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ON ORDERED Λ-BOUNDED VARIATION

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ABSTRACT. An example is given of a continuous real function that is of ordered Λ-bounded variation but not of Λ-bounded variation. No special assumptions on Λ are required.

Preliminaries and motivation. Let $\Lambda = (\lambda_i)$ be a Λ-sequence, i.e. a nonincreasing convergent-to-0 sequence of positive numbers with $\sum \lambda_i = \infty$. For $m = 0, 1, 2, \ldots$ we will denote the Λ-sequence $(\lambda_{m+i})_{i=1}^{\infty}$ by $\Lambda_{(m)}$. Let $\langle \alpha, \beta \rangle$ be a closed interval and x be a real function defined on $\langle \alpha, \beta \rangle$. For a subinterval $I = \langle s, t \rangle$ of $\langle \alpha, \beta \rangle$ we set $r(x, I) = x(t) - x(s)$, $x(I) = |x(t) - x(s)|$ and $\Lambda - \mathrm{Var}(x; \langle s, t \rangle) = \sup \sum \lambda_i x(I_i)$, where the supremum is taken over all sequences (I_i) of nonoverlapping closed subintervals of $\langle s, t \rangle$. If $\Lambda - \mathrm{var}(x; \langle \alpha, \beta \rangle)$ is finite then we say x is of Λ-bounded variation (on $\langle \alpha, \beta \rangle$) and write $x \in \Lambda\mathrm{BV}$.

A finite collection $\{I_i : i = 1, \ldots, n\}$ of nonoverlapping closed subintervals of $\langle s, t \rangle$ will be called left-ordered (right-ordered) if $\sup I_{i+1} \leq \inf I_i (\sup I_i \leq \inf I_{i+1})$. Set $\Lambda - \mathrm{lovar}(x; \langle s, t \rangle) = \sup_1 \sum \lambda_i x(I_i)$ and $\Lambda - \mathrm{rovar}(x; \langle s, t \rangle) = \sup_2 \sum \lambda_i x(I_i)$, where the supremum $\sup_1 (\sup_2)$ is taken over all left-ordered (right-ordered) finite collections of nonoverlapping subintervals of $\langle s, t \rangle$. If $\Lambda - \mathrm{ovar}(x; \langle \alpha, \beta \rangle) = \max\{\Lambda - \mathrm{lovar}(x; \langle \alpha, \beta \rangle), \quad \Lambda - \mathrm{rovar}(x; \langle \alpha, \beta \rangle)\}$ is finite, then we say x is of ordered Λ-bounded variation and write $x \in O\Lambda\mathrm{BV}$. It is easy to see that the above definition is equivalent to the classical one ([4], p. 75).

Clearly, $\Lambda\mathrm{BV} \subset O\Lambda\mathrm{BV}$. In [4] D. Waterman asked: Is this inclusion proper? C. L. Belna has constructed in [1] a continuous function $f \in \mathrm{OHBV}$ such that $f \notin \mathrm{HBV}$ for the Λ-sequence $H = (1/i)$. Since the method of C. L. Belna is applicable only to OHBV and not to $O\Lambda\mathrm{BV}$ defined in the obvious way, D. Waterman has repeated his question in [6] and in the present paper we give the positive answer in the general case.

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Results. Let $n, k_1, \dots, k_n$ be positive integers and $(a_i)_{i=1}^n$ be a finite decreasing sequence of positive numbers. For a function $x\colon \langle \alpha, \beta\rangle \to \mathbf{R}$ we will write $x \in F_{k_1,\dots,k_n}^{a_1,\dots,a_n}\langle \alpha, \beta\rangle$ if there exists a finite decreasing sequence $(t_i)_{i=1}^{r_n+1}$ of points of $\langle \alpha, \beta\rangle$ such that $t_1 = \beta$, $t_{r_n+1} = \alpha$ and

$$\begin{aligned} &x(t_i) = 0 \qquad \text{for } i = 1, 3, 5, \dots, r_n + 1;\\ &x(t_i) = a_j \qquad \text{for } j = 1, \dots, n \quad i = r_{j-1} + 2, r_{j-1} + 4, \dots, r_j;\\ &x \text{ is linear on } \langle t_{i+1}, t_i\rangle \quad \text{for } i = 1, \dots, r_n, \end{aligned}$$

where $r_j = 2\sum_{i=1}^{j} k_i$ for $j = 0, 1, \dots, n$.

