THE EXISTENCE OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR $C^{1+\varepsilon}$ JABLONSKI TRANSFORMATION IN $\mathbb{R}^n$

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ABSTRACT. Using a result of Rychlik, we present a sufficient condition for the existence of an absolutely continuous invariant measure for $C^{1+\varepsilon}$ Jablonski transformation in $\mathbb{R}^n$.

1. Introduction

In 1973, Lasota and Yorke [1] proved a general sufficient condition for the existence of an absolutely continuous invariant measure for expanding piecewise $C^2$ transformation $\tau$ on the interval. Their proof uses the fact that the Frobenius–Perron operator induced by $\tau$ is quasicompact in the space of function of bounded variation.

For $I^n = [0, 1]^n$, let $\beta = \{D_1, D_2, \ldots, D_p\}$ be a partition of $I^n$ such that

$$\bigcup_{j=1}^p D_j = I^n, \quad D_j \cap D_k = \emptyset \quad \text{for } j \neq k.$$

A partition $\beta$ is called a Jablonski partition if for any $1 < j < p$, $D_j$ is a rectangle. A transformation $\tau: I^n \rightarrow I^n$ is called a Jablonski transformation if it is given by the formula

$$\tau(x_1, \ldots, x_n) = \varphi_{ij}(x_1), \ldots, \varphi_{nj}(x_n),$$

for $(x_1, \ldots, x_n) \in D_j, \quad j = 1, \ldots, p, \quad p < \infty,$

where $D_j = \prod_{i=1}^n (a_{ij}, b_{ij})$, $\varphi_{ij} : [a_{ij}, b_{ij}] \rightarrow [0, 1]$ and we write

$$[a_{ij}, b_{ij}) = \begin{cases} [a_{ij}, b_{ij}), \quad \text{if } b_{ij} < 1 \\ [a_{ij}, b_{ij}], \quad \text{if } b_{ij} = 1. \end{cases}$$

In 1983, Jablonski [2] proved the existence of an absolutely continuous invariant measure for the above $\tau$ under the conditions that for any $1 \leq i \leq n$,
1 \leq j \leq p, \phi_{ij} \text{ is a } C^2\text{-function and }
\lambda = \inf_{i,j} \left\{ \inf \left[ \left[ a_{ij}, b_{ij} \right] \right] \right\} > 1 .

The proof in [2] uses bounded variation, which is a complicated notion in higher dimension.

In this paper we present a sufficient condition for the existence of an absolutely continuous invariant measure for Jablonski transformations, which does not use bounded variation in \( R^n \). The proof is much easier and allows us to weaken the \( \phi_{ij} \in C^2 \) condition to \( \phi_{ij} \in C^{1+\varepsilon} \) for some \( 0 < \varepsilon < 1 \). There is, however, a new condition \( \bar{r}^{(t)}/\lambda^{nt} < 1 \) for some integer \( t \geq 1 \) in this paper, where \( \bar{r}^{(t)} \) is a number which depends on the defining partition of the transformation.

2. Rychlik's sufficient condition

Let \((X, \Sigma, m)\) be a Lebesgue space with a \( \sigma \)-algebra \( \Sigma \) and a probability measure \( m \). Let \( \tau: X \to X \) be a measurable, nonsingular transformation and \( g_x \) be the absolute value of the reciprocal of the Jacobian of \( \tau \). Now we can express the Frobenius-Perron operator in the following way:

\[ P_\tau f(x) = \sum_{x' \in \tau^{-1}(x)} g_{x'}(y) f(y), \quad x \in X . \]

Let \( \beta \) be a partition of \( X \) which is a generator for \( \tau \), i.e. \( \bigcup_{k=0}^{\infty} \tau^{-k}(\beta) = X \), where \( \varepsilon \) is a partition into points. For any positive integer \( l \), let \( \beta_l \) be a partition of \( X \) which is a generator for \( \tau^l \). Let \( g = g_1 = g_\tau \) and \( g_l = g_{\tau^l} \).

