LARGE COMPACT SEPARABLE SPACES MAY ALL CONTAIN $\beta N$

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Abstract. In the Cohen model any compact separable space that does not contain $\beta N$ has cardinality at most of the continuum.

In this paper we prove the theorem stated in the abstract. By the Cohen model we mean any model obtained by adjoining $\aleph_2$ Cohen reals to a model of GCH. We shall, in fact, show that any compact separable space of cardinality greater than the continuum $c$ will map onto the space $I^c$. We will see that this is equivalent to containing $\beta N$. We freely use any notation and conventions which we believe to be standard enough to justify this usage.

Suppose that $V$ is a model of GCH, $G$ is $Fn(\omega_2, 2)$-generic over $V$, and that in $V[G]$, $X$ is a compact separable space with cardinality greater than $c$. We will assume (and show later that we may assume) that $X$ has a countable discrete dense set $\omega$.

In $V$, let $\theta$ be a large enough regular cardinal and let $M$ be an elementary submodel of $H(\theta)$ such that $X \in M[G]$, $|M| = \aleph_2$ and such that $M^{\aleph_1} \subseteq M$. It follows that $M[G]^{\aleph_1} \subseteq M[G]$. Still in $V$, choose an elementary chain $\{M_{\alpha}|\alpha < \omega_2\}$ of elementary submodels of $M$ whose union is $M$ and so that each submodel is closed under $\omega$-sequences and each has cardinality $\aleph_1$. It is remarked in [2] that since $M \prec H = H(\theta)$, it will be the case that $M[G] \prec H[G]$ in $V[G]$. This is easily proven using names, forcing, and the definition

$M[G] = \{\text{val}(\dot{t}, G) | \dot{t} \in M \text{ is a } Fn(\omega_2, 2)\text{-name}\}$.

It will also be the case here (although not in general) that $H[G]$ will be $H(\theta)$ as computed in $V[G]$.

Passing to $V[G]$, choose a point $x \in X \setminus M$. For each $\alpha < \omega_2$, define

$x^\alpha = \{y \in P(\omega) \cap M_\alpha[G] | x \in \text{int}_{x} \text{cl}_{x} Y\}$.

Although $x$ is not in $M[G]$, it is in $H[G]$, and the set $x^\alpha$ is just an $\omega_1$-sized subset of $M_\alpha[G]$ and is therefore in $M[G]$. By elementarity, it follows that...
there is some point \( x_n \in M[G] \) such that

\[
x'' = \{ Y \in \mathcal{P}(\omega) \cap M_n[G] \mid x_n \in \text{int}_x \text{cl}_x Y \}.
\]

Now choose a continuous function \( f_n : X \to I \) so that

\[
x_n \in \text{int}_x \text{cl}_x [f_n^{-1}(0) \cap \omega] \text{ and } x \in \text{int}_x \text{cl}_x [f_n^{-1}(1) \cap \omega].
\]

Let \( A_n = f_n^{-1}(0) \cap \omega \) and \( B_n = f_n^{-1}(1) \cap \omega \). Since \( M \) contains (up to forcing isomorphism) all \( Fn(\omega_2, 2) \)-names of subsets of \( \omega \), it follows that \( \{ A_n, B_n \} \subset M[G] \).

Now, returning to \( V \), we may fix names for all of the elements of \( V[G] \) already mentioned. By thinning out and reindexing, we may assume that:

1. \( A \) and \( B \) are \( F_{\omega_1} \)-names for some \( I, n \in [\omega_2]^\omega \);
2. \( \{ I_n \mid n < \omega_2 \} \) forms a \( \Delta \)-system with root \( I \);
3. for each \( \alpha < \beta < \omega_2 \) there is an isomorphism \( \phi_{\alpha, \beta} \) taking \( A_n \), \( B_n \) to \( A_\beta \) and \( B_\beta \), respectively, such that \( \phi_{\alpha, \beta} \) is induced by a bijection from \( I \) to \( I_\beta \) which is the identity on \( I \);
4. each of \( x_n \), \( A_n \), and \( B_n \) are in \( M_{n+1} \); and
5. \( x'' \) is equal \( \{ Y \in M_n[G] \mid x_n \in \text{cl}_x \text{int}_x Y \} \).