Lemma 1. *Let $\Lambda = (\lambda_i)$ be a Λ-sequence and $x \in F_{k_1,\dots,k_n}^{a_1,\dots,a_n}\langle \alpha, \beta\rangle$. Then*

$$\Lambda - \operatorname{var}(x; \langle \alpha, \beta\rangle) = \Lambda - \operatorname{lovar}(x; \langle \alpha, \beta\rangle) = \sum_{j=1}^{n} a_j \sum_{i=r_{j-1}+1}^{r_j} \lambda_i.$$

Proof. Setting $P_i = \langle t_{i+1}, t_i\rangle$ for $i = 1, \dots, r_n$, we have $\sup P_{i+1} = \inf P_i$. Thus

$$\begin{aligned} \Lambda - \operatorname{var}(x; \langle \alpha, \beta\rangle) &\geq \Lambda - \operatorname{lovar}(x; \langle \alpha, \beta\rangle)\\ &\geq \sum_{i=1}^{r_n} \lambda_i x(P_i) = \sum_{j=1}^{n} a_j \sum_{i=r_{j-1}+1}^{r_j} \lambda_i. \end{aligned}$$

Hence, it is enough to show that

$$\Lambda - \operatorname{var}(x; \langle \alpha, \beta\rangle) \leq \sum_{i=1}^{r_n} \lambda_i x(P_i).$$

Observe that for every finite collection $\{I_i\colon i = 1, \dots, s\}$ of nonoverlapping subintervals of $\langle \alpha, \beta\rangle$ there exists a collection $\{I_i'\colon i = 1, \dots, s\}$ of subintervals of $\langle \alpha, \beta\rangle$ such that $I_i' \subset I_i$, $x(I_i') = x(I_i)$ and $\{t_i\colon i = 1, \dots, r_n+1\} \cap \operatorname{int} I_j' = \varnothing$ for $j = 1, \dots, s$. Without loss of generality we may assume that the sets $Z_i = \{j = 1, \dots, s\colon I_j' \subset \langle t_{i+1}, t_i\rangle\}$ are nonempty for $i = 1, \dots, r_n$. Setting $m_i = \min Z_i$ and arranging all m_i in the decreasing order $w_1 > \dots > w_{r_n}$, we have $\lambda_{w_i} \leq \lambda_i$ for $i = 1, \dots, r_n$. Therefore, by ([2], Theorem 368)

$$\begin{aligned} \sum_{j=1}^{s} \lambda_j x(I_j) &= \sum_{i=1}^{r_n} \sum_{j \in Z_i} \lambda_j x(I_j') \leq \sum_{i=1}^{r_n} \lambda_{m_i} \sum_{j \in Z_i} x(I_j')\\ &\leq \sum_{i=1}^{r_n} \lambda_{m_i} x(P_i) \leq \sum_{i=1}^{r_n} \lambda_{w_i} x(P_i) \leq \sum_{i=1}^{r_n} \lambda_i x(P_i). \end{aligned}$$

So by ([3], Theorem 1) the proof is complete. □

For a Λ-sequence $\Lambda = (\lambda_i)$, a function $x \in F^{a_1,\dots,a_n}_{k_1,\dots,k_n}\langle\alpha, \beta\rangle$ and $s = 1, \dots, r_n$ we set $j_s = \max\{j = 1, \dots, n: r_{j-1} < s\}$, $p_s = s - r_{j_s-1}$ and

$$v(\Lambda, x, s) = a_{j_s}\sum_{i=1}^{p_s}\lambda_i + \sum_{m=1}^{j_s-1} a_m \sum_{i=p_s+r_{j_s-1}-r_{m+1}}^{p_s+r_{j_s-1}-r_{m-1}} \lambda_i.$$

Lemma 2. *Let $\Lambda = (\lambda_i)$ be a Λ-sequence and $x \in F^{a_1,\dots,a_n}_{k_1,\dots,k_n}\langle\alpha, \beta\rangle$. Then* $\Lambda - \operatorname{rovar}(x : \langle\alpha, \beta\rangle) = \max\{v(\Lambda, x, s): s = 1, \dots, r_n\}$.

