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THE EXISTENCE OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR $C^{1+\varepsilon}$ JABLONSKI TRANSFORMATION IN R^n

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ABSTRACT. Using a result of Rychlik, we present a sufficient condition for the existence of an absolutely continuous invariant measure for $C^{1+\varepsilon}$ Jablonski transformation in R^n.

1. INTRODUCTION

In 1973, Lasota and Yorke [1] proved a general sufficient condition for the existence of an absolutely continuous invariant measure for expanding piecewise C^2 transformation τ on the interval. Their proof uses the fact that the Frobenius–Perron operator induced by τ is quasicompact in the space of function of bounded variation.

For $I^n = [0, 1]^n$, let $\beta = \{D_1, D_2, \dots, D_p\}$ be a partition of I^n such that

$$\bigcup_{j=1}^{p} D_j = I^n, \qquad D_j \cap D_k = \phi \quad \text{for } j \neq k .$$

A partition β is called a Jablonski partition if for any $1 \leq j \leq p$, D_j is a rectangle. A transformation $\tau: I^n \to I^n$ is called a Jablonski transformation if it is given by the formula

$$\tau(x_1, \dots, x_n) = \varphi_{1j}(x_1), \dots, \varphi_{nj}(x_n)),$$
$$\text{for } (x_1, \dots, x_n) \in D_j, \quad j = 1, \dots, p, \quad p < \infty,$$

where $D_j = \prod_{i=1}^{n} [a_{ij}, b_{ij})$, $\varphi_{ij}: [a_{ij}, b_{ij}] \to [0, 1]$ and we write

$$[a_{ij}, b_{ij}) = \begin{cases} [a_{ij}, b_{ij}), & \text{if } b_{ij} < 1 \\ [a_{ij}, b_{ij}], & \text{if } b_{ij} = 1 . \end{cases}$$

In 1983, Jablonski [2] proved the existence of an absolutely continuous invariant measure for the above τ under the conditions that for any $1 \leq i \leq n$,

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$1 \le j \le p$, φ_{ij} is a C^2-function and

$$\lambda = \inf_{i,j}\left\{\inf_{[a_{ij},b_{ij}]}|\varphi'_{ij}|\right\} > 1.$$

The proof in [2] uses bounded variation, which is a complicated notion in higher dimension.

In this paper we present a sufficient condition for the existence of an absolutely continuous invariant measure for Jablonski transformations, which does not use bounded variation in R^n. The proof is much easier and allows us to weaken the $\varphi_{ij} \in C^2$ condition to $\varphi_{ij} \in C^{1+\varepsilon}$ for some $0 < \varepsilon < 1$. There is, however, a new condition $\bar{r}^{(t)}/\lambda^{nt} < 1$ for some integer $t \ge 1$ in this paper, where $\bar{r}^{(t)}$ is a number which depends on the defining partition of the transformation.

2. Rychlik's sufficient condition

Let (X, Σ, m) be a Lebesgue space with a σ-algebra Σ and a probability measure m. Let $\tau\colon X \to X$ be a measurable, nonsingular transformation and g_τ be the absolute value of the reciprocal of the Jacobian of τ. Now we can express the Frobenius–Perron operator in the following way:

$$P_\tau f(x) = \sum_{y\in\tau^{-1}(x)} g_\tau(y)f(y), \qquad x \in X.$$

Let β be a partition of X which is a generator for τ, i.e. $V_{k=0}^{\infty}\tau^{-k}(\beta) = \varepsilon$, where ε is a partition into points. For any positive integer l, let β_l be a partition of X which is a generator for τ^l. Let $g = g_1 = g_\tau$ and $g_l = g_{\tau^l}$. For every $A \in \Sigma$, we define $\beta(A) = \{B \in \beta\colon m(B \cap A) > 0\}$.

Condition 1 (Distortion condition). *There exists a positive number b such that for any $l \ge 1$ and any $B \in \beta_l$, we have* $\sup_B g_l \le b \inf_B g_l$.

