

ON EIGENVALUE PROBLEMS OF THE p -LAPLACIAN WITH NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. We study the nonlinear eigenvalue problem

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad \text{where } p > 1, \lambda \in \mathbf{R}.$$

For $\int_{\Omega} m(x) < 0$, we prove that the first positive eigenvalue λ_1 exists and is simple and unique, in the sense that it is the only eigenvalue with a positive eigenfunction. In the case $\int_{\Omega} m(x) = 0$, we prove that $\lambda_0 = 0$ is the only eigenvalue with a positive eigenfunction.

1. INTRODUCTION

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. For a given number $p > 1$ and a function $m(x) \in C(\Omega)$, we consider the following nonlinear eigenvalue problem with Neumann boundary condition

$$(1) \quad -\Delta_p u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega$$

$$(2) \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian and $\lambda \in \mathbf{R}$.

The operator $-\Delta_p$ with $p \neq 2$ arises from a variety of physical phenomena. It is used in non-Newtonian fluids, in some reaction-diffusion problems, as well as in flow through porous media. It also appears in nonlinear elasticity, glaciology, and petroleum extraction. Diaz [4] collected detailed references on physical background and presented mathematical treatments of free boundary problem associated with $-\Delta_p$. Recently, the eigenvalue problems of $-\Delta_p$ with indefinite weight with respect to Dirichlet boundary conditions were investigated by Otani and Teshima [9] and Anane [1], while Guedda and Veron [5] studied the one-dimensional bifurcation phenomena of $-\Delta_p$. We refer to [1, 4, 5, 9]

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for more references and for other aspects of $-\Delta_p$. In the case $p = 2$, eigenvalue problems of second-order elliptic operator with indefinite weight were studied by many authors. Senn and Hess [10], Hess and Senn [7] studied eigenvalue problems with Neumann boundary conditions. Bandle, Pozio, and Tesei [2] studied the existence and uniqueness of positive solutions to some nonlinear Neumann problems. The purpose of this paper is to study the existence of a positive real number λ_1 , the first eigenvalue, such that (1)-(2) has a positive solution u_1 if and only if $\lambda = \lambda_1$ and such that λ_1 is simple in the sense that such solution u_1 is unique up to a constant multiplication. Our approach in this paper follows the technique of [10]. However, since the problem we study is nonlinear and the associated operator is not selfadjoint, the Krein-Rutman theory used extensively in [10] is not applicable here.

The rest of the paper is organized as follows. In §2, we introduce some known results for the p -Laplacian and set up the framework for later sections. In §3, the existence, uniqueness, and simplicity of the first positive eigenvalue λ_1 are proved for the case $\int_{\Omega} m(x) dx < 0$. The exceptional case $\int_{\Omega} m(x) dx = 0$ is discussed in §4, where we prove that $\lambda_0 = 0$ is the only eigenvalue with a corresponding positive eigenfunction.

2. PRELIMINARIES

In this section, we will set up an appropriate functional analysis framework for our problem. We first state some fundamental properties of the solutions of the following Neumann boundary value problem,

$$(3) \quad -\Delta_p u + \varepsilon |u|^{p-2} u = f \quad \text{in } \Omega$$

$$(4) \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary, $p > 1$ and $\varepsilon > 0$.

By Theorem 2.1 of Lions [8, p. 171], and Theorem 2.3 of [8, p. 176], (3)-(4) has a unique solution $u \in W^{1,p}(\Omega)$ for any given $f \in L^q(\Omega)$ where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. Regularity theorems (cf. Tolksdorf [11, p. 806] or Di Benedetto [3]), further show that if $f \in L^\infty(\Omega)$, then $u \in C^{1+\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$.

Let $Ju := |u|^{p-2}u$, and

$$\mathcal{D} \equiv \mathcal{D}(L) = \{u \in L^\infty(\Omega) \mid u \text{ solves (3)-(4) for some } f \in L^\infty(\Omega)\}.$$

On \mathcal{D} define the operator L by $Lu = -\varepsilon Ju + f$, where $f \in L^\infty(\Omega)$ is such that u solves (3)-(4) for this f .

Then for any $\varepsilon > 0$, $L + \varepsilon J$ is injective and maps \mathcal{D} onto $L^\infty(\Omega)$. Thus its inverse $R_\varepsilon = (L + \varepsilon J)^{-1} : L^\infty(\Omega) \rightarrow \mathcal{D} \subset L^\infty(\Omega)$ exists. Since $C^{1+\alpha}(\bar{\Omega})$ is embedded in $L^\infty(\Omega)$ compactly and densely, $R_\varepsilon : L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ is compact. We now show that R_ε is strictly positive, i.e. for $f \geq 0$ and $f \neq 0$, $R_\varepsilon f > 0$. Indeed, let $u = R_\varepsilon f$, then $-\Delta_p u + \varepsilon |u|^{p-2} u = f$. Let $\tilde{u} = \min(u, 0)$.