For every \( A \in \Sigma \), we define \( \beta(A) = \{ B \in \beta : m(B \cap A) > 0 \} \).

**Condition 1** (Distortion condition). There exists a positive number \( b \) such that for any \( l \geq 1 \) and any \( B \in \beta_l \), we have \( \sup_B g_l \leq b \inf_B g_l \).

**Condition 2** (Localization condition). There exist \( \varepsilon > 0 \) and \( 0 < \gamma < 1 \) such that for any \( l \geq 1 \) and \( B \in \beta_l \), \( m(\tau^l B) < \varepsilon \) implies \( \sum_{B' \in \beta(\tau^l B)} \sup_{B'} g \leq \gamma \).

**Condition 3** (Boundedness condition). \( \sum_{B \in \beta} \sup_B g < \infty \).

Under these three conditions Rychlik [3] proved the existence of fixed point for \( P_\tau \).

**Theorem 1.** Assume Conditions 1, 2, and 3 are satisfied. Then the sequence \( \{ P_l \} \) is bounded in \( L^\infty(m) \), and the averages \( (1/l) \sum_{j=0}^{l-1} P_j^l \) converge in \( L^1(m) \) to some \( \varphi \in L^\infty(m) \) such that \( P_\tau \varphi = \varphi \).
3. MAIN RESULT

Let $X = I^n$. For any $x, y \in I^n$, we define
\[
d(x, y) = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2},
\]
where $x = (x_1, \ldots, x_n), \ y = (y_1, \ldots, y_n)$.

**Lemma 1.** If there exists a positive number $M$ such that for any $l \geq 1$, $B \in \beta_l$ and $x, y \in B$ we have $g_l(x)/g_l(y) \leq M$, then Condition 1 is satisfied.

**Lemma 2.** Let $\tau : I^n \to I^n$ be a nonsingular piecewise $C^{1+\epsilon}$ transformation, i.e. there exists a constant $\bar{c}$ such that for any $x, y \in D_j, \ j = 1, \ldots, p, |J(x) - J(y)| \leq \bar{c}d(x, y)^\epsilon$, where $J$ is the absolute value of the Jacobian of $\tau$. If there exist constants $\lambda > 1$ and $c > 0$ such that for $l \geq 1$, $B \in \beta_l$ and $x, y \in B$ we have $d(\tau^l x, \tau^l y) \geq c\lambda^l d(x, y)$, then there exists a constant $M$ such that $g_l(x)/g_l(y) \leq M$ and Condition 1 is satisfied.

**Proof.** For $x, y \in B \in \beta_l, \ k = 0, 1, \ldots, l - 1$, we have
\[
d(\tau^k x, \tau^k y) \leq c^{-1} \lambda^{-(l-k)} d(\tau^l x, \tau^l y) \leq c^{-1} \lambda^{-(l-k)} \sqrt{n}.
\]
Then, using the fact that $\tau$ is piecewise $C^{1+\epsilon}$, we have
\[
\frac{J(\tau^k y)}{J(\tau^k x)} = \left| 1 + \frac{J(\tau^k y) - J(\tau^k x)}{J(\tau^k x)} \right| \leq 1 + \frac{|J(\tau^k y) - J(\tau^k x)|}{J(\tau^k x)} \leq 1 + \frac{\bar{c}d(\tau^k x, \tau^k y)^\epsilon}{c^n \lambda^n} \leq 1 + \frac{\bar{c}\sqrt{n}^\epsilon}{c^{n+\epsilon} \lambda^n} \lambda^{-(l-k)\epsilon}, \quad k = 0, 1, \ldots, l - 1,
\]
and
\[
g_l(x) = \frac{J(\tau^{-1} y)J(\tau^{-2} y) \cdots J(\tau y)J(y)}{J(\tau^{-1} x)J(\tau^{-2} x) \cdots J(\tau x)J(x)} \leq \prod_{k=0}^{l-1} \left( 1 + \frac{\bar{c}\sqrt{n}^\epsilon}{c^{n+\epsilon} \lambda^n} \lambda^{-(l-k)\epsilon} \right) = \prod_{i=1}^{l} \left( 1 + \frac{\bar{c}\sqrt{n}^\epsilon}{c^{n+\epsilon} \lambda^n} \lambda^{-\epsilon} \right) \leq \prod_{i=1}^{\infty} \left( 1 + \frac{\bar{c}\sqrt{n}^\epsilon}{c^{n+\epsilon} \lambda^n} \lambda^{-\epsilon} \right) = M. \quad \Box
\]

