PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 109, Number 1, May 1990

A LINEARIZATION OF THE CIRCULAR MAXIMAL OPERATOR

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(Communicated by J. Marshall Ash)

ABSTRACT. An interesting linearization of the circular maximal operator is of restricted weak type (2,2).

The spherical maximal operator $\mathscr{M}$ on $\mathbf{R}^n$ is defined by

$$\mathscr{M} f(x) = \sup_{t>0} \int_{|y|=1} |f|(x - ty)\, dy .$$

Here f is an appropriate function and dy denotes normalized Lebesgue measure on the unit sphere in $\mathbf{R}^n$. In [4] Stein proved that $\mathscr{M}$ is bounded on $L^p(\mathbf{R}^n)$ if $p > n/(n-1)$ and $n \geq 3$. More recently Bourgain [1] established the same result for $n = 2$. Bourgain [2] also noted that when $n \geq 3$ and $p = n/(n-1)$, $\mathscr{M}$ is restricted weak type (p, p)—that is, $\mathscr{M}$ maps $L^{p,1}(\mathbf{R}^n)$ into $L^{p,\infty}(\mathbf{R}^n)$. This result implies that of [4]. It is then natural to ask if $\mathscr{M}$ maps $L^{2,1}(\mathbf{R}^2)$ into $L^{2,\infty}(\mathbf{R}^2)$. Leckband [3] provided a partial result: the answer is yes if one restricts to the subspace of radial functions. The purpose of this note is to give a partial result with a different flavor. We restrict the operator instead of its domain and define, as in [1, p. 70], a linearization T of $\mathscr{M}$ by

$$Tf(x) = \int_{|y|=1} f(x - |x|y)\, dy .$$

Theorem. *The operator T maps $L^{2,1}(\mathbf{R}^2)$ into $L^{2,\infty}(\mathbf{R}^2)$.*

In all known cases the mapping properties of T are as bad as those of $\mathscr{M}$, and so this theorem lends support to the conjecture that $\mathscr{M}$ is of restricted weak type $(2, 2)$ on $\mathbf{R}^2$.

For $x \in \mathbf{R}^2$, let L_x be the line through x perpendicular to the radial segment from the origin to x. Define an operator S by

$$Sf(x) = \frac{1}{|x|} \int_{L_x} f(y)\, dy ,$$

Received by the editors April 14, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42B25.
Partially supported by a grant from the National Science Foundation.

where dy is one-dimensional Lebesgue measure on L_x. Then S is a weighted version of the Radon transform on $\mathbf{R}^2$ and (see Lemma 2) is equivalent to the adjoint of T. Our theorem follows from the boundedness of S from $L^{2,1}(\mathbf{R}^2)$ to $L^{2,\infty}(\mathbf{R}^2)$, which is proved as Lemma 4. To simplify notation we regard points in $\mathbf{R}^2$ as complex numbers.

Lemma 1. *If $z \neq 0$ is a complex number, define $z' = ze^{-i\pi/2}/|z|$. If g and h are nonnegative and measurable, then*

$$\int_0^{2\pi}\int_0^{2\pi} h(e^{i\theta}+e^{it})g(e^{i\theta})\,dt\,d\theta$$
$$= \int_{|z|<2} h(z)\left[g\left(\frac{z}{2}+z'\sqrt{1-\frac{|z|^2}{4}}\right)+g\left(\frac{z}{2}-z'\sqrt{1-\frac{|z|^2}{4}}\right)\right]\frac{dz}{|z|\sqrt{1-\frac{|z|^2}{4}}},$$

where dz denotes two-dimensional Lebesgue measure.