Multiplying the previous equation by \tilde{u} on both sides and integrating, we get $u \geq 0$. Lemma 4 of Otani and Teshima [9] further shows that $u > 0$ on $\overline{\Omega}$. Thus R_ϵ is strictly positive.

Let $M: L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ denote the multiplication operator induced by the weight function $m(x) \in C(\Omega)$, i.e. $Mu = m(x)u$. Then the eigenvalue problem (1)–(2) is equivalent to the following operator equation,

$$(5) \quad Lu = \lambda MJu,$$

in $L^\infty(\Omega)$, for $\lambda \in \mathbf{R}$.

A simple integration shows that if $m(x)$ is of constant sign, there is no nonzero eigenvalue λ of (1)–(2) with a positive eigenfunction. Thus from now on we always assume that $m(x)$ changes sign in Ω . Also, without loss of generality, we assume that $\|m(x)\|_\infty < 1$.

For $\alpha > 0$, define

$$K_\alpha = R_\alpha(M + 1)J.$$

Then K_α is compact and strictly positive. We note that u is a solution of (5) if and only if u is a solution of

$$(6) \quad u = \lambda^{\frac{1}{p-1}} K_\lambda u.$$

3. MAIN RESULTS

We start with the construction of a subsolution to (6). The same technique was employed in [2, 6, 10] for the same purpose.

Lemma 1. *There exist $\alpha_0 > 0$ and $w_0 \in L^\infty(\Omega)$, $w_0 \geq 0$, such that*

$$(7) \quad w_0 \leq \alpha_0^{\frac{1}{p-1}} K_{\alpha_0} w_0,$$

i.e., w_0 is a subsolution to (6) for $\lambda = \alpha_0$.

Proof. Let $\Omega^+ = \{x \in \Omega | m(x) > 0\}$ and let $U \subset \Omega^+$ be an open connected domain. Set $\delta = \min_{x \in U} m(x)$. It is well known that there exist $\gamma_1 > 0$, and $\tilde{w}_1 \in L^\infty(U)$, $\tilde{w}_1 > 0$, such that

$$-\Delta_p \tilde{w}_1 = \gamma_1 J \tilde{w}_1 \quad \text{in } U, \quad \tilde{w}_1 = 0 \quad \text{on } \partial U.$$

Set $\alpha_0 = \gamma_1 / \delta$, then

$$(-\Delta_p + \alpha_0 J) \tilde{w}_1 = \alpha_0 (\delta + 1) J \tilde{w}_1.$$

Now let

$$w_0 = \begin{cases} \tilde{w}_1 & \text{in } U \\ 0 & \text{in } \overline{\Omega} \setminus U. \end{cases}$$

We claim that

$$(8) \quad w_0 \leq \alpha_0^{\frac{1}{p-1}} (L + \alpha_0 J)^{-1} (\delta + 1) J w_0.$$

Indeed, let $v = \alpha_0^{\frac{1}{p-1}}(L + \alpha_0 J)^{-1}(\delta + 1)Jw_0$. Since $(\delta + 1)Jw_0 \geq 0$, $v > 0$ on $\bar{\Omega}$. Thus $v > w_0$ in $\bar{\Omega} \setminus U$.

Since w_0 and v solve

$$\begin{aligned} (-\Delta_p + \alpha_0 J)w_0 &= \alpha_0(\delta + 1)Jw_0 \text{ in } \Omega, & w_0 &= 0 \text{ on } \partial U \\ (-\Delta_p + \alpha_0 J)v &= \alpha_0(\delta + 1)Jw_0 \text{ in } \Omega, & v &> 0 \text{ on } \partial U \end{aligned}$$

respectively, by maximum principle (cf. Lemma 3 of [9], or Proposition 3.2.1 of [11]), we conclude that $v \geq w_0$ in U . Thus (8) follows.

Since $(\delta + 1)Jw_0 \leq (M + 1)Jw_0$, (8) and maximum principle imply that $w_0 \leq \alpha_0^{\frac{1}{p-1}}K_{\alpha_0}w_0$. This completes the proof of the lemma.

