A LINEARIZATION OF THE CIRCULAR MAXIMAL OPERATOR

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Abstract. An interesting linearization of the circular maximal operator is of restricted weak type \((2, 2)\).

The spherical maximal operator \(\mathcal{M}\) on \(\mathbb{R}^n\) is defined by

\[
\mathcal{M}f(x) = \sup_{t > 0} \int_{|y| = 1} |f|(x - ty) \, dy.
\]

Here \(f\) is an appropriate function and \(dy\) denotes normalized Lebesgue measure on the unit sphere in \(\mathbb{R}^n\). In [4] Stein proved that \(\mathcal{M}\) is bounded on \(L^p(\mathbb{R}^n)\) if \(p > n/(n-1)\) and \(n \geq 3\). More recently Bourgain [1] established the same result for \(n = 2\). Bourgain [2] also noted that when \(n \geq 3\) and \(p = n/(n-1)\), \(\mathcal{M}\) is restricted weak type \((p, p)\)—that is, \(\mathcal{M}\) maps \(L^{p,1}(\mathbb{R}^n)\) into \(L^{p,\infty}(\mathbb{R}^n)\). This result implies that of [4]. It is then natural to ask if \(\mathcal{M}\) maps \(L^{2,1}(\mathbb{R}^2)\) into \(L^{2,\infty}(\mathbb{R}^2)\). Leckband [3] provided a partial result: the answer is yes if one restricts to the subspace of radial functions. The purpose of this note is to give a partial result with a different flavor. We restrict the operator instead of its domain and define, as in [1, p. 70], a linearization \(T\) of \(\mathcal{M}\) by

\[
Tf(x) = \int_{|y| = 1} f(x - |x|y) \, dy.
\]

Theorem. The operator \(T\) maps \(L^{2,1}(\mathbb{R}^2)\) into \(L^{2,\infty}(\mathbb{R}^2)\).

In all known cases the mapping properties of \(T\) are as bad as those of \(\mathcal{M}\), and so this theorem lends support to the conjecture that \(\mathcal{M}\) is of restricted weak type \((2, 2)\) on \(\mathbb{R}^2\).

For \(x \in \mathbb{R}^2\), let \(L_x\) be the line through \(x\) perpendicular to the radial segment from the origin to \(x\). Define an operator \(S\) by

\[
Sf(x) = \frac{1}{|x|} \int_{L_x} f(y) \, dy,
\]

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where $dy$ is one-dimensional Lebesgue measure on $L_x$. Then $S$ is a weighted version of the Radon transform on $\mathbb{R}^2$ and (see Lemma 2) is equivalent to the adjoint of $T$. Our theorem follows from the boundedness of $S$ from $L^{2,1}(\mathbb{R}^2)$ to $L^{2,\infty}(\mathbb{R}^2)$, which is proved as Lemma 4. To simplify notation we regard points in $\mathbb{R}^2$ as complex numbers.

**Lemma 1.** If $z \neq 0$ is a complex number, define $z' = ze^{-i\pi/2}/|z|$. If $g$ and $h$ are nonnegative and measurable, then

$$
\int_0^{2\pi} \int_0^{2\pi} h(e^{i\theta} + e^{i\theta'})g(e^{i\theta'}) \, dt \, d\theta = \int_{|z|<2} h(z) \left[ g \left( \frac{z'}{2} + z' \sqrt{1 - \frac{|z|^2}{4}} \right) + g \left( \frac{z}{2} - z' \sqrt{1 - \frac{|z|^2}{4}} \right) \right] \frac{dz}{|z| \sqrt{1 - \frac{|z|^2}{4}}},
$$

where $dz$ denotes two-dimensional Lebesgue measure.

