m \in \mathbb{N}$,

$$\|A_n - A_m\|_1 \leq \|P_{\tau_0}(A_n - A_m)P_{\tau_0}\|_1 + \alpha_{\tau_0}(A_n) + \alpha_{\tau_0}(A_m)$$

$$\leq \|P_{\tau_0}(A_n - A_m)P_{\tau_0}\|_1 + 2\sqrt{3}(1 - (k - \eta)^2)^{1/2},$$

from inequality (2); and hence using (3) we see that

$$\inf_{n \neq m} \|A_n - A_m\|_1 \leq 2\sqrt{3}(1 - (k - \eta)^2)^{1/2}.$$ 

It is clear now that

$$\gamma(S[C, k]) \leq 2\sqrt{3}(1 - (k - \eta)^2)^{1/2},$$

and since $\eta \in (0, k)$ is arbitrary, we have

$$\gamma(S[C, k]) \leq 2\sqrt{3}(1 - k^2)^{1/2}. \tag{4}$$

Finally, if $\varepsilon \in (0, 2\sqrt{3})$ is given, choose

$$k = (1 - (\varepsilon/2\sqrt{3})^2)^{1/2}.$$ 

$k \in (0, 1)$ and inequality (4) gives us that for each $C \in S_{\mathcal{C}}$,

$$\gamma(S[C, k]) \leq \varepsilon. \quad \square$$

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