Proof. If $z = e^{i\theta}+e^{it}$ with $\theta < t < \theta+\pi$, then a sketch shows that

$$e^{i\theta} = \frac{z}{2}+z'\sqrt{1-\frac{|z|^2}{4}}.$$

Also the Jacobian of the map

$$(t,\theta) \mapsto z = e^{i\theta}+e^{it}$$

is

$$|\sin(t-\theta)| = |z|\sqrt{1-\frac{|z|^2}{4}}.$$

Thus the formula

$$\int_0^{2\pi}\int_\theta^{\theta+\pi} h(e^{i\theta}+e^{it})g(e^{it})\,dt\,d\theta$$
$$= \int_{|z|<2} h(z)g\left(\frac{z}{2}+z'\sqrt{1-\frac{|z|^2}{4}}\right)\frac{dz}{|z|\sqrt{1-\frac{|z|^2}{4}}}$$

is just a change of variable. A similar formula for the range $\theta+\pi < t < \theta+2\pi$ completes the proof. □

Lemma 2. *If g and h are nonnegative and measurable, then*

$$\int_{\mathbf{R}^2} Tf(x)g(x)\,dx = \tfrac{1}{2}\int_{\mathbf{R}^2} f(x)Sg\left(\frac{x}{2}\right)\,dx.$$

Proof.

$$\int_{\mathbf{R}^2} Tf(x)g(x)\,dx = \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} f(re^{i\theta} + re^{it})g(re^{i\theta})r\,dt\,d\theta\,dr$$

$$= \int_0^\infty \int_0^2 \int_0^{2\pi} f(rue^{i\phi}) \left\{ g\left(r\left[\frac{ue^{i\phi}}{2} + e^{i(\phi-\frac{\pi}{2})}\sqrt{1-\frac{u^2}{4}} \right]\right) \right.$$

$$\left. + g\left(r\left[\frac{ue^{i\phi}}{2} - e^{i(\phi-\frac{\pi}{2})}\sqrt{1-\frac{u^2}{4}} \right]\right) \right\} \frac{r\,d\phi\,du\,dr}{\sqrt{1-\frac{u^2}{4}}}$$

by Lemma 1, with $z = eu^{i\phi}$. To this last expression apply Fubini's theorem, let $s = ur$, and apply Fubini's theorem again. The result is

$$\int_0^\infty \int_0^2 \int_0^{2\pi} f(se^{i\phi}) \left\{ g\left(\frac{s}{2}e^{i\phi} + \frac{s}{u}e^{i(\phi-\frac{\pi}{2})}\sqrt{1-\frac{u^2}{4}} \right) \right.$$

$$\left. + g\left(\frac{s}{2}e^{i\phi} - \frac{s}{u}e^{i(\phi-\frac{\pi}{2})}\sqrt{1-\frac{u^2}{4}} \right) \right\} \frac{s\,d\phi\,du\,ds}{u^2\sqrt{1-\frac{u^2}{4}}}$$

$$= \int_0^\infty \int_0^\infty \int_0^{2\pi} f(se^{i\phi}) \left\{ g\left(\frac{s}{2}e^{i\phi} + sve^{i(\phi-\frac{\pi}{2})} \right) \right.$$

$$\left. + g\left(\frac{s}{2}e^{i\phi} - sve^{i(\phi-\frac{\pi}{2})} \right) \right\} s\,d\phi\,dv\,ds$$

$$= \int_0^\infty \int_0^\infty \int_0^{2\pi} f(se^{i\phi}) \left\{ g\left(\frac{s}{2}e^{i\phi} + te^{i(\phi-\frac{\pi}{2})} \right) \right.$$

$$\left. + g\left(\frac{s}{2}e^{i\phi} - te^{i(\phi-\frac{\pi}{2})} \right) \right\} d\phi\,dt\,ds,$$

where the equalities are from the changes of variable

$$v = \frac{\sqrt{1-\frac{u^2}{4}}}{u} \quad \text{and} \quad t = sv.$$

This last integral is

$$\frac{1}{2}\int_{\mathbf{R}^2} f(x)Sg\left(\frac{x}{2}\right)\,dx.$$

In what follows, $|\cdot|$ will denote Lebesgue measure on either $\mathbf{R}$ or $\mathbf{R}^2$, the exact meaning being clear from the context. Also, $\chi(x, E)$ will stand for the characteristic function of the set E evaluated at x.