**Proof.** If $z = e^{i\theta} + e^{i\theta'}$ with $\theta < \theta < \theta + \pi$, then a sketch shows that

$$
e^{i\theta} = \frac{z}{2} + z' \sqrt{1 - \frac{|z|^2}{4}}.
$$

Also the Jacobian of the map

$$(t, \theta) \mapsto z = e^{i\theta} + e^{i\theta'}$$

is

$$|\sin(t - \theta)| = |z| \sqrt{1 - \frac{|z|^2}{4}}.
$$

Thus the formula

$$
\int_0^{\theta + \pi} \int_\theta^{\theta + \pi} h(e^{i\theta} + e^{i\theta'})g(e^{i\theta'}) \, dt \, d\theta = \int_{|z|<2} h(z) g \left( \frac{z}{2} + z' \sqrt{1 - \frac{|z|^2}{4}} \right) \frac{dz}{|z| \sqrt{1 - \frac{|z|^2}{4}}}
$$

is just a change of variable. A similar formula for the range $\theta + \pi < t < \theta + 2\pi$ completes the proof. \square

**Lemma 2.** If $g$ and $h$ are nonnegative and measurable, then

$$
\int_{\mathbb{R}^2} T f(x) g(x) \, dx = \frac{1}{2} \int_{\mathbb{R}^2} f(x) S g \left( \frac{x}{2} \right) \, dx.
$$
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Proof.

\[ \int_{\mathbb{R}^2} T f(x) g(x) \, dx = \int_{0}^{\infty} \int_{0}^{2\pi} f(r e^{i\theta} + re^{it}) g(r e^{i\theta}) r \, dt \, d\theta \, dr \]

\[ = \int_{0}^{\infty} \int_{0}^{2\pi} f(r_e^{i\phi}) \left\{ g \left( r \left[ \frac{ue^{i\phi}}{2} + e^{i(\phi-\frac{\pi}{4})} \sqrt{1 - \frac{u^2}{4}} \right] \right) + g \left( r \left[ \frac{ue^{i\phi}}{2} - e^{i(\phi-\frac{\pi}{4})} \sqrt{1 - \frac{u^2}{4}} \right] \right) \right\} \frac{r \, d\phi \, du \, dr}{\sqrt{1 - \frac{u^2}{4}}} \]

by Lemma 1, with \( z = eu^{i\phi} \). To this last expression apply Fubini’s theorem, let \( s = ur \), and apply Fubini’s theorem again. The result is

\[ \int_{0}^{\infty} \int_{0}^{2\pi} f(se^{i\phi}) \left\{ g \left( \frac{s}{2} e^{i\phi} + \frac{s}{u} e^{i(\phi-\frac{\pi}{4})} \sqrt{1 - \frac{u^2}{4}} \right) + g \left( \frac{s}{2} e^{i\phi} - \frac{s}{u} e^{i(\phi-\frac{\pi}{4})} \sqrt{1 - \frac{u^2}{4}} \right) \right\} \frac{sd\phi \, du \, ds}{u^2 \sqrt{1 - \frac{u^2}{4}}} \]

where the equalities are from the changes of variable

\[ v = \sqrt{1 - \frac{u^2}{4}} \quad \text{and} \quad t = su. \]

This last integral is

\[ \frac{1}{2} \int_{\mathbb{R}^2} f(x) S g \left( \frac{x}{2} \right) \, dx. \]

In what follows, \( | \cdot | \) will denote Lebesgue measure on either \( \mathbb{R} \) or \( \mathbb{R}^2 \), the exact meaning being clear from the context. Also, \( \chi(x, E) \) will stand for the characteristic function of the set \( E \) evaluated at \( x \).

Lemma 3. Suppose \( h \) and \( k \) are measurable functions on \([0, \infty]\) with \( 0 \leq h, k \leq 1 \). Then

\[ \int_{0}^{\infty} \int_{0}^{\infty} \min(h(s), k(r)) \, ds \, dr \leq 4 \left[ \int_{0}^{\infty} sh(s) \, ds \right]^\frac{1}{2} \left[ \int_{0}^{\infty} rk(r) \, dr \right]^\frac{1}{2}. \]

Proof. The left hand side of the conclusion is not affected by measure-preserving rearrangements of \( h \) and \( k \) while the right hand side will be least when \( h \) and \( k \)
are decreasing. So replacing $h$ and $k$ by suitable approximations shows that it is enough to establish the lemma under the additional hypotheses that $h$ and $k$ are continuous positive strictly decreasing functions satisfying $h(0) = k(0) = 1$.