Proof. Set $P_i = \langle t_{i+1}, t_i\rangle$ for $i = 1, \dots, r_n$. Since

$$\begin{aligned}\Lambda - \operatorname{rovar}(x; \langle\alpha, \beta\rangle) &\geq \max\left\{\sum_{i=1}^{s}\lambda_i x(P_{s+1-i}): s = 1, \dots, r_n\right\}\\ &= \max\{v(\Lambda, x, s): s = 1, \dots, r_n\},\end{aligned}$$

it is enough to show that

$$\Lambda - \operatorname{rovar}(x, \langle\alpha, \beta\rangle) \leq \max\{v(\Lambda, x, s): s = 1, \dots, r_n\}.$$

Let $\{I_j: j = 1, \dots, k\}$ be a finite right-ordered collection of nonoverlapping subintervals of $\langle\alpha, \beta\rangle$. Without loss of generality we may assume that $\{t_i: i = 1, \dots, r_n + 1\} \cap \operatorname{int} I_j = \varnothing$ for $j = 1, \dots, k$. Let $Z_i = \{j = 1, \dots, k: I_j \subset \langle t_{i+1}, t_i\rangle\}$ for $i = 1, \dots, r_n$. Setting $L = \{i = 1, \dots, r_n: Z_i \neq \varnothing\}$ and $c = \operatorname{card} L$, we have $c \leq r_n$. Further, set $m_i = \min Z_i$ for $i \in L$. Then $\lambda_j \leq \lambda_{m_i}$ for $j \in Z_i$. Arranging all elements of L in the decreasing order $w_1 > \cdots > w_c$, we have $m_{w_i} \geq i$ and $w_i \geq c + 1 - i$ for $i = 1, \dots, c$. Thus $\lambda_{m_{w_i}} \leq \lambda_i$ and $x(P_{w_i}) \leq x(P_{c+1-i})$ for $i = 1, \dots, c$. Finally,

$$\begin{aligned}\sum_{j=1}^{k}\lambda_j x(I_j) &= \sum_{i=1}^{c}\sum_{j\in Z_{w_i}}\lambda_j x(I_j) \leq \sum_{i=1}^{c}\lambda_{m_{w_i}}\sum_{j\in Z_{w_i}} x(I_j)\\ &\leq \sum_{i=1}^{c}\lambda_{m_{w_i}} x(P_{w_i}) \leq \sum_{i=1}^{c}\lambda_i x(P_{c+1-i})\\ &= v(\Lambda, x, c) \leq \max\{v(\Lambda, x, s): s = 1, \dots, r_n\}. \quad \square\end{aligned}$$

Lemma 3. *Let n be a positive integer. Then for arbitrary numbers $a, \gamma, \varepsilon > 0$ and arbitrary Λ-sequence $\Lambda = (\lambda_i)$ there exist positive numbers $a_1 > \cdots > a_n$ with $a_1 \leq a$ and positive integers $k_1, \dots, k_n$ such that* $\Lambda - \operatorname{var}(x; \langle\alpha, \beta\rangle) = n\gamma$ *and* $\Lambda - \operatorname{rovar}(x; \langle\alpha, \beta\rangle) \leq \gamma + \varepsilon$ *for $x \in F^{a_1,\dots,a_n}_{k_1,\dots,k_n}\langle\alpha, \beta\rangle$.*

Proof. Let $n = 1$. Given positive numbers a, γ, ε and a Λ-sequence $\Lambda = (\lambda_i)$ we set

$$k_1 = \min\left\{m = 1, 2, \dots: a\sum_{i=1}^{2m}\lambda_i \geq \gamma\right\} \quad \text{and} \quad a_1 = \gamma\left(\sum_{i=1}^{2k_1}\lambda_i\right)^{-1}.$$