Lemma 2. *For this α_0 , there exist α_1 such that $0 < \alpha_1 \leq \alpha_0$ and $w_1 \in L^\infty(\Omega)$, $w_1 > 0$, such that*

$$(9) \quad w_1 = \alpha_1^{\frac{1}{p-1}}K_{\alpha_0}w_1 = \alpha_1^{\frac{1}{p-1}}(L + \alpha_0 J)^{-1}(M + 1)Jw_1.$$

Proof. Consider the following eigenvalue problem

$$\begin{aligned} -\Delta_p u + \alpha_0 |u|^{p-2}u &= \lambda(m(x) + 1)|u|^{p-2}u, & \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0, & \text{on } \partial\Omega. \end{aligned}$$

Then the arguments in the proof of Theorem 1 in [9] yield the existence of the first eigenvalue $\tilde{\lambda}_1 > 0$ with a positive eigenfunction $w_1 > 0$. It is easy to see that w_1 satisfies (9) with $\alpha_1 = \tilde{\lambda}_1$. Since $\tilde{\lambda}_1$ can be expressed as

$$\tilde{\lambda}_1 = \inf_{u \in W^{1,p}} \frac{\int |\nabla u|^p + \alpha_0 \int |u|^p}{\int (m + 1)|u|^p},$$

we conclude that $\alpha_0 \geq \tilde{\lambda}_1 = \alpha_1$. This completes the proof of the lemma.

Now, since w_1 satisfies (9),

$$(L + \alpha_0 J)w_1 = \alpha_1(M + 1)Jw_1.$$

Thus

$$(L + \alpha_1 J)w_1 = \alpha_1(M + 1)Jw_1 - (\alpha_0 - \alpha_1)Jw_1.$$

Hence we conclude that

$$w_1 \leq \alpha_1^{\frac{1}{p-1}}K_{\alpha_1}w_1,$$

since $(\alpha_0 - \alpha_1) \geq 0$ and $(L + \alpha_1 J)^{-1}$ is strictly positive.

Repeated application of Lemma 2 yields a decreasing sequence of real numbers $\alpha_n \geq 0$ and a sequence of functions $w_n \in L^\infty(\Omega)$, $w_n \geq 0$, such that

$$(10) \quad (L + \alpha_n J)w_n = \alpha_n(M + 1)w_n - (\alpha_{n-1} - \alpha_n)Jw_n.$$

We will normalize w_n so that $\|w_n\|_\infty = 1$.

Lemma 3. *Assume that*

$$(11) \quad \int_{\Omega} m(x) dx < 0.$$

Then there exist λ_1 with $0 < \lambda_1 \leq \alpha_0$, and $u > 0$, such that $Lu = \lambda_1 MJu$.

Proof. From the above we know that $\alpha_n \rightarrow \lambda_1 \in [0, \alpha_0]$. If $\lambda_1 = 0$, then since

$$Lw_n = \alpha_n(M + 1)Jw_n - \alpha_{n-1}Jw_n,$$

we obtain that $\|Lw_n\|_{\infty} \leq 3\alpha_{n-1}\|w_n\|_{\infty} = 3\alpha_{n-1}$. Hence $Lw_n \rightarrow 0$ in L^{∞} , i.e. $\|(L + J)w_n\|_{\infty} \leq \text{const}$. The compactness of $(L + J)^{-1}$ implies that $w_n \rightarrow u$ for some $u \in L^{\infty}(\Omega)$, and $u > 0$, $\|u\|_{\infty} = 1$. We further have $Lu = 0$, and thus $u = 1$. Since

$$0 \leq \int |\nabla w_n|^p = \int \alpha_n m |w_n|^p + (\alpha_n - \alpha_{n-1}) \int |w_n|^p$$

we conclude that

$$\alpha_n \int m |w_n|^p > 0.$$

This implies that $\int_{\Omega} m(x) dx > 0$, which is a contradiction. Hence $\lambda_1 > 0$.

Then $(L + \alpha_n J)^{-1} \rightarrow (L + \lambda_1 J)^{-1}$ and $K_{\alpha_n} \rightarrow K_{\lambda_1}$. Thus we have

$$(12) \quad w_n - \alpha_n^{\frac{1}{p-1}} K_{\alpha_n} w_n \rightarrow 0.$$

Since K_{α_n} is compact, $\{w_n\}$ lies in a compact subset of $L^{\infty}(\Omega)$. Therefore by going to subsequence, if necessary, we can assume that $w_n \rightarrow u$ in L^{∞} and $u > 0$. From (12) we conclude that

$$u = \lambda_1^{\frac{1}{p-1}} K_{\lambda_1} u, \quad \text{i.e. } Lu = \lambda_1 MJu.$$

This ends the proof of the lemma.

Lemma 4. λ_1 is simple and unique.