Condition 2 (Localization condition). *There exist $\varepsilon > 0$ and $0 < \gamma < 1$ such that for any $l \ge 1$ and $B \in \beta_l$, $m(\tau^l B) < \varepsilon$ implies* $\sum_{B'\in\beta(\tau^l B)} \sup_{B'} g \le \gamma$.

Condition 3 (Boundedness condition). $\sum_{B\in\beta} \sup_B g < \infty$.

Under these three conditions Rychlik [3] proved the existence of fixed point for P_τ.

Theorem 1. *Assume Conditions* 1, 2, *and* 3 *are satisfied. Then the sequence* $\{P_\tau^l 1\}_{l=1}^{\infty}$ *is bounded in* $L^\infty(m)$, *and the averages* $(1/l)\Sigma_{j=0}^{l-1}P_\tau^j 1$ *converge in* $L^1(m)$ *to some* $\varphi \in L^\infty(m)$ *such that* $P_\tau\varphi = \varphi$.

3. Main result

Let $X = I^n$. For any $x, y \in I^n$, we define

$$d(x, y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2},$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$.

Lemma 1. *If there exists a positive number M such that for any $l \geq 1$ $B \in \beta_l$ and $x, y \in B$ we have $g_l(x)/g_l(y) \leq M$, then Condition 1 is satisfied.*

Lemma 2. *Let $\tau: I^n \to I^n$ be a nonsingular piecewise $C^{1+\varepsilon}$ transformation, i.e. there exists a constant $\bar{c}$ such that for any $x, y \in D_j$, $j = 1, \dots, p$, $|J(x) - J(y)| \leq \bar{c}d(x, y)^\varepsilon$, where J is the absolute value of the Jacobian of τ. If there exist constants $\lambda > 1$ and $c > 0$ such that for $l \geq 1$, $B \in \beta_l$ and $x, y \in B$ we have $d(\tau^l x, \tau^l y) \geq c\lambda^l d(x, y)$, then there exists a constant M such that $g_l(x)/g_l(y) \leq M$ and Condition 1 is satisfied.*

Proof. For $x, y \in B \in \beta_l$, $k = 0, 1, \dots, l-1$, we have

$$d(\tau^k x, \tau^k y) \leq c^{-1}\lambda^{-(l-k)}d(\tau^l x, \tau^l y) \leq c^{-1}\lambda^{-(l-k)}\sqrt{n}.$$

Then, using the fact that τ is piecewise $C^{1+\varepsilon}$, we have

$$\begin{aligned}\frac{J(\tau^k y)}{J(\tau^k x)} &= \left|1 + \frac{J(\tau^k y) - J(\tau^k x)}{J(\tau^k x)}\right| \leq 1 + \frac{|J(\tau^k y) - J(\tau^k x)|}{J(\tau^k x)} \\ &\leq 1 + \frac{\bar{c}d(\tau^k x, \tau^k y)^\varepsilon}{c^n\lambda^n} \leq 1 + \frac{\bar{c}\sqrt{n^\varepsilon}}{c^{n+\varepsilon}\lambda^n}\lambda^{-(l-k)\varepsilon}, \qquad k = 0, 1, \dots, l-1,\end{aligned}$$

and

$$\begin{aligned}\frac{g_l(x)}{g_l(y)} &= \frac{J(\tau^{l-1}y)J(\tau^{l-2}y)\cdots J(\tau y)J(y)}{J(\tau^{l-1}x)J(\tau^{l-2}x)\cdots J(\tau x)J(x)} \\ &\leq \prod_{k=0}^{l-1}\left(1 + \frac{\bar{c}\sqrt{n^\varepsilon}}{c^{n+\varepsilon}\lambda^n}\lambda^{-(l-k)\varepsilon}\right) = \prod_{i=1}^{l}\left(1 + \frac{\bar{c}\sqrt{n^\varepsilon}}{c^{n+\varepsilon}\lambda^n}\lambda^{-\varepsilon i}\right) \\ &\leq \prod_{i=1}^{\infty}\left(1 + \frac{\bar{c}\sqrt{n^\varepsilon}}{c^{n+\varepsilon}\lambda^n}\lambda^{-\varepsilon i}\right) = M. \quad \square\end{aligned}$$