For $x \in F_{k_1}^{a_1}\langle \alpha, \beta\rangle$ we have by Lemma 1

$$\Lambda - \operatorname{var}(x; \langle \alpha, \beta\rangle) = a_1 \sum_{i=1}^{2k_1} \lambda_i = \gamma$$

and by Lemma 2

$$\Lambda - \operatorname{rovar}(x; \langle \alpha, \beta\rangle) \le \max\left\{ a_1 \sum_{i=1}^{s} \lambda_i : s = 1, \dots, 2k_1 \right\} = \gamma < \gamma + \varepsilon.$$

Now assume that n is a positive integer such that for every positive number a, γ, ε and every Λ-sequence Λ there exist positive numbers $a_1 > \cdots > a_n$ with $a_1 \le a$ and positive integers $k_1, \dots, k_n$ such that $\Lambda-\operatorname{var}(x; \langle \alpha, \beta\rangle) = n\gamma$ and $\Lambda - \operatorname{rovar}(x; \langle \alpha, \beta\rangle) \le \gamma + \varepsilon$ for $x \in F_{k_1, \dots, k_n}^{a_1, \dots, a_n}\langle \alpha, \beta\rangle$. It is easy to see that if $x \in F_{k_1, \dots, k_n}^{a_1, \dots, a_n}\langle \alpha, \beta\rangle$ then $x \circ h \in F_{k_1, \dots, k_n}^{a_1, \dots, a_n}\langle \alpha', \beta\rangle$, where $\alpha' = (\alpha + \beta)/2$ and $h(t) = 2t - \beta$ for $t \in \langle \alpha', \beta\rangle$. Hence, given positive numbers a, γ, ε and a Λ-sequence $\Lambda = (\lambda_i)$, there exist positive numbers $a_1 > \cdots > a_n$ with $a_1 \le a$ and positive integers $k_1, \dots, k_n$ such that $\Lambda-\operatorname{var}(x; \langle \alpha', \beta\rangle) = n\gamma$ and $\Lambda-\operatorname{rovar}(x; \langle \alpha', \beta\rangle) \le \gamma + \varepsilon/2$ for $x \in F_{k_1, \dots, k_n}^{a_1, \dots, a_n}\langle \alpha', \beta\rangle$. Set $k = \sum_{i=1}^{n} k_i$ and

$$(+) \quad i_0 = \min\left\{ i = 1, 2, \dots : \frac{\lambda_{2k+i}}{\lambda_{2k}} \le \frac{\varepsilon}{4\gamma + 2\varepsilon} \right\},$$

$$(++) \quad d = \min\left\{ \frac{a_n}{2}, \varepsilon\left(4\sum_{i=1}^{2k} \lambda_i\right)^{-1}, \varepsilon\left(4\sum_{i=2k+1}^{2k+i_0} \lambda_i\right)^{-1} \right\},$$

$$k_{n+1} = \min\left\{ m = 1, 2, \dots : \sum_{i=2k+1}^{2k+2m} \lambda_i \ge \frac{\gamma}{d} \right\},$$

$$a_{n+1} = \gamma\left(\sum_{i=2k+1}^{2k+2k_{n+1}} \lambda_i\right)^{-1}.$$

Then $a_{n+1} \le d < a_n$ and $\Lambda_{(2k)} - \operatorname{var}(x; \langle \alpha, \alpha'\rangle) = \gamma$ for $x \in F_{k_{n+1}}^{a_{n+1}}\langle \alpha, \alpha'\rangle$. Given functions $x_1 \in F_{k_1, \dots, k_n}^{a_1, \dots, a_n}\langle \alpha', \beta\rangle$, $x_2 \in F_{k_{n+1}}^{a_{n+1}}\langle \alpha, \alpha'\rangle$, we set

$$y(t) = \begin{cases} x_1(t) & \text{for } t \in (\alpha', \beta\rangle, \\ x_2(t) & \text{for } t \in \langle \alpha, \alpha'\rangle. \end{cases}$$