Let $\tau : I^n \to I^n$ be a Jablonski transformation:
\[
\tau(x) = (\phi_{l_1}(x_1), \ldots, \phi_{l_n}(x_n)), \quad x \in D_j,
\]
where for any $1 \leq j \leq p$, $D_j$ is a rectangle, $D_j \cap D_k = \emptyset$ for $j \neq k$ and $\bigcup_{j=1}^{p} D_j = I^n$. $\beta = \{D_1, \ldots, D_P\}$. For any $l \geq 1$, let $\beta_l = \{D_{l_1}, \ldots, D_{l_n}\}$. Then every $D_{l_1}^{(l)}$ is a rectangle and $\tau^l$ is a Jablonski transformation.

Let $L_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ be the straight line parallel to the $i$th axis and having all the coordinates but the $i$th one fixed and equal to $x_1, \ldots, x_{i-1}, \ldots, x_n$. Let $r_i^{(l)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ be the number of $D_{l_1}^{(l)}$'s
Figure 1.

for which \( D_j^{(l)} \cap L_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \neq \emptyset \). For any \( 1 \leq i \leq n \) let

\[
    r_i^{(l)} = \sup_{0 \leq x_i \leq 1, 1 \leq k \leq n, k \neq i} r_i^{(l)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \quad r_i = r_i^{(1)},
\]

\[
    r^{(l)} = r_1^{(l)} r_2^{(l)} \cdots r_n^{(l)} = \prod_{i=1}^{n} r_i^{(l)}, \quad r = r^{(1)},
\]

\[
    \overline{r}_i^{(l)} = r_i^{(l)}/r_i^{(1)} = r_1^{(l)} r_2^{(l)} \cdots r_{i-1}^{(l)} r_{i+1}^{(l)} \cdots r_n^{(l)} = \prod_{j=1}^{n} r_j^{(l)}, \quad \overline{r}_i = \overline{r}_i^{(1)},
\]

\[
    \overline{r}^{(l)} = \max_{1 \leq i \leq n} \overline{r}_i^{(l)}, \quad \overline{r} = \overline{r}^{(1)},
\]

i.e. \( r_i^{(l)} \) is the maximal number of parts of \( \beta_i \) in \( x_i \) direction. For example \( n = 2 \), \( \beta_i \) is shown in Figure 1 and we have: \( r_1^{(l)} = 4 \), \( r_2^{(l)} = 5 \), \( r^{(l)} = r_1^{(l)} r_2^{(l)} = 20 \), \( \overline{r}_1^{(l)} = r_1^{(l)}/r_1^{(1)} = 5 \), \( \overline{r}_2^{(l)} = r_2^{(l)}/r_2^{(1)} = 4 \), \( \overline{r}^{(l)} = \max(\overline{r}_1^{(l)}, \overline{r}_2^{(l)}) = 5 \).

**Lemma 3.** Let \( \tau : I^n \to I^n \) be a Jablonski transformation and assume that \( \lambda = \inf_{i,j} \inf_{0 \leq x_i \leq 1} |\phi'_{ij}| > 1 \). If \( \tau \lambda^{-n} < 2^{-1} \), then Condition 2 is satisfied.