Let $\tau: I^n \to I^n$ be a Jablonski transformation:

$$\tau(x) = (\varphi_{1j}(x_1), \dots, \varphi_{nj}(x_n)), \qquad x \in D_j,$$

where for any $1 \leq j \leq p$, D_j is a rectangle, $D_j \cap D_k = \varnothing$ for $j \neq k$ and $\bigcup_{j=1}^p D_j = I^n$. $\beta = \{D_1, \dots, D_p\}$. For any $l \geq 1$, let $\beta_l = \{D_1^{(l)}, \dots, D_{p_l}^{(l)}\}$. Then every $D_j^{(l)}$ is a rectangle and τ^l is a Jablonski transformation.

Let $L_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ be the straight line parallel to the ith axis and having all the coordinates but the ith one fixed and equal to $x_1, \dots, x_{i-1}, \dots, x_n$. Let $r_i^{(l)}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ be the number of $D_j^{(l)}$'s

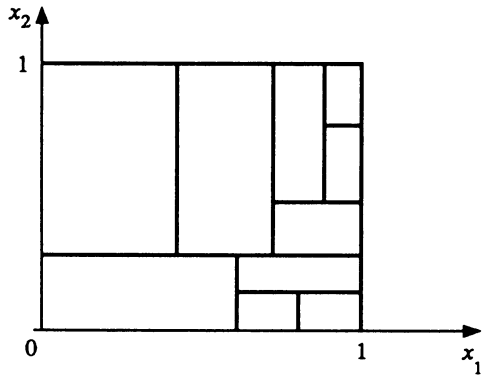

FIGURE 1.

for which $D_j^{(l)} \cap L_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \neq \varnothing$. For any $1 \le i \le n$ let

$$r_i^{(l)} = \sup_{0 \le x_k \le 1,\, 1 \le k \le n,\, k \ne i} r_i^{(l)}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \qquad r_i = r_i^{(1)},$$

$$r^{(l)} = r_1^{(l)} r_2^{(l)} \cdots r_n^{(l)} = \prod_{i=1}^{n} r_i^{(l)}, \qquad r = r^{(1)},$$

$$\bar{r}_i^{(l)} = r^{(l)}/r_i^{(l)} = r_1^{(l)} r_2^{(l)} \cdots r_{i-1}^{(l)} r_{i+1}^{(l)} \cdots r_n^{(l)} = \prod_{j=1,\ j \ne i}^{n} r_j^{(l)}, \quad \bar{r}_i = \bar{r}_i^{(1)},$$

$$\bar{r}^{(l)} = \max_{1 \le i \le n} \bar{r}_i^{(l)}, \qquad \bar{r} = \bar{r}^{(1)},$$

i.e. $r_i^{(l)}$ is the maximal number of parts of β_l in x_i direction. For example $n = 2$, β_l is shown in Figure 1 and we have: $r_1^{(l)} = 4$, $r_2^{(l)} = 5$, $r^{(l)} = r_1^{(l)} r_2^{(l)} = 20$, $\bar{r}_1^{(l)} = r^{(l)}/r_1^{(l)} = 5$, $\bar{r}_2^{(l)} = r^{(l)}/r_2^{(l)} = 4$, $\bar{r}^{(l)} = \max(\bar{r}_1^{(l)}, \bar{r}_2^{(l)}) = 5$.

Lemma 3. *Let $\tau\colon I^n \to I^n$ be a Jablonski transformation and assume that $\lambda = \inf_{i,j} \inf_{0 \le x_i \le 1} |\varphi'_{ij}| > 1$. If $\bar{r}\lambda^{-n} < 2^{-1}$, then Condition 2 is satisfied.*

Proof. Take $\varepsilon > 0$ such that $\varepsilon^{1/n} < d = \min_{1 \le j \le p} d_j$, where d_j is the minimal width of D_j, and $0 < \gamma < 1$ such that $0 < 2\bar{r}\lambda^{-n} < \gamma < 1$. Since $|\varphi'_{ij}| \ge \lambda$, we have $g \le \lambda^{-n}$. Since $d > \varepsilon^{1/n}$, for any $B \in \beta_l$, $m(\tau^l B) < \varepsilon$ means at least one of the widths of $\tau^l B$ is smaller than $\varepsilon^{1/n} < d$ and the number of $\beta(\tau^l B)$ is at most $2\bar{r}$. Hence

$$\sum_{B' \in \beta(\tau^l B)} \sup_{B'} g < \frac{\text{the number of } \beta(\tau^l B)}{\lambda^n} \le \frac{2\bar{r}}{\lambda^n} < \gamma. \quad \square$$