We complete the proof by showing that

Claim. \( V[G] \models \{ (A_n, B_n) \mid \alpha < \omega_2 \} \) is a dyadic family. (I.e., for any disjoint finite \( F, H \subset \omega_2, \cap_{n \in F} A_n \cap \cap_{n \in H} B_n \neq \emptyset \).)

The main idea here is that since \( x_\beta \) is chosen to be like \( x \), we can often force that \( x_\beta \in \text{cl}_x B_\alpha \) for \( (\beta > \alpha) \) and by thinning out, we have that \( x_\beta \) is like \( x_n \), so we can often force that \( x_\beta \in \text{cl}_x A_n \).

**Definition 1.** Let \( G_1 = G \cap M_1 \). For any \( Fn(\omega_2, 2) \)-name \( \dot{Y} \) and condition \( r \in Fn(\omega_2, 2) \), let

\[
\dot{Y}_r = \{ n \in \omega \mid (\exists q < r)(q \cap I \in G_1 \text{ and } q \vDash n \in \dot{Y}) \}.
\]

Although it is an abuse of notation, we will think of \( \dot{Y}_r \) as being an element of \( M[G_1] \), as well as being a \( Fn(\omega_2 \cap M_1, 2) \)-name of itself.

**Proposition 1.** For any \( r \in Fn(\omega_2, 2) \) such that \( r \cap M_1 \in G_1 \) and any \( \alpha \in \omega_2 \),

\[
V[G] \vDash \dot{A}_{n,r} \in x^1.
\]

**Proof.** It should be clear from Definition 1 that \( \dot{A}_{n,r} = \dot{A}_{\alpha, (r \cap I_n)} \) (since we may assume \( q \in Fn(I_n, 2) \)). Now let \( r' \in Fn(I_0, 2) \) be such that \( \phi_{0,n}(r') = r \cap I_n \) (if \( \alpha = 0 \), let \( \phi_{0,n} \) be the identity function). Now observe that, since \( \dot{A}_{n} \) is isomorphic to \( \dot{A}_0 \) by \( \phi_{0,n} \),

\[
\dot{A}_{n,r} = \dot{A}_{0,r'} \in M[G_1].
\]

Note that \( \{ \phi_{0,\beta}(r') \mid \beta \in \omega_2 \} \) is what one might call a \( \Delta \)-system of conditions with root \( r'I_0 \). i.e., they each extend \( r'I_0 \) and their domains form a \( \Delta \)-system.
with root equal to the domain of \( r'|I_0 \). Such a \( \Delta \)-system is always "dense-below" the root. Since \( r'|I_0 \in G \), we may choose \( \beta \) such that \( r_\beta = \phi_{0, \beta}(r') \in G \).

Again, note that \( \dot{A}_{\beta, r_\beta} = \dot{A}_{0, r'} \). Now it is clear that (in \( V[G] \))

\[
x_\beta \in \text{cl}_X \text{int}_X[\dot{A}_{\beta, r_\beta}]
\]

since

\[
r_\beta \in G \text{ and } r_\beta \models \dot{A}_\beta \subseteq \dot{A}_{\beta, r_\beta}.
\]

Therefore, we have \( x_\beta \in \text{cl}_X \text{int}_X[\dot{A}_{0, r}] \), from which it follows that \( \dot{A}_{\alpha, r} = \dot{A}_{0, r} \in x_\beta \). The result now follows from the fact that \( x_\beta \cap M_1[G_1] = x^1 \).

**Proposition 2.** For any \( r \in Fn(\omega_2, 2) \) such that \( r|I \in G_1 \) and any \( \alpha < \omega_2 \),

\[
V[G] \models \dot{B}_{\alpha, r} \in x^1.
\]

**Proof.** As shown earlier, we know that \( \dot{B}_{\alpha, r} \in M_1[G_1] \) and we may find a \( \beta > \alpha \) so that \( r_\beta = \phi_{\alpha, \beta}(r|I_\alpha) \in G \). Now, by the isomorphism, it follows that \( \dot{B}_{\alpha, r} = \dot{B}_{\beta, r_\beta} \). Since \( \alpha < \beta \), we also have that \( \dot{B}_{\alpha, r} \in M_\beta[G_\beta] \). Again,

\[
r_\beta \in G \text{ and } r_\beta \models \dot{x} \in \text{cl}_X(\dot{B}_\beta) \subseteq \text{cl}_X \text{int}_X[\dot{B}_{\beta, r_\beta}].
\]