Lemma 3. *Suppose h and k are measurable functions on $[0, \infty]$ with $0 \le h, k \le 1$. Then*

$$\int_0^\infty \int_0^\infty \min(h(s), k(r))\,ds\,dr \le 4\left[\int_0^\infty sh(s)\,ds\right]^{\frac{1}{2}} \left[\int_0^\infty rk(r)\,dr\right]^{\frac{1}{2}}.$$

Proof. The left hand side of the conclusion is not affected by measure-preserving rearrangements of h and k while the right hand side will be least when h and k

are decreasing. So replacing h and k by suitable approximations shows that it is enough to establish the lemma under the additional hypotheses that h and k are continuous positive strictly decreasing functions satisfying $h(0) = k(0) = 1$. Now

$$\int_0^\infty \int_0^\infty \min(h(s), k(r))\, ds\, dr = \int_0^\infty h(s)|\{r: k(r) > h(s)\}|\, ds + \int_0^\infty k(r)|\{s: h(s) > k(r)\}|\, dr = I_1 + I_2.$$

Since $|\{r: k(r) > h(s)\}| = k^{-1}(h(s))$,

$$\begin{aligned} I_1 &= \int_0^\infty h(s)k^{-1}(h(s))\, ds = \int_0^\infty \int_0^{k^{-1}(h(s))} \int_0^{k(x)} \chi(y, [0, h(s)])\, dy\, dx\, ds \\ &= \int_0^\infty \int_0^{k(x)} \int_0^\infty \chi(x, [0, k^{-1}(h(s))])\chi(y, [0, h(s)])\, ds\, dy\, dx \\ &\le \int_0^\infty \int_0^{k(x)} h^{-1}(y)\, dy\, dx \\ &= \int_0^1 \int_0^{k^{-1}(y)} h^{-1}(y)\, dx\, dy = \int_0^1 h^{-1}(y)k^{-1}(y)\, dy \\ &\le \left(\int_0^1 [h^{-1}(y)]^2\, dy\right)^{1/2} \left(\int_0^1 [k^{-1}(y)]^2\, dy\right)^{1/2} \\ &= 2\left[\int_0^\infty sh(s)\, ds\right]^{1/2} \left[\int_0^\infty rk(r)\, dr\right]^{1/2}, \end{aligned}$$

where the last equality can be verified by comparing two methods for computing the volume of a solid of revolution. Now interchanging h and k completes the proof of the lemma. □

Lemma 4. *There is a positive number C such that if E and F are measurable subsets of $\mathbf{R}^2$, then*

$$\int_{\mathbf{R}^2} \chi(x, E) S\chi(\cdot, F)(x)\, dx \le C|E|^{1/2}|F|^{1/2}.$$

Proof. The operator S is given by the formula

$$Sg(re^{i\theta}) = \frac{1}{r}\int_{-\infty}^\infty g(re^{i\theta} + te^{i(\theta+\frac{\pi}{2})})\, dt,$$

so we will show that

$$\int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} \chi(re^{i\theta}, E)\chi(re^{i\theta} + te^{i(\theta+\frac{\pi}{2})}, F)\, d\theta\, dt\, dr \le C|E|^{1/2}|F|^{1/2}.$$

We will actually consider only the integral $\int_0^\infty \int_0^\infty \int_0^{2\pi}$—the integral $\int_0^\infty \int_{-\infty}^0 \int_0^{2\pi}$ is treated analogously. The change of variable

$$x = x_r(\theta, t) = re^{i\theta} + te^{i(\theta+\frac{\pi}{2})}$$

shows that

$$\int_0^\infty \int_0^\infty \int_0^{2\pi} \chi(re^{i\theta}, E)\chi(re^{i\theta} + te^{i(\theta+\frac{\pi}{2})}, F)\, d\theta\, dt\, dr$$
$$= \int_0^\infty \int_{\{|x|>r\}} \chi(p(r, x), E)\chi(x, F) \frac{dx}{\sqrt{|x|^2 - r^2}}\, dr,$$