Lemma 4. *Let $\beta = \{D_1, \dots, D_p\}$, $p < \infty$. If for any D_j, $\sup_{D_j} g < \infty$, then Condition 3 is satisfied.*

Proof. Let $M = \max_{1 \le j \le p} \sup_{D_j} g$. Then we have $\sum_{B \in \beta} \sup_B \le pM < \infty$, which is Condition 3. $\square$

By Lemma 2, 3, and 4, and Theorem 1, we have:

Theorem 2. *Let* $\tau\colon I^n \to I^n$ *be a Jablonski transformation with a finite partition* $\beta = \{D_1, \dots, D_p\}$, *and let* $\lambda = \inf_{i,j} \inf_{0 \le x_i \le 1} |\varphi'_{ij}| > 1$. *If for any* i *and* j, $\varphi_{ij} \in C^{1+\varepsilon}$, $0 < \varepsilon < 1$ *and* $\bar{r}/\lambda^n < \frac{1}{2}$ *then the sequence* $\{P_\tau^l 1\}_{l=1}^\infty$ *is bounded in* $L^\infty(m)$ *and the averages* $(1/l)\sum_{j=0}^{l-1} P_\tau^j$ *converge in* $L^1(m)$ *to some* $\varphi \in L^\infty(m)$ *such that* $P_\tau \varphi = \varphi$.

Lemma 5. *If* φ *is a fixed point of* $P_{\tau^t} = P_\tau^t$ *for some positive integer* t *then* $\psi = (1/t)(\varphi + P_\tau \varphi + \cdots + P_\tau^{t-1}\varphi)$ *is a fixed point of* P_τ.

For a Jablonski transformation τ and any $l \ge 1$ we have $r_i^{(l)} \le r_i^l$, $r^{(l)} \le r^l$, $\bar{r}_i^{(l)} \le \bar{r}_i^l$, $\bar{r}^{(l)} \le \bar{r}^l$, $g_l < \lambda^{-nl}$. We therefore have:

Theorem 3. *Let* $\tau\colon I^n \to I^n$ *be a Jablonski transformation with a finite partition* $\beta = \{D_1, \dots, D_p\}$, *and assume* $\lambda = \inf_{i,j} \inf_{0 \le x_i \le 1} |\varphi'_{ij}| > 1$. *If for any* i *and* j $\varphi_{ij} \in C^{1+\varepsilon}$, *with* $0 < \varepsilon < 1$ *and* $\bar{r} \cdot \lambda^{-n} < 1$, *then there exists a function* $\psi \in L^\infty(m)$ *such that* $P_\tau \psi = \psi$.

Proof. Since $\bar{r} \cdot \lambda^{-n} < 1$, we can take an integer $t \ge 1$ such that $(\bar{r} \cdot \lambda^{-n})^t < \frac{1}{2}$. Since τ^t satisfies all the conditions of Theorem 2, there exists a function $\varphi \in L^\infty(m)$ such that $P_\tau^t \varphi = P_{\tau^t} \varphi = \varphi$. By Lemma 5

$$\psi = \frac{1}{t} \sum_{s=0}^{t-1} P_\tau^s \varphi$$

satisfies $P_\tau \psi = \psi$. □

For $n = 1$, we have $\bar{r} = 1$. By virtue of Theorem 3, we have:

Theorem 4. *Let* $\tau\colon I \to I$ *be a piecewise expanding and piecewise* $C^{1+\varepsilon}$ *transformation with* $0 < \varepsilon < 1$. *Then there exists a function* $\varphi \in L^\infty(m)$ *such that* $P_\tau \varphi = \varphi$.

