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RESTRICTIONS OF OPEN MAPPINGS OF CONTINUA

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ABSTRACT. We give an example of an open map $f : X \to Y$, where X is a locally connected one-dimensional continuum, such that there is a subcontinuum K of Y for which the restriction of f to a certain component of $f^{-1}(K)$ is not open.

E. Duda had recently asked the first author whether the following question has a negative answer: Let $f : X \xrightarrow{onto} Y$ be an open mapping of a continuum X (i.e. a compact connected Hausdorff space) onto a Hausdorff space Y. If L is a continuum in X such that L is a component of $f^{-1}(f(L))$, is $f|L$, the restriction of f to L, open ? Duda's question has a positive answer if the preimage of every continuum in X has only finitely many components. This is the case, for example, if the map f is finite–to–one, or, if the domain of f is hereditarily locally connected. Below (Theorem) we give a one-dimensional counterexample with a locally connected domain. We start by constructing an example with non-locally connected domain. It will serve to provide motivation for the more complicated main construction. Let us recall that a map $f : X \longrightarrow Y$ is *open at* $x \in X$ if it carries neighborhoods of x to neighborhoods of $f(x)$.

Example. Let $v = (0, 1)$, $a_0 = (0, 0)$ and $a_n = (1/n, 0)$ for each positive integer n. For each non-negative integer n let L_n be the line segment in the plane with endpoints v and a_n. Then $X = \bigcup\{L_n : n = 0, 1, 2, \dots\}$, the harmonic fan, is the cone with vertex v over the convergent sequence $\{a_0, a_1, a_2, \dots\}$. Let $B = \{]p_n, q_n[: n = 1, 2, \dots\}$ be a basis of open intervals for the open interval $]0, 1[$. Define a map f_0 of the closed interval $[0, 1]$ to itself by letting $f_0(0) = 0$, $f_0(1/3) = f_0(2/3) = 1/2$, $f_0(1) = 1$ and extending f_0 linearly. Let $\{f_n : n = 1, 2, \dots\}$ be a sequence of homeomorphisms of $[0, 1]$ onto $[0, 1]$ such that for each $n = 1, 2, \dots$

(1) $||f_n - f_0|| < 1/n$,
(2) $f_n(p_n) < f_0(p_n)$, and
(3) $f_0(q_n) < f_n(q_n)$.

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Define $f : X \xrightarrow{onto} [0, 1]$ by $f(x, y) = f_n(y)$ for $(x, y) \in L_n$. Then f is continuous and f is clearly open at every point of $X \setminus \{0\} \times [1/3, 2/3]$. Let $y \in [1/3, 2/3]$. Let $B_y = \{]p_{n(i)}, q_{n(i)}[: i = 1, 2, \dots\} \subset B$ be a neighborhood base at y such that $p_{n(i)} < p_{n(i+1)} < y < q_{n(i+1)} < q_{n(i)}$ for each i. Then the sequence of open intervals $]0, 1[\times]p_{n(i)}, q_{n(i)}[\cap L_{n(i)}$ is eventually in every neighborhood of $(0, y)$. Hence, $f(0, y) = f_0(y)$ is in the interior of $f(U)$ for each neighborhood U of $(0, y)$ and f is open at $(0, y)$. Now $K = \{0\} \times [0, 2/3]$ is a component of $f^{-1} \circ f(\{0\} \times [0, 2/3]) = f^{-1}([0, 1/2])$ but $f(\{0\} \times [1/3, 2/3]) = \{1/2\}$ so $f|K$ is not open.

A closed subset K of a space X is said to be a *Z-set* in X if for each $\varepsilon > 0$ there exists a mapping $f : X \longrightarrow X \setminus K$ such that f moves no point of X more than ε (see [2, p.151]. We denote the closure [resp., the boundary] of a set A by ClA [resp., BdA].

Theorem. *There exists an open map $h : X \xrightarrow{onto} N$ of a one-dimensional locally connected metric continuum X onto a Menger curve N such that there is a continuum K in X which is a component of $f^{-1} \circ f(K)$ and such that $f|K$ is not open.*

Proof. Let M denote the Menger curve. Let K be an arc which is a Z-set in M and let $A \subset K$ be an arc which does not contain an endpoint of K. Let $N = M/A$ and let $q : M \xrightarrow{onto} M/A$ be the natural quotient map. Observe that N is homeomorphic to M by Anderson's characterization of the Menger curve ([1]; see also [3, Theorem 4.11]) and let $a = q(A)$.