Then $y \in F^{a_1, \dots, a_{n+1}}_{k_1, \dots, k_{n+1}}\langle \alpha, \beta \rangle$ and by Lemma 1

$$\begin{aligned}
\Lambda - \operatorname{var}(y; \langle \alpha, \beta \rangle) &= \sum_{j=1}^{n+1} a_j \sum_{i=r_{j-1}+1}^{r_j} \lambda_i \\
&= a_{n+1} \sum_{i=2k+1}^{2k+2k_{n+1}} \lambda_i + \sum_{j=1}^{n} a_j \sum_{i=r_{j-1}+1}^{r_j} \lambda_i \\
&= \Lambda_{(2k)} - \operatorname{var}(x_2; \langle \alpha, \alpha' \rangle) + \Lambda - \operatorname{var}(x_1; \langle \alpha', \beta \rangle) \\
&= (n+1)\gamma.
\end{aligned}$$

For $s = 1, \dots, r_n$ we have

(a) if $1 \le s \le 2k = r_n$ then

$$v(\Lambda, y, s) = v(\Lambda, x_1, s) \le \Lambda - \operatorname{rovar}(x_1; \langle \alpha', \beta \rangle) \le \gamma + \varepsilon/2;$$

(b) if $2k + 1 \le s$ then

(b_1) if $p_s \le 2k + i_0$ then by (++) and by (a)

$$\begin{aligned}
v(\Lambda, y, s) &= a_{n+1} \sum_{i=1}^{p_s} \lambda_i + \sum_{m=1}^{n} a_m \sum_{i=p_s+r_n-r_{m-1}+1}^{p_s+r_n-r_m} \lambda_i \\
&\le a_{n+1} \sum_{i=1}^{2k} \lambda_i + a_{n+1} \sum_{i=2k+1}^{2k+i_0} \lambda_i + \sum_{m=1}^{n} a_m \sum_{i=r_n-r_m+1}^{r_n-r_{m-1}} \lambda_{p_s+i} \\
&\le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \sum_{m=1}^{n} a_m \sum_{i=r_n-r_m+1}^{r_n-r_{m-1}} \lambda_i \\
&= \frac{\varepsilon}{2} + v(\Lambda, y, 2k) \le \gamma + \varepsilon;
\end{aligned}$$

(b_2) if $p_s > 2k + i_0$ then by (+)

$$\frac{\lambda_{p_s+i}}{\lambda_1} \le \frac{\lambda_{2k+i_0}}{\lambda_{2k}} \le \frac{\varepsilon}{4\gamma + 2\varepsilon} \qquad \text{for } i = 1, \dots, 2k$$

and thus, by (a),

$$\begin{aligned}
v(\Lambda, y, s) &= a_{n+1} \sum_{i=1}^{2k} \lambda_i + a_{n+1} \sum_{i=2k+1}^{p_s} \lambda_i + \sum_{m=1}^{n} a_m \sum_{i=r_n-r_m+1}^{r_n-r_{m-1}} \lambda_{p_s+i} \\
&\le \frac{\varepsilon}{4} + a_{n+1} \sum_{i=2k+1}^{2k+2k_{n+1}} \lambda_i + \sum_{m=1}^{n} a_m \sum_{i=r_n-r_m+1}^{r_n-r_{m-1}} \lambda_{2k+i_0+i} \\
&\le \frac{\varepsilon}{4} + \Lambda_{(2k)} - \operatorname{var}(x_2; \langle \alpha, \alpha' \rangle) + \sum_{m=1}^{n} a_m \sum_{i=r_n-r_m+1}^{r_n-r_{m-1}} \frac{\lambda_i \varepsilon}{4\gamma + 2\varepsilon} \\
&= \frac{\varepsilon}{4} + \gamma + \frac{\varepsilon}{4\gamma + 2\varepsilon} v(\Lambda, y, 2k).
\end{aligned}$$