**Proposition 3.** \( V[G] \models x^1 \) contains all the cofinite sets and has the finite intersection property.

**Proposition 4.** \( V[G] \models \) for any \( r \in Fn(\omega_2, 2) \) such that \( r|I \in G \) and any disjoint \( F, H \in [\omega_2]^{<\omega} \).

\[
\bigcap_{\alpha \in F} \dot{A}_{\alpha, r} \cap \bigcap_{\alpha \in H} \dot{B}_{\alpha, r} \neq \emptyset.
\]

**Proof.** By Propositions 1 and 2, each of these sets are in \( x^1 \). Now apply Proposition 3.

**Proof of claim.** Suppose that \( r \in G \) and \( F, H \in [\omega_2]^{<\omega} \) are such that

\[
F \cap H = \emptyset \text{ and } r \models \bigcap_{\alpha \in F} \dot{A}_{\alpha} \cap \bigcap_{\alpha \in H} \dot{B}_{\alpha} = \emptyset.
\]

By Proposition 4, we may choose an integer \( k \) so that

\[
k \in \bigcap_{\alpha \in F} \dot{A}_{\alpha, r} \cap \bigcap_{\alpha \in H} \dot{B}_{\alpha, r}.
\]

For each \( \alpha \in F \), we may choose \( r_\alpha \in Fn(I_\alpha, 2) \) so that

\[
r_\alpha < r|I_\alpha, r_\alpha|I \in G \text{ and } r_\alpha \models k \in \dot{A}_{\alpha}.
\]

Similarly for each \( \alpha \in H \), we may choose \( r_\alpha \in Fn(I_\alpha, 2) \) so that

\[
r_\alpha < r|I_\alpha, r_\alpha|I \in G \text{ and } r_\alpha \models k \in \dot{B}_{\alpha}.
\]
Since \( \{ I_n \mid \alpha \in \omega_2 \} \) is a \( \Delta \)-system with root \( I \), it follows that \( q = r \cup \bigcup_{n \in F \cup H} r_n \) is in \( Fn(\omega_2, 2) \). Furthermore,

\[
q \models k \in \bigcap_{n \in F} A_n \cap \bigcap_{n \in H} B_n,
\]

which is a contradiction.

Therefore we have established the following theorem.

**Theorem.** In \( V[G] \), if \( X \) is a compact separable space, then

\[
|X| > c \iff X \text{ maps onto } I^c.
\]

(If we assume that \( X \) is 0-dimensional, then we can replace \( I^c \) by \( 2^c \) in the statement.)

**Proof.** Since \( X \) is separable, it can be embedded in a compactification \( b\omega \) of \( \omega \) in such a way that \( [b\omega - \omega] = X \). Let \( \{ f_n \mid \alpha \in \omega_2 \} \) be the family of maps chosen as above, i.e., \( \{(f_n^{-1}(0) \cap \omega, f_n^{-1}(1) \cap \omega) \mid \alpha \in \omega_2 \} \) form a dyadic family. Therefore, the function \( \prod_{n \in \omega_2} f_n \) is a continuous function from \( b\omega \) into \( I^c \).

Since the family \( \{(f_n^{-1}(0) \cap \omega, f_n^{-1}(1) \cap \omega) \mid \alpha \in \omega_2 \} \) is dyadic, it follows that the image of \( b\omega \) contains \( 2^c \). Also, there is a continuous map from \( I^c \) to itself which takes the subspace \( 2^c \) onto \( I^c \). Now it follows easily that the restriction of the composition of the two maps to \( X \) is an onto map.

Our primary motivation for proving the above result was to show that there need not be a "psi-space" with a compactification of cardinality greater than \( c \). This follows from the above result by a result of Baumgartner and Weese that is equivalent to the result that, in the above model, no (zero-dimensional) compactification of a \( \psi \)-space can be mapped onto \( 2^c \). A \( \psi \)-space is a space containing a countable dense set of isolated points such that the set of nonisolated points is a discrete set and every infinite set of isolated points contains a converging subsequence. In [1] a Boolean algebra, \( \mathcal{B} \), is called representable if there is a maximal almost-disjoint family \( \mathcal{A} \) of subsets of \( \omega \) so that \( \mathcal{B} \) is isomorphic to the quotient of \( \mathcal{P}(\omega) \) by the ideal generated by \( \mathcal{A} \). This corresponds to the fact that the Stone space of \( \mathcal{B} \) is a compactification of a \( \psi \)-space. The following lemma is the last part of their Theorem 4.1 (modified so as to include spaces which are not zero-dimensional).