Now

$$\int_0^\infty \int_0^\infty \min(h(s), k(r)) \, ds \, dr = \int_0^\infty h(s) \{r \colon k(r) > h(s)\} \, ds$$

$$+ \int_0^\infty k(r) \{s \colon h(s) > k(r)\} \, dr = I_1 + I_2.$$

Since $|\{r \colon k(r) > h(s)\}| = k^{-1}(h(s))$, 

$$I_1 = \int_0^\infty h(s)k^{-1}(h(s)) \, ds = \int_0^\infty \int_0^{k^{-1}(h(s))} \chi(y, [0, h(s)]) \, dy \, dx \, ds$$

$$= \int_0^\infty \int_0^{k^{-1}(x)} \chi(x, [0, k^{-1}(h(s))]) \chi(y, [0, h(s)]) \, ds \, dy \, dx$$

$$\leq \int_0^\infty \int_0^{k^{-1}(x)} h^{-1}(y) \, dy \, dx$$

$$= \int_0^1 \int_0^{k^{-1}(y)} h^{-1}(y) \, dx \, dy = \int_0^1 h^{-1}(y)k^{-1}(y) \, dy$$

$$\leq \left( \int_0^1 [h^{-1}(y)]^2 \, dy \right)^{1/2} \left( \int_0^1 [k^{-1}(y)]^2 \, dy \right)^{1/2}$$

$$= 2 \left[ \int_0^\infty sh(s) \, ds \right]^{1/2} \left[ \int_0^\infty rk(r) \, dr \right]^{1/2},$$

where the last equality can be verified by comparing two methods for computing the volume of a solid of revolution. Now interchanging $h$ and $k$ completes the proof of the lemma. \hfill \square

**Lemma 4.** There is a positive number $C$ such that if $E$ and $F$ are measurable subsets of $\mathbb{R}^2$, then

$$\int_{\mathbb{R}^2} \chi(x, E)S\chi(\cdot, F)(x) \, dx \leq C|E|^{1/2}|F|^{1/2}.$$

**Proof.** The operator $S$ is given by the formula

$$Sg(re^{i\theta}) = \frac{1}{r} \int_{-\infty}^\infty g(re^{i\theta} + te^{i(\theta + \xi)}) \, dt,$$

so we will show that

$$\int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} \chi(re^{i\theta}, E)\chi(re^{i\theta} + te^{i(\theta + \xi)}, F) \, d\theta \, dt \, dr \leq C|E|^{1/2}|F|^{1/2}.$$

We will actually consider only the integral $\int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi}$—the integral $\int_0^\infty \int_{-\infty}^0 \int_0^{2\pi}$ is treated analogously. The change of variable

$$x = x_r(\theta, t) = re^{i\theta} + te^{i(\theta + \xi)}$$
shows that

\[
\int_0^\infty \int_0^\infty \int_0^{2\pi} \chi(re^{i\theta}, E)\chi(re^{i\theta} + te^{i(\theta+\frac{\pi}{2})}, F) \, d\theta \, dt \, dr
= \int_0^\infty \int_0^\infty \int_{\{x > r\}} \chi(p(r, x), E)\chi(x, F) \frac{dx}{\sqrt{|x|^2 - r^2}} \, dr,
\]

where \( p(r, x) \) is the point \( re^{i\theta} \) such that \( x \) can be written \( x = re^{i\theta} + te^{i(\theta+\frac{\pi}{2})} \) for some \( t > 0 \). We write the last integral as \( I_1 + I_2 \) where