**Proof.** Take \( \varepsilon > 0 \) such that \( \varepsilon^{1/n} < d = \min_{1 \leq j \leq p} d_j \), where \( d_j \) is the minimal width of \( D_j \), and \( 0 < \gamma < 1 \) such that \( 0 < 2\tau \lambda^{-n} < \gamma < 1 \). Since \( |\phi'_{ij}| \geq \lambda \), we have \( g \leq \lambda^{-n} \). Since \( d > \varepsilon^{1/n} \), for any \( B \in \beta_j \), \( m(\tau' B) < \varepsilon \) means at least one of the widths of \( \tau' B \) is smaller than \( \varepsilon^{1/n} < d \) and the number of \( \beta(\tau' B) \) is at most \( 2\tau \). Hence

\[
    \sum_{B' \in \beta(\tau' B)} \sup_{B' \in B} g < \frac{\text{the number of } \beta(\tau' B)}{\lambda^n} \leq \frac{2\tau}{\lambda^n} < \gamma. \quad \square
\]

**Lemma 4.** Let \( \beta = \{D_1, \ldots, D_p\} \), \( p < \infty \). If for any \( D_j \), \( \sup_{D_j} g < \infty \), then Condition 3 is satisfied.

**Proof.** Let \( M = \max_{1 \leq j \leq p} \sup_{D_j} g \). Then we have \( \sum_{B \in \beta} \sup_{B} g \leq pM < \infty \), which is Condition 3. \( \square \)
By Lemma 2, 3, and 4, and Theorem 1, we have:

**Theorem 2.** Let \( \tau: I^n \to I^n \) be a Jablonski transformation with a finite partition \( \beta = \{D_1, \ldots, D_p\} \), and let \( \lambda = \inf_{i,j} \inf_{0 \leq i, j \leq 1} |\varphi_{ij}| > 1 \). If for any \( i \) and \( j \), \( \varphi_{ij} \in C^{1+\varepsilon}, \ 0 < \varepsilon < 1 \) and \( \bar{r}/\lambda^n < 1/2 \) then the sequence \( \{P^i \}_{i=1}^{\infty} \) is bounded in \( L^\infty(m) \) and the averages \( (1/\lambda^n) \sum_{s=0}^{\infty} P^s \) converge in \( L^1(m) \) to some \( \varphi \in L^\infty(m) \) such that \( P_\varphi = \varphi \).

**Lemma 5.** If \( \varphi \) is a fixed point of \( P_{t^t} = P_{t^t}^t \) for some positive integer \( t \) then \( \psi = (1/t)(\varphi + P_t \varphi + \cdots + P^{t-1}_t \varphi) \) is a fixed point of \( P_t \).

For a Jablonski transformation \( \tau \) and any \( l \geq 1 \) we have \( r_i^{(l)} \leq r_i^{(l)} \leq r_i^{(l)} \), \( \bar{r}_i^{(l)} \leq \bar{r}_i^{(l)} \leq \bar{r}_i^{(l)} \), \( g_i \leq \lambda^{-nl} \). We therefore have:

**Theorem 3.** Let \( \tau: I^n \to I^n \) be a Jablonski transformation with a finite partition \( \beta = \{D_1, \ldots, D_p\} \), and assume \( \lambda = \inf_{i,j} \inf_{0 \leq i, j \leq 1} |\varphi_{ij}| > 1 \). If for any \( i \) and \( j \) \( \varphi_{ij} \in C^{1+\varepsilon}, \ 0 < \varepsilon < 1 \) and \( \bar{r} \cdot \lambda^n < 1 \), then there exists a function \( \psi \in L^\infty(m) \) such that \( P_\varphi = \psi \).

**Proof.** Since \( \bar{r} \cdot \lambda^n < 1 \), we can take an integer \( t \geq 1 \) such that \( (\bar{r} \cdot \lambda^{-n})^t < t \). Since \( \tau^t \) satisfies all the conditions of Theorem 2, there exists a function \( \varphi \in L^\infty(m) \) such that \( P_\tau \varphi = \varphi \). By Lemma 5

\[
\psi = \frac{1}{t} \sum_{s=0}^{t-1} P_s \varphi
\]

satisfies \( P_\tau \psi = \psi \).

