INSTANTANEOUS SHRINKING
IN NONLINEAR DIFFUSION-CONVECTION

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ABSTRACT. The Cauchy problem for a nonlinear diffusion-convection equation is studied. The equation may be classified as being of degenerate parabolic type with one spatial derivative and a time derivative. It is shown that under certain conditions solutions of the initial-value problem exhibit instantaneous shrinking. This is to say, at any positive time the spatial support of the solution is bounded above, although the support of the initial data function is not. This is a phenomenon which is normally only associated with nonlinear diffusion with strong absorption. In conjunction, a previously unreported phenomenon