\[
I_1 = \int_0^\infty \int_{\{4r > |x| > r\}} \chi(p(r, x), E)\chi(x, F) \frac{dx}{\sqrt{|x|^2 - r^2}} \, dr,
\]

and we begin by considering \( I_1 \):

\[
\int_0^\infty \int_{\{4r > |x| > r\}} \chi(p(r, x), E)\chi(x, F) \frac{dx}{\sqrt{|x|^2 - r^2}} \, dr
\leq C \int_0^\infty (r^{-\frac{1}{2}} |F \cap \{r < |x| < 4r\}|^{1/2}) \times (r^{-\frac{1}{2}} \chi(p(r, x), E)(|x|^2 - r^2)^{-\frac{1}{2}} \chi(|x|, (r, \infty)))_{L^2, \infty} \, dr,
\]

since \( L^2, \infty \) is the dual of \( L^{2,1} \). Applying Hölder's inequality shows that this last integral is dominated by

\[
C \left( \int_0^\infty |F \cap \{r < |x| < 4r\}| \frac{dr}{r} \right)^{1/2} \times \left( \int_0^\infty \| \chi(p(r, x), E)(|x|^2 - r^2)^{-\frac{1}{2}} \chi(|x|, (r, \infty)) \|_{L^2, \infty}^2 \, rd\alpha \right)^{1/2}.
\]

The first term in parentheses is just \((\log 4) \cdot |F|\). We will now prove that the second parenthesized integral is bounded by \( C|E| \) and hence that \( I_1 \leq C|E|^{1/2} |F|^{1/2} \). A sketch shows that if \( se^{i\phi} = re^{i\theta} + te^{i(\theta + \frac{\pi}{2})} \), then

\[
\exp(i\theta) = \exp(i[\phi - \cos^{-1}(r/s)]).
\]

Thus

\[
|\{\phi \in [0, 2\pi) : p(r, se^{i\phi}) \in E\}| = |\{\phi \in [0, 2\pi) : re^{i\phi} \in E\}|,
\]

and so, for \( \lambda > 0 \),

\[
|\{x : |x| > r, p(r, x) \in E, (|x|^2 - r^2)^{-\frac{1}{2}} > \lambda\}|
= \int_r^{\sqrt{r^2 + \lambda^{-2}}} |\{\phi \in [0, 2\pi) : p(r, se^{i\phi}) \in E\}| \, ds
= |\{\phi \in [0, 2\pi) : re^{i\phi} \in E\}|/2\lambda^2.
\]
It follows that
\[ \int_0^\infty \| \chi(p(r, x), E)(|x|^2 - r^2)^{-\frac{1}{2}} \chi(|x|, (r, \infty)) \|^2_{L^2_0} r \, dr \leq C|E| \]
as claimed. Thus the proof will be complete when we see that
\[ I_2 \leq C|E|^{1/2}|F|^{1/2}. \]
Now if \(|x| > 4r\), then
\[ \frac{1}{\sqrt{|x|^2 - r^2}} \leq \frac{2}{|x|}, \]
so
\[ \frac{I_2}{4\pi} \leq \int_0^\infty \int_0^\infty \frac{1}{2\pi} \int_0^{2\pi} \chi(p(r, se^{i\phi}), E)\chi(se^{i\phi}, F) \, d\phi \, ds \, dr \]
\[ \leq \int_0^\infty \int_0^\infty \min \left\{ \frac{1}{2\pi} \int_0^{2\pi} \chi(p(r, se^{i\phi}), E) \, d\phi, \frac{1}{2\pi} \int_0^{2\pi} \chi(se^{i\phi}, F) \, d\phi \right\} ds \, dr. \]
As noted earlier,
\[ \int_0^{2\pi} \chi(p(r, se^{i\phi}), E) \, d\phi = \int_0^{2\pi} \chi(re^{i\phi}, E) \, d\phi. \]
Thus an application of Lemma 3 completes the proof. □

References


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