Theorem 5. *Let* $\tau\colon I^n \to I^n$ *be a Jablonski transformation with a finite partition* $\beta = \{D_1, \dots, D_p\}$, *and* $\lambda = \inf_{i,j} \inf_{0 \le x_i \le 1} |\varphi'_{ij}| > 1$. *If for any* i *and* j, $\varphi_{ij} \in C^{1+\varepsilon}$, $0 < \varepsilon < 1$, *and there exists a positive integer* $t \ge 1$ *such that* $\bar{r}^{(t)}/\lambda^{nt} < 1$ *then there exists a function* $\psi \in L^\infty(m)$ *such that* $P_\tau \psi = \psi$.

Proof. Since τ^t satisfies all the conditions of Theorem 3, there exists a function $\varphi \in L^\infty(m)$ such that $P_\tau^t \varphi = P_{\tau^t} = \varphi$. By Lemma 5

$$\psi = \frac{1}{t} \sum_{s=0}^{t-1} P_\tau^s \varphi \qquad \text{satisfies } P_\tau \psi = \psi. \quad \square$$

4. Generalized-Jablonski transformations

A transformation $\tau: I^n \to I^n$ is called a generalized-Jablonski transformation if it is given by the formula

$$\tau(x_1, \dots, x_n) = (\varphi_{1j}(x_1), \varphi_{2j}(x_1, x_2), \dots, \varphi_{nj}(x_1, x_2, \dots, x_n)),$$
$$(x_1, \dots, x_n) \in D_j, \qquad j = 1, \dots, p, \quad p < \infty,$$

the image of any hyperplane in I^n is a collection of segments of hyperplanes in I^n and the inverse image of any hyperplane in I^n is a collection of segments of hyperplanes in I^n, where D_j is the same as for a Jablonski transformation and $\varphi_{ij}(x_1, \dots, x_i): \prod_{k=1}^{i}[a_{kj}, b_{kj}] \to [0, 1]$.

Taking $c = 1$ in Lemma 2, we have:

Lemma 6. *Let $\tau: I^n \to I^n$ be a nonsingular piecewise $C^{1+\varepsilon}$ transformation. If there exists a constant $\lambda > 1$ such that for any $B \in \beta$ and $x, y \in B$ we have*

$$(*) \qquad d(\tau x, \tau y) \geq \lambda d(x, y),$$

then Condition 1 *holds.*

Lemma 7. *Let $\tau: I^n \to I^n$ be a generalized-Jablonski transformation. If $(*)$ is satisfied with $\lambda > 1$ and $2\bar{r}/\lambda^n < 1$, then Condition* 2 *holds.*

Proof. Let $\varepsilon > 0$ such that $\varepsilon^{1/n} < d/2^{n-1}$, where $d = \min_{1 \leq j \leq p} d_j$ and d_j is the minimal width of D_j, and $0 < \gamma < 1$ such that $0 < 2\bar{r}\lambda^{-n} < \gamma < 1$. Since for any $x, y \in B \in \beta$, $d(\tau x, \tau y) \geq \lambda d(x, y)$, we have $g \leq \lambda^{-n}$. Since $d > 2^{n-1} \cdot \varepsilon^{1/n}$, for any $B \in \beta_l$, $m(\tau^l B) < \varepsilon$ means at least one of the maximal length of $\tau^l B$ on x_i direction is smaller than $2^{n-1} \cdot \varepsilon^{1/n} < d$ and the number of $\beta(\tau^l B)$ is at most $2\bar{r}$. Hence

$$\sum_{B' \in \beta(\tau^l B)} \sup_{B'} g \leq \frac{(\text{the number of } \beta(\tau^l B))}{\lambda^n} \leq \frac{2\bar{r}}{\lambda^n} < \gamma. \quad \square$$

By Lemmas 4, 6, 7, and Theorem 1, we have:

Theorem 6. *Let $\tau: I^n \to I^n$ be a nonsingular, piecewise $C^{1+\varepsilon}$ generalized-Jablonski transformation with a finite partition $\beta = \{D_1, \dots, D_p\}$. If for any $x, y \in B \in \beta$, $d(\tau x, \tau y) \geq \lambda d(x, y)$ for some constant $\lambda > 1$ and $2\bar{r}/\lambda^n < 1$ then the sequence $\{P_\tau^l 1\}_{l=1}^{\infty}$ is bounded in $L^\infty(m)$ and the averages $(1/l)\sum_{j=0}^{l-1} P_\tau^j 1$ converge in $L^1(m)$ to some $\varphi \in L^\infty(m)$ such that $P_\tau \varphi = \varphi$.*

5. Example

In Theorem 5 we need a condition $\bar{r}^{(t)}/\lambda^{nt} < 1$ for some integer $t \geq 1$. Now we show that, in general, $\lambda > 1$ does not imply this condition.

