INDUCTIVE LIMITS
OF NORMED ALGEBRAS AND \( m \)-CONVEX STRUCTURES

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Abstract. In this note, we give a negative answer to a question raised by A. Arosio in 1974. It concerns the multiplicativity of the inductive limit topology associated to a family of normed algebras.

1. Introduction

We consider the following situation: \( E \) is an algebra and \( F = (E_i)_{i \in I} \) a family of subalgebras of \( E \) such that

(1) \( E = \bigcup E_i \), (2) \( E_i \) is a normed algebra for every \( i \), (3) \( F \) is a net with continuous injections.

We consider the finest locally convex topology on \( E \) making continuous the canonical mappings \( f_i \) ([2]). We denote it by \( t_L \). The question is to know if \( t_L \) is always \( m \)-convex.

Actually the question arose implicitly in 1956 when S. Warner ([9]) considered the finest locally \( m \)-convex topology making the \( f_i \) continuous.

A. Arosio ([2]) showed that the answer is positive if \( F \) is a chain. He did the same if \( F \) is countable and \( E \) commutative. Using a result of M. Raouyane ([8]), Nacir showed ([7]) that commutativity is not necessary for this result.

By considering the algebra of strongly bounded operators ([1]) and using a result of J. Esterle ([4]) we show that the answer is not always positive in the non-commutative case. The counter-example also contradicts an assertion given without proof in ([5]).

2. Preliminaries

Let \( E \) be a complex algebra endowed with a locally convex topology \( t \) given by a family \( (P_s)_{s \in S} \) of semi-norms, where \( S \) is a directed set; \( (E, t) \) is said to be:

(1) A locally convex algebra (l.c.a.) if the product \( (x, y) \mapsto x \cdot y \) is separately continuous.
(2) An $m$-convex algebra (l.m.c.a. [6]) if, for every $s$, $P_s(xy) \leq P_s(x) \cdot P_s(y)$ for every $x, y$.

(3) An $A$-convex algebra (l.A.c.a. [3]) if, for every $s$ and every $x$, there exist scalars $M(s, x) > 0$ and $N(s, x) > 0$ such that

$$P_s(xy) \leq M(s, x) \cdot P_s(y) \quad \text{for every } y,$$

$$P_s(yx) \leq N(s, x) \cdot P_s(y) \quad \text{for every } y.$$

3. Counter example

In the sequel $E$ will designate a Frechet locally convex space. Recall that an operator $T : E \to E$ is said to be strongly bounded if there exists a neighborhood $V$ of zero such that $T(V)$ is bounded. We also consider the collection $\mathcal{B}_0$ of closed absolutely convex bounded sets of $E$ and $\mathcal{V}$ a fundamental system of absolutely convex neighborhoods of zero such that $\cap\{V, V \in \mathcal{V}\} = \emptyset$. We denote by $\mathcal{B}(E)$ the set of strongly bounded operators of $E$. It is a subalgebra of the algebra of bounded operators of $E$. If, for every $V$ in $\mathcal{V}$ and every $B$ in $\mathcal{B}_0$ such that $B \subset V$, we put

$$E(V, B) = \{T \in \mathcal{B}(E) | \exists \alpha > 0 : T(V) \subset \alpha B\}$$

then

(1) the $E(V, B)$ are subalgebras the union of which is $\mathcal{B}(E)$.

(2) Each $E(V, B)$ is endowed with an algebra norm $\|T\| = \sup\{\|T(x)\|_B : x \in V\}$ where $\| \cdot \|_B$ is the gauge of $B$ in the space $E_B$ spanned by $B$.

(3) The family $(E(V, B))$ is directed if we put $(V_1, B_1) \leq (V_2, B_2)$, for $V_1 \supset V_2$ and $B_1 \subset \alpha B_2$ for some $\alpha > 0$. And the canonical mappings are continuous.

Now if we put $\Omega_{BV} = \{T \in \mathcal{B}(E) | T(B) \subset V\}$, $B \in \mathcal{B}_0$ and $V \in \mathcal{V}$, we get a fundamental system of a locally convex topology $t$ on $\mathcal{B}(E)$. It is Hausdorff and coarser than $t_L$. Hence $t_L$ is Hausdorff. Let us notice that $A(E) \subset \mathcal{B}(E)$, where $A(E)$ is the algebra of finite rank and continuous operators of $E$. At last we conclude to the non-$m$-convexity of $t_L$ by the fact that if $E$ is not normable then the algebra $A(E)$ admits no Hausdorff algebra topology for which the product is continuous ([3, Corollary 2, p. 1159]).

Remark 3.1. In the counter example the product is not continuous on $(\mathcal{B}(E), t_L)$. Also $(\mathcal{B}(E), t_L)$ cannot be a l.A.c.a. since then we could endow it with a Hausdorff $m$-convex topology by setting

$$q_s(x) = \sup\{P_s(xy) : P_s(y) \leq 1\}$$

Actually we first adjoin a unit to $\mathcal{B}(E)$, then consider the restriction of such a topology.

Remark 3.2. The counter example contradicts an affirmation given without proof in ([5, 1.4, p. 57]); it is asserted there that an inductive limit of l.c.a. with continuous products has also a continuous product.
REFERENCES


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