For \( n = 1 \), we have \( \bar{r} = 1 \). By virtue of Theorem 3, we have:

**Theorem 4.** Let \( \tau: I \to I \) be a piecewise expanding and piecewise \( C^{1+\varepsilon} \) transformation with \( 0 < \varepsilon < 1 \). Then there exists a function \( \varphi \in L^\infty(m) \) such that \( P_\tau \varphi = \varphi \).

**Theorem 5.** Let \( \tau: I^n \to I^n \) be a Jablonski transformation with a finite partition \( \beta = \{D_1, \ldots, D_p\} \), and \( \lambda = \inf_{i,j} \inf_{0 \leq i, j \leq 1} |\varphi_{ij}'| > 1 \). If for any \( i \) and \( j \), \( \varphi_{ij} \in C^{1+\varepsilon}, \ 0 < \varepsilon < 1 \), and there exists a positive integer \( t \geq 1 \) such that \( \bar{r}^{(t)} / \lambda^{nt} < 1 \) then there exists a function \( \psi \in L^\infty(m) \) such that \( P_\tau \psi = \psi \).

**Proof.** Since \( \tau^t \) satisfies all the conditions of Theorem 3, there exists a function \( \varphi \in L^\infty(m) \) such that \( P_\tau \varphi = \varphi \). By Lemma 5

\[
\psi = \frac{1}{t} \sum_{s=0}^{t-1} P_s \varphi
\]

satisfies \( P_\tau \psi = \psi \).
4. Generalized-Jablonski Transformations

A transformation \( \tau : I^n \rightarrow I^n \) is called a generalized-Jablonski transformation if it is given by the formula

\[
\tau(x_1, \ldots, x_n) = (\phi_{1j}(x_1), \phi_{2j}(x_1, x_2), \ldots, \phi_{nj}(x_1, x_2, \ldots, x_n)),
\]

\((x_1, \ldots, x_n) \in D_j, \quad j = 1, \ldots, p, \quad p < \infty,\)

the image of any hyperplane in \( I^n \) is a collection of segments of hyperplanes in \( I^n \) and the inverse image of any hyperplane in \( I^n \) is a collection of segments of hyperplanes in \( I^n \), where \( D_j \) is the same as for a Jablonski transformation and \( \phi_{ij}(x_1, \ldots, x_i) : \prod_{k=1}^{j} [a_{kj}, b_{kj}] \rightarrow [0, 1] \).

Taking \( c = 1 \) in Lemma 2, we have:

**Lemma 6.** Let \( \tau : I^n \rightarrow I^n \) be a nonsingular piecewise \( C^{1+\varepsilon} \) transformation. If there exists a constant \( \lambda > 1 \) such that for any \( B \in \beta \) and \( x, y \in B \) we have

\[
d(\tau x, \tau y) \geq \lambda d(x, y),
\]

then Condition 1 holds.

**Lemma 7.** Let \( \tau : I^n \rightarrow I^n \) be a generalized-Jablonski transformation. If (*) is satisfied with \( \lambda > 1 \) and \( 2T/\lambda^n < 1 \), then Condition 2 holds.

**Proof.** Let \( \varepsilon > 0 \) such that \( \varepsilon^{1/n} < d/2^{n-1} \), where \( d = \min_{1 \leq j \leq p} d_j \) and \( d_j \) is the minimal width of \( D_j \), and \( 0 < \gamma < 1 \) such that \( 0 < 2T\lambda^{-n} < \gamma < 1 \).

Since for any \( x, y \in B \in \beta \), \( d(\tau x, \tau y) \geq \lambda d(x, y) \), we have \( d \leq \lambda^{n-1} \cdot d^{1/n} \), for any \( B \in \beta \), \( m(\tau'B) < \varepsilon \) means at least one of the maximal length of \( \tau'B \) on \( x \) direction is smaller than \( 2^{n-1} \cdot d^{1/n} < d \) and the number of \( \beta(\tau'B) \) is at most \( 2T \). Hence

\[
\sum_{B' \in \beta(\tau'B)} \sup_{B'} g \leq \frac{\text{(the number of } \beta(\tau'B))}{\lambda^n} \leq \frac{2T}{\lambda^n} < \gamma. \quad \square
\]