where $p(r, x)$ is the point $re^{i\theta}$ such that x can be written $x = re^{i\theta} + te^{i(\theta+\frac{\pi}{2})}$ for some $t > 0$. We write the last integral as $I_1 + I_2$ where

$$I_1 = \int_0^\infty \int_{\{4r>|x|>r\}}, \qquad I_2 = \int_0^\infty \int_{\{|x|>4r\}},$$

and we begin by considering I_1:

$$\int_0^\infty \int_{\{4r>|x|>r\}} \chi(p(r, x), E)\chi(x, F) \frac{dx}{\sqrt{|x|^2 - r^2}}\, dr$$
$$\leq C \int_0^\infty (r^{-\frac{1}{2}} |F \cap \{r < |x| < 4r\}|^{1/2})$$
$$\times (r^{\frac{1}{2}} \|\chi(p(r, x), E)(|x|^2 - r^2)^{-\frac{1}{2}} \chi(|x|, (r, \infty))\|_{L_x^{2,\infty}}\, dr,$$

since $L^{2,\infty}$ is the dual of $L^{2,1}$. Applying Hölder's inequality shows that this last integral is dominated by

$$C \left(\int_0^\infty |F \cap \{r < |x| < 4r\}| \frac{dr}{r} \right)^{1/2}$$
$$\times \left(\int_0^\infty \|\chi(p(r, x), E)(|x|^2 - r^2)^{-\frac{1}{2}} \chi(|x|, (r, \infty))\|_{L_x^{2,\infty}}^2 r dr \right)^{1/2}.$$

The first term in parentheses is just $(\log 4) \cdot |F|$. We will now prove that the second parenthesized integral is bounded by $C|E|$ and hence that $I_1 \leq C|E|^{1/2}|F|^{1/2}$. A sketch shows that if

$$se^{i\phi} = re^{i\theta} + te^{i(\theta+\frac{\pi}{2})},$$

then

$$\exp(i\theta) = \exp(i[\phi - \cos^{-1}(r/s)]).$$

Thus

$$|\{\phi \in [0, 2\pi) : p(r, se^{i\phi}) \in E\}| = |\{\phi \in [0, 2\pi) : re^{i\phi} \in E\}|,$$

and so, for $\lambda > 0$,

$$|\{x : |x| > r, p(r, x) \in E, (|x|^2 - r^2)^{-\frac{1}{2}} > \lambda\}|$$
$$= \int_r^{\sqrt{r^2+\lambda^{-2}}} |\{\phi \in [0, 2\pi) : p(r, se^{i\phi}) \in E\}| s ds$$
$$= |\{\phi \in [0, 2\pi) : re^{i\phi} \in E\}| / 2\lambda^2.$$

It follows that

$$\int_0^\infty \|\chi(p(r, x), E)(|x|^2 - r^2)^{-\frac{1}{2}}\chi(|x|, (r, \infty))\|_{L_x^{2,\infty}}^2 r\,dr \le C|E|$$

as claimed. Thus the proof will be complete when we see that

$$I_2 \le C|E|^{1/2}|F|^{1/2}.$$

Now if $|x| > 4r$, then

$$\frac{1}{\sqrt{|x|^2 - r^2}} \le \frac{2}{|x|},$$

so

$$\frac{I_2}{4\pi} \le \int_0^\infty \int_0^\infty \frac{1}{2\pi} \int_0^{2\pi} \chi(p(r, se^{i\phi}), E)\chi(se^{i\phi}, F)\,d\phi\,ds\,dr$$

$$\le \int_0^\infty \int_0^\infty \min\left\{\frac{1}{2\pi}\int_0^{2\pi} \chi(p(r, se^{i\phi}), E)\,d\phi, \frac{1}{2\pi}\int_0^{2\pi} \chi(se^{i\phi}, F)\,d\phi\right\} ds\,dr.$$

As noted earlier,

$$\int_0^{2\pi} \chi(p(r, se^{i\phi}), E)\,d\phi = \int_0^{2\pi} \chi(re^{i\phi}, E)\,d\phi.$$

Thus an application of Lemma 3 completes the proof. □

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