**Lemma [1].** In \( V[G] \), if \( \omega \) is densely embedded into \( I^c \) by a function \( f \), then there is a set \( Y \subset \omega \) such that \( f[Y] \) contains no converging sequences. Hence, no \( \psi \)-space maps continuously to a dense subset of \( I^c \).

**Proof.** Suppose that \( \check{f} \) is the name of \( f \). For each \( \alpha \in \omega_2 \), let \( \check{A}_n \) be the name of \( \{ n \in \omega \mid f(n)(\alpha) < \frac{1}{4} \} \). Similarly let \( \check{B}_n \) be the name for \( \{ n \in \omega \mid f(n)(\alpha) > \frac{3}{4} \} \). Also for each \( \alpha \), fix a countable set \( I_n \subset \omega_2 \), so that both \( \check{A}_n \) and \( \check{B}_n \) are \( Fn(I_n, 2) \)-names. Note that we may assume that

\[
I \models \{ \langle \check{A}_n, \check{B}_n \rangle \mid \alpha \in \omega_2 \} \text{ is a dyadic family.}
\]
Let $M$ be an $\aleph_1$-sized elementary submodel of a sufficiently large $H(\theta)$ such that $\{\check{f}\} \cup \{(\check{A}_n, \check{B}_n, I_n) | \alpha \in \omega_2\} \in M$ and $M$ is closed under $\omega$-sequences. Let $M \cap \omega_2 = \lambda$ and let $I = I_\lambda \cap M$. Since $M$ is an elementary submodel of $H(\theta)$, we can inductively choose $\{\alpha_n | n \in \omega\} \subset \lambda$ so that:

1. $\check{A}_{\alpha_n}$ and $\check{B}_{\alpha_n}$ are isomorphic to $\check{A}_\lambda$ and $\check{B}_\lambda$ by an isomorphism $\phi_{\alpha_n}$ induced by the order preserving isomorphism from $I_{\alpha_n}$ to $I_\lambda$, which is the identity on $I$.
2. $I_{\alpha_{n+1}} \cap \text{sup}(I_{\alpha_n}) = I$ for each $n \in \omega$.

We can choose in $M[G \cap M]$ an infinite $Y \subset \omega$ and a name $\hat{Y} \in M$ for $Y$ so that $Y$ is almost contained in $A_{\alpha_{2n}} \cap B_{\alpha_{2n+1}}$ for each $n \in \omega$ (recall that the family is dyadic). If $Y$ contains a subset whose image under $f$ converged, there would be such a set in $M[G]$. Therefore we may assume that $f[Y]$ converges and we finish the proof by showing that $A_\lambda \cap Y$ and $B_\lambda \cap Y$ are both infinite.

Indeed, suppose $p \in G$ and $j \in \omega$ are such that

$$p \Vdash \check{A}_\lambda \cap \hat{Y} \subset j.$$ 

Choose $n$ large enough so that $[I_{\alpha_{2n}} - I] \cap \text{dom}(p) = \emptyset$. Let $r \in Fn(I_{\alpha_{2n}})$ exist such that $\phi_{2n}(r) = p|I_\lambda$; note that $r$ is compatible with $p$. Now

$$(r \cup p) \Vdash \hat{Y} \cap A_{\alpha_{2n}}$$

so we may choose $k > j$ and $r' < r \cup (p \cap M)$ with $r' \in M$ so that $r' \Vdash k \in \hat{Y} \cap A_{\alpha_{2n}}$. But now $\phi_{2n}(r'|I_{2n}) = p'|\Vdash k \in \check{A}_\lambda$ and $p'$ is compatible with $r'$. To see this last fact we note that $r' < r$, $\phi_{2n}(r'|I_{2n}) < p$, and $r' < p' \cap M = p' \cap \text{dom}(r')$.

The case $p \Vdash \check{B}_\lambda \cap \hat{Y} \subset j$ is obviously similar.

REFERENCES


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