Proof. Set

$$(13) \quad I(u, v) = \int (-\Delta_p u) \cdot \frac{u^p - v^p}{u^{p-1}} - \int (-\Delta_p v) \cdot \frac{u^p - v^p}{v^{p-1}}$$

for $u, v \in W^{1,p}$ and $u, v > 0$. Then Proposition 1 of Anane [1] ensures that $I(u, v) \geq 0$, and $I(u, v) = 0$ if and only if $u = cv$ for some constant c .

If u_1, u_2 are two normalized positive eigenfunctions (i.e. $\|u_i\|_{\infty} = 1$, $i = 1, 2$) associated with λ_1 , then

$$\begin{aligned} I(u_1, u_2) &= \int \lambda_1 m \cdot (u_1^p - u_2^p) - \int \lambda_1 m \cdot (u_1^p - u_2^p) \\ &= 0. \end{aligned}$$

Thus $u_1 \equiv u_2$, and λ_1 is simple. Similarly one can prove that λ_1 is unique. Lemma 4 is proved.

Now, summarizing the above we have

Theorem 1. *Assume that $\int_{\Omega} m < 0$. Then the eigenvalue problem (1)–(2) admits a positive eigenvalue λ_1 with a positive eigenfunction; and such a λ_1 is unique and simple.*

4. THE EXCEPTIONAL CASE

In this section we consider the exceptional case

$$(14) \quad \int_{\Omega} m(x) dx = 0.$$

Obviously, $\lambda_0 = 0$ is an eigenvalue with $u_0 = 1$ as the eigenfunction to (1)–(2). Furthermore, we have the following

Theorem 2. *Assume that $\int_{\Omega} m(x) dx = 0$. Then $\lambda_0 = 0$ is the only eigenvalue of (1)–(2) having a positive eigenfunction.*

Proof. Suppose not, i.e. assume that there exist a $\lambda_1 \neq 0$ and $u_1 > 0$ satisfying

$$-\Delta_p u_1 = \lambda_1 m u_1^{p-1} \text{ in } \Omega \quad \text{and} \quad \frac{\partial u_1}{\partial n} = 0 \text{ on } \partial\Omega.$$

For any $\delta > 0$, let

$$h(u_1) = \int_{\delta}^{u_1} \frac{dt}{t}.$$

Then

$$\begin{aligned} \nabla h(u_1) &= \frac{1}{u_1} \nabla u_1, \\ \frac{\partial h(u_1)}{\partial n} &= \frac{1}{u_1} \frac{\partial u_1}{\partial n} = 0 \quad \text{on } \partial\Omega \end{aligned}$$

and

$$(15) \quad \begin{aligned} -\Delta_p h(u_1) &= -\nabla \left(\frac{1}{u_1^{p-1}} |\nabla u_1|^{p-2} \nabla u_1 \right) \\ &= -u_1^{-(p-1)} \Delta_p u_1 + (p-1) u_1^{-p} |\nabla u_1|^p \\ &= \lambda_1 m + (p-1) u_1^{-p} |\nabla u_1|^p. \end{aligned}$$

Integrating (15) yields

$$-\int \lambda_1 m = \int (p-1) u_1^{-p} |\nabla u_1|^p.$$

Combining with (14) we conclude that $|\nabla u_1| = 0$, i.e. u_1 is a constant. Thus from (1)–(2) we conclude that $\lambda_1 = 0$, which is a contradiction. Hence the theorem is proved.

By the same technique, we have the following more general

Corollary 1. Assume that u is a positive solution to

$$(16) \quad -\Delta_p u = m(x)f(u) \quad \text{in } \Omega$$

$$(17) \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

where $f'(s) > 0$ for $s > 0$, $f(s) > 0$ for $s > 0$. Then $\int_{\Omega} m(x) \leq 0$.

Proof. Let

$$h(u) = \int_{\delta}^u \frac{dt}{[f(t)]^{1/p-1}},$$

where $\delta > 0$ is fixed. Then $h(u)$ satisfies

$$-\Delta_p h(u) = m(x) + \frac{f'(u)}{f^2(u)} |\nabla u|^p \quad \text{in } \Omega$$

$$\frac{\partial h(u)}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Integration yields

$$-\int_{\Omega} m(x) = \int_{\Omega} \frac{f'(u)}{f^2(u)} |\nabla u|^p \geq 0,$$

and hence the result.

Remark. We note that this corollary holds for nonnegative solutions: we need only consider the integration over the support of u in this case. A direct consequence of this corollary is that if $\int_{\Omega} m(x) > 0$, (16)–(17) has no positive solutions.

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