Therefore, by Lemma 2, $\Lambda - \mathrm{rovar}(y; \langle \alpha, \beta \rangle) \leq \gamma + \varepsilon$. Since for every $x \in F^{a_1, \dots, a_{n+1}}_{k_1, \dots, k_{n+1}} \langle \alpha, \beta \rangle$ we have $\Lambda - \mathrm{var}(x; \langle \alpha, \beta \rangle) = \Lambda - \mathrm{var}(y; \langle \alpha, \beta \rangle)$ and $\Lambda - \mathrm{rovar}(x; \langle \alpha, \beta \rangle) = \Lambda - \mathrm{rovar}(y; \langle \alpha, \beta \rangle)$, the proof is complete. □

Theorem. *For every Λ-sequence $\Lambda = (\lambda_i)$ there exists a continuous $x \in O\Lambda\mathrm{BV} - \Lambda\mathrm{BV}$.*

Proof. Let (t_n) be an increasing convergent-to-β sequence of points of $\langle \alpha, \beta \rangle$ with $t_1 = \alpha$ and let $\sum \varepsilon_i$ be a convergent series of positive numbers. By Lemmas 3 and 1 for $n = 1, 2, \dots$ there exist positive numbers $a^n_1 > \dots > a^n_{2^n}$ with

$$a^n_1 \leq d_n, \tag{1}$$

positive integers $k^n_1, \dots, k^n_{2^n}$ and functions $x_n \in F^{a^n_1, \dots, a^n_{2^n}}_{k^n_1, \dots, k^n_{2^n}} \langle t_n, t_{n+1} \rangle$ such that

$$\Lambda_{(w_{n-1})} - \mathrm{var}(x_n; \langle t_n, t_{n+1} \rangle) = \Lambda_{(w_{n-1})} - \mathrm{lovar}(x_n; \langle t_n, t_{n+1} \rangle) = 2^n 2^{-n} \tag{2}$$

and

$$\Lambda_{(w_{n-1})} - \mathrm{rovar}(x_n; \langle t_n, t_{n+1} \rangle) \leq 2^{-n} + \varepsilon_n/2 \tag{3}$$

for $n = 1, 2, \dots,$ where

$$d_1 = 1 \qquad \text{and for } n = 1, 2, \dots \tag{4}$$

$$d_{n+1} = \min \left\{ \frac{a^n_{2^n}}{2}, \varepsilon_{n+1} \left(2 \sum_{i=1}^{w_n} \lambda_i \right)^{-1}, \qquad \varepsilon_{n+1} \left(2 \sum_{i=w_n+1}^{w_n+v_{n+1}} \lambda_i \right)^{-1} \right\};$$

$$w_n = 2 \sum_{j=1}^{n} \sum_{i=1}^{2^j} k^j_i \qquad \text{for } n = 0, 1, 2, \dots; \tag{5}$$

$$v_{n+1} = \min \left\{ i = 1, 2, \dots : 4 \sum_{j=1}^{n} \sum_{i=1}^{2^j} k^j_i a^j_i < \frac{\varepsilon_{n+1}}{\lambda_{w_n+i}} \right\} \qquad \text{for } n = 1, 2, \dots. \tag{6}$$

For $n = 1, 2, \dots$ we define a continuous function $y_n \colon \langle t_1, t_{n+1} \rangle \to \mathbf{R}$ by $y_n(t) = x_i(t)$ for $t \in \langle t_i, t_{i+1} \rangle$, $i = 1, \dots, n$. We shall show that for $n = 1, 2, \dots$ the following inequality holds:

$$\Lambda - \mathrm{lovar}(y_n; \langle t_1, t_{n+1} \rangle) \leq 1 + \sum_{i=1}^{n} \varepsilon_i. \tag{7}$$

By (2) it is clear for $n = 1$, because $x_1 = y_1$. Assume that n is a positive integer such that (7) holds. Let $\{I_i \colon i = 1, \dots, l\}$ be a finite left-ordered collection of nonoverlapping subintervals of $\langle t_1, t_{n+2} \rangle$. Without loss of generality we may assume $t_{n+1} \notin \mathrm{int}\ I_i$ for $i = 1, \dots, l$. Set $z = \max\{i = 1, \dots, l \colon I_i \subset \langle t_{n+1}, t_{n+2} \rangle\}$ and $h = \min\{z, w_n\}$. If $z \leq w_n + v_{n+1}$ then we have by the

inductional assumption and by (1), (4)