(1) Consider the partition $\beta = \{D_1, D_2, \dots, D_9\}$ shown in Figure 2 for that unit square in two-dimensions. Let τ be a Jablonski transformation defined as

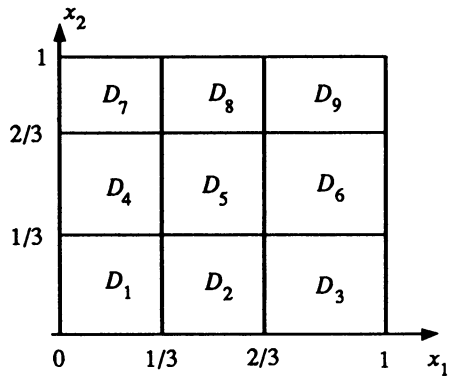

Figure 2.

follows. For $j = 1, 2, 4$ and $i = 1, 2$, φ_{ij} is a linear function which is onto $[0, 1]$ with $|\varphi'_{ij}| = 3$. For $j = 3, 5, 6, 7, 8, 9$ and $i = 1, 2$, $\varphi_{ij} \in C^{1+\varepsilon}$ with $\inf_{x_i} |\varphi'_{ij}| = 1.1$. Therefore, $\lambda = 1.1$, $\lambda^2 = 1.21$, $\bar{r} = 3$, $\bar{r}^{(2)} \geq 2\bar{r} + 1 > 2\bar{r}$. For $l > 1$, $\bar{r}^{(l)} > 2^{l-1}\bar{r} = 3 \cdot 2^{l-1} = 1.5 \cdot 2^l$. This means that for any positive integer $t \geq 1$,

$$\frac{\bar{r}^{(t)}}{\lambda^{nt}} = \frac{\bar{r}^{(t)}}{\lambda^{2t}} = \frac{\bar{r}^{(t)}}{1.21^t} > 1.5 \cdot \left(\frac{2}{1.21}\right)^t > 1.$$

(2) For $n = 2$, and the partition $\beta = \{D_1, \dots, D_9\}$ is the same as in (1), assume that for every i and j, $\varphi_{ij} \in C^{1+\varepsilon}$, $\lambda = \inf_{i,j} \inf_{x_i} |\varphi'_{ij}| > \sqrt{3}$. Then we have

$$\frac{\bar{r}}{\lambda^n} < \frac{3}{(\sqrt{3})^2} = 1.$$

For example, let $\varphi(x) = x^{3/2} + 1.74x$. Then

$$\varphi'(x) = \tfrac{3}{2}x^{1/2} + 1.74 \geq 1.74 > \sqrt{3}$$

for $x \geq 0$ and $\varphi(x)$ maps $[0, \frac{1}{3}]$ onto $[0, 0.77245]$. We let

$$\varphi_{ij}(x_i) = \begin{cases} \varphi(x_i), & \text{if } [a_{ij}, b_{ij}] = [0, \frac{1}{3}], \\ 1.2\varphi(x_i - \frac{1}{3}), & \text{if } [a_{ij}, b_{ij}] = [\frac{1}{3}, \frac{2}{3}], \\ \varphi(x_i - \frac{2}{3}) + .2, & \text{if } [a_{ij}, b_{ij}] = [\frac{2}{3}, 1], \end{cases}$$

and we have $\varphi_{ij}(x_i) \in C^{1+1/2}$, $\lambda = 1.74 > \sqrt{3}$.

By Theorem 3, there exists a function $\psi \in L^\infty(m)$ such that $P_\tau \psi = \psi$.

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