By Lemmas 4, 6, 7, and Theorem 1, we have:

**Theorem 6.** Let \( \tau : I^n \rightarrow I^n \) be a nonsingular, piecewise \( C^{1+\varepsilon} \) generalized-Jablonski transformation with a finite partition \( \beta = \{D_1, \ldots, D_p\} \). If for any \( x, y \in B \in \beta \), \( d(\tau x, \tau y) \geq \lambda d(x, y) \) for some constant \( \lambda > 1 \) and \( 2T/\lambda^n < 1 \) then the sequence \( \{P^{1}_t\}_{t=1}^{\infty} \) is bounded in \( L^\infty(m) \) and the averages \( (1/t) \sum_{j=0}^{t-1} P^1_t \) converge in \( L^1(m) \) to some \( \varphi \in L^\infty(m) \) such that \( P_t \varphi = \varphi \).

5. Example

In Theorem 5 we need a condition \( \frac{T}{\lambda^n} < 1 \) for some integer \( t \geq 1 \). Now we show that, in general, \( \lambda > 1 \) does not imply this condition.

(1) Consider the partition \( \beta = \{D_1, D_2, \ldots, D_9\} \) shown in Figure 2 for that unit square in two-dimensions. Let \( \tau \) be a Jablonski transformation defined as
follows. For \( j = 1, 2, 4 \) and \( i = 1, 2, \varphi_{ij} \) is a linear function which is onto \([0, 1]\) with \(|\varphi'_{ij}| = 3\). For \( j = 3, 5, 6, 7, 8, 9 \) and \( i = 1, 2, \varphi_{ij} \in C^{1+\varepsilon} \) with \(\inf_{x_i} |\varphi'_{ij}| = 1.1\). Therefore, \(\lambda = 1.1, \lambda^2 = 1.21, \bar{r} = 3, \bar{r}^{(2)} \geq 2\bar{r} + 1 > 2\bar{r}\).

For \( l > 1, \bar{r}^{(l)} > 2^{l-1}\bar{r} = 3 \cdot 2^{l-1} = 1.5 \cdot 2^l\). This means that for any positive integer \( t \geq 1, \frac{\bar{r}^{(l)}}{\lambda^t} = \frac{\bar{r}^{(l)}}{1.21^t} > \frac{1.5 \cdot 2^l}{1.21^t} > 1\).

(2) For \( n = 2 \), and the partition \( \beta = \{D_1, \ldots, D_9\} \) is the same as in (1), assume that for every \( i \) and \( j, \varphi_{ij} \in C^{1+\varepsilon}, \lambda = \inf_{i,j} \inf_{x_i} |\varphi'_{ij}| > \sqrt{3} \). Then we have

\[
\frac{\bar{r}}{\lambda^n} < \frac{3}{(\sqrt{3})^2} = 1.
\]

For example, let \( \varphi(x) = x^{3/2} + 1.74x \). Then

\[
\varphi'(x) = \frac{3}{2}x^{1/2} + 1.74 \geq 1.74 > \sqrt{3}
\]

for \( x \geq 0 \) and \( \varphi(x) \) maps \([0, 1]\) onto \([0, 0.77245]\). We let

\[
\varphi_{ij}(x_i) = \begin{cases} 
\varphi(x_i), & \text{if } [a_{ij}, b_{ij}] = [0, \frac{1}{3}], \\
1.2\varphi(x_i - \frac{1}{3}), & \text{if } [a_{ij}, b_{ij}] = [\frac{1}{3}, \frac{2}{3}], \\
\varphi(x_i - \frac{2}{3}) + .2, & \text{if } [a_{ij}, b_{ij}] = [\frac{2}{3}, 1],
\end{cases}
\]

and we have \( \varphi_{ij}(x_i) \in C^{1+1/2}, \lambda = 1.74 > \sqrt{3} \).

By Theorem 3, there exists a function \( \psi \in L^\infty(m) \) such that \( P_\epsilon \psi = \psi \).

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