Let $B = q(K)$. Then B is an arc in N which is a Z-set in N and such that a is not an endpoint of B. Let $\{x_n : n = 1, 2, \dots\}$ be a dense set of non-endpoints of A.

Let $\{U_n : n = 1, 2, \dots\}$ be a sequence of open neighborhoods of A such that $M \supset ClU_1 \supset U_1 \supset ClU_2 \supset \cdots \supset ClU_n \supset U_n \supset ClU_{n+1} \supset \cdots$, $M \neq ClU_1$, $\bigcap\{U_n : n = 1, 2, \dots\} = A$, each ClU_n is a Menger curve and $Bd(U_n)$ is a Cantor set which is a Z-set in both ClU_n and $M \setminus U_n$. Notice that $q(Bd(U_n))$ is a Z-set in both $N \setminus q(U_n)$ and $q(ClU_n)$ for each n.

We claim that there exists a sequence of homeomorphisms $q_n : M \xrightarrow{onto} N$ such that

(4) $\lim_{n\to\infty} q_n = q$,
(5) $q_n(K) = B$,
(6) $q_n(x_n) = a$, and
(7) $q_n|M \setminus U_n = q|M \setminus U_n$.

By Corollary 6.5 of [3], there exists for each n a homeomorphism $q_n : M \xrightarrow{onto} N$ such that $q_n|M \setminus U_n = q|M \setminus U_n$, $q_n(K) = B$ and $q_n(x_n) = a$.

For each $n = 1, 2, \dots$ let $A_n = \{x_{n,1}, \dots, x_{n,m_n}\} \subset M \setminus K)$ be a $1/n$-mesh for M. This is possible since K is nowhere dense in M. For each $i = 1, \dots, m_n$ let $U_{n,i}$ be an open neighborhood of $x_{n,i}$ of diameter less

than $1/n$ and such that $ClU_{n,i} \cap (ClU_{n,j} \cup K) = \varnothing$ for $i \neq j$. By choosing U_n to be sufficiently small we can also insure that $Cl(U_{n,i}) \cap Cl(U_n) = \varnothing$.

Let $X = M \times Z/\sim$, where Z is the convergent sequence $\{0\} \cup \{1/n : n = 1, 2, \ldots\}$ and $\sim$ is the smallest upper semi-continuous equivalence relation on $M \times Z$ such that $(x, 1/n) \sim (x, 0)$ for $x \in U_{n,i}$, where $n = 1, 2, \ldots$ and $i \in \{1, \ldots, m_n\}$. Let $\pi_1 : M \times Z \xrightarrow{onto} X$ be the natural quotient map. Then X is a continuum. We prove that X is locally connected. Let $x \in X$. Since M is locally connected, let V be a small connected neighborhood of x in M. There exists an integer n such that the $1/n$–ball centered at x is in V. Hence, $A_m \cap V \neq \varnothing$ for each $m \geq n$. For n large $\pi_1(\bigcup\{V \times \{z\} | z \in Z$ and $z \leq 1/n$ or $\pi_1(x, z) = \pi_1(x, 0)\})$ is a small connected neighborhood of $\pi_1(x, 0)$. If $z \in Z$ is such that $\pi_1(x, z) \neq \pi_1(x, 0)$ then $\pi_1(V \times \{z\})$ is a small connected neighborhood of $\pi_1(x, z)$. Define a mapping $h : M \times Z \xrightarrow{onto} N \times Z$ by $h(x, 0) = q(0)$ and $h(x, 1/n) = q_n$ for $n = 1, 2, \ldots$. Let $\sim_2$ be the smallest upper semi-continuous relation on $N \times Z$ such that there is a commutative diagram of continuous maps and spaces

$$\begin{array}{ccc} M \times Z & \xrightarrow{h} & N \times Z \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ X & \xrightarrow{F} & N \times Z/\sim_2 \end{array}$$

where π_2 is the natural quotient map of $N \times Z$ onto $N \times Z/\sim_2$. There is a natural projection $\pi : N \times Z/\sim_2 \xrightarrow{onto} N$ such that $\pi \circ \pi_2 : N \times Z \xrightarrow{onto} N$ is the first coordinate projection. Let $f = \pi \circ F : X \xrightarrow{onto} N$. That f is open at points of $X \setminus \pi_1(A \times \{0\})$ is clear. It follows from (6) and the openness of q_n that f is open at points of $\pi_1(A \times \{0\})$. Clearly $\pi_1(K \times \{0\})$ is a component of $f^{-1}(B)$ and $f|\pi_1(K \times \{0\})$ is not open.

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