$$\begin{aligned}\sum_{i=1}^{l}\lambda_i y_{n+1}(I_i) &\le \sum_{i=1}^{h}\lambda_i x_{n+1}(I_i) + \sum_{i=w_n+1}^{z}\lambda_i x_{n+1}(I_i) + \sum_{i=z+1}^{l}\lambda_i y_n(I_i)\\ &\le \sum_{i=1}^{h}\lambda_i a_1^{n+1} + \sum_{i=w_n+1}^{w_n+v_{n+1}}\lambda_i a_1^{n+1} + \sum_{i=z+1}^{l}\lambda_{i-z} y_n(I_i)\\ &\le \frac{\varepsilon_{n+1}}{2} + \frac{\varepsilon_{n+1}}{2} + \Lambda - \operatorname{lovar}(y_n\,;\langle t_1, t_{n+1}\rangle)\\ &\le 1 + \sum_{i=1}^{n+1}\varepsilon_i.\end{aligned}$$

If $z > w_n + v_{n+1}$ then by (6)

$$\begin{aligned}\sum_{i=1+z}^{l}\lambda_i y_{n+1}(I_i) &\le \lambda_{z+1}\sum_{i=z+1}^{l} y_n(I_i)\\ &\le 2\lambda_{w_n+v_{n+1}}\sum_{j=1}^{n}\sum_{i=1}^{2^j} k_i^j a_i^j \le \frac{\varepsilon_{n+1}}{2}\end{aligned}$$

and by (1), (4), (2)

$$\begin{aligned}\sum_{i=1}^{z}\lambda_i y_{n+1}(I_i) &\le \sum_{i=1}^{w_n}\lambda_i x_{n+1}(I_i) + \sum_{i=w_n+1}^{z}\lambda_i x_{n+1}(I_i)\\ &\le a_1^{n+1}\sum_{i=1}^{w_n}\lambda_i + \Lambda_{(w_n)} - \operatorname{var}(x_{n+1}\,;\langle t_{n+1}, t_{n+2}\rangle)\\ &\le \frac{\varepsilon_{n+1}}{2} + 1.\end{aligned}$$

Therefore,

$$\Lambda - \operatorname{lovar}(y_{n+1}\,;\langle t_1, t_{n+2}\rangle) \le 1 + \sum_{i=1}^{n+1}\varepsilon_i.$$

Now, we shall show that

$$\Lambda - \operatorname{rovar}(y_n\,;\langle t_1, t_{n+1}\rangle) \le \sum_{i=1}^{n} 2^{-i} + \sum_{i=1}^{n}\varepsilon_i \tag{8}$$

for $n = 1, 2, \ldots$. For $n = 1$ it is clear by (2). Assume that n is a positive integer such that (8) holds. Let $\{I_i : i = 1, \ldots, l\}$ be a finite right-ordered collection of nonoverlapping subintervals of $\langle t_1, t_{n+2}\rangle$. Without loss of generality we may assume that $t_{n+1} \notin \operatorname{int} I_i$ for $i = 1, \ldots, l$. Setting $z = \max\{i = 1, \ldots, l : I_i \subset \langle t_1, t_{n+1}\rangle\}(\max\varnothing = 0)$ and $h = \min\{l, \max\{x, w_n\}\}$,

we have by (1), (4), (3) and by the inductional assumption

$$\begin{aligned}
&\sum_{i=1}^{l} \lambda_i y_{n+1}(I_i) \\
&\quad \le \sum_{i=1}^{z} \lambda_i y_n(I_i) + \sum_{i=z+1}^{h} \lambda_i x_{n+1}(I_i) + \sum_{i=h+1}^{l} \lambda_i x_{n+1}(I_i) \\
&\quad \le \Lambda - \operatorname{rovar}(y_n ; \langle t_1 , t_{n+1} \rangle) + a_1^{n+1} \sum_{i=1}^{w_n} \lambda_i + \Lambda_{(w_n)} - \operatorname{var}(x_{n+1} ; \langle t_{n+1} , t_{n+2} \rangle) \\
&\quad \le \sum_{i=1}^{n} 2^{-i} + \sum_{i=1}^{n} \varepsilon_i + \frac{\varepsilon_{n+1}}{2} + 2^{-n-1} + \frac{\varepsilon_{n+1}}{2}.
\end{aligned}$$

Therefore,

$$\Lambda - \operatorname{rovar}(y_{n+1} ; \langle t_1 , t_{n+2} \rangle) \le \sum_{i=1}^{n+1} 2^{-i} + \sum_{i=1}^{n+1} \varepsilon_i .$$

Finally, we define the function $y\colon \langle \alpha , \beta \rangle \to \mathbf{R}$ by $y(\beta) = 0$ and $y(t) = x_n(t)$ for $t \in \langle t_n , t_{n+1})$, $n = 1, 2, \dots$. The function y is continuous, since $a_1^n \le 2^{-n+1}$ for $n = 1, 2, \dots$. It is easy to see that $\Lambda - \operatorname{var}(y ; \langle \alpha , \beta \rangle) \ge \sum_n \Lambda_{(w_{n-1})} - \operatorname{var}(x_n ; \langle t_n , t_{n+1} \rangle)$. Thus, by (2) x is not of Λ-bounded variation.

Observe that for every finite left-ordered collection $\{I_i\colon i = 1, \dots, l\}$ of nonoverlapping subintervals of $\langle \alpha , \beta \rangle$ there exists a positive integer n and a finite left-ordered collection $\{I_i'\colon i = 1, \dots, l\}$ such that $I_i' \subset I_i$, $x(I_i') = x(I_i)$ and $\sup I_1' \le t_{n+1}$. Thus

$$\sum_{i=1}^{l} \lambda_i y(I_i) \le \Lambda - \operatorname{lovar}(y_n ; \langle t_1 , t_{n+1} \rangle)$$

and by (7) we have

$$\Lambda - \operatorname{lovar}(y ; \langle \alpha ; \beta \rangle) \le 1 + \sum_i \varepsilon_i . \tag{9}$$

It may be proved in a similar manner that

$$\Lambda - \operatorname{rovar}(y ; \langle \alpha , \beta \rangle) \le 1 + \sum_i \varepsilon_i .$$

This fact together with (9) implies that $y \in O\Lambda\mathrm{BV}$. □

The remarks from §§2 and 3 of [5] are true for arbitrary Λ-sequence Λ. In particular, $O\Lambda\mathrm{BV}$ is a Banach space with norm

$$\|g\| = |g(\alpha)| + \Lambda - \operatorname{ovar}(g ; \langle \alpha , \beta \rangle)$$

and one has

Corollary. $\Lambda\mathrm{BV}$ *is a first category subset of* $O\Lambda\mathrm{BV}$ *for arbitrary* Λ*-sequence* Λ.

Observe that the knowledge of construction of $x \in O\Lambda\mathrm{BV} - \Lambda\mathrm{BV}$ is not necessary to prove the above corollary. Namely, let $x \in O\Lambda\mathrm{BV} - \Lambda\mathrm{BV}$. Then there exists a sequence $\{I_n\}$ of nonoverlapping subintervals of $\langle \alpha, \beta \rangle$ such that $\sum \lambda_n x(I_n) = \infty$. It follows that there exists a subsequence $\{I_{n_i}\}$ such that $r(x, I_{n_i}) > 0$ and $\sum \lambda_i r(x, I_{n_i}) = \sum \lambda_i x(I_{n_i}) = \infty$ (we replace x by $-x$, if necessary). Set

$$m_k = \min\left\{ m = 1, 2, \ldots : \sum_{i=1}^{m} \lambda_i r(x, I_{n_i}) \geq k \right\}$$

and

$$F_k(g) = \sum_{i=1}^{m_k} \lambda_i r(g, I_{n_i})$$

for $k = 1, 2, \ldots$; $g \in O\Lambda\mathrm{BV}$. It is easy to see that for $k = 1, 2, \ldots$, F_k is a continuous linear functional on $O\Lambda\mathrm{BV}$. Using D. Waterman's arguments ([5], proof of Proposition 3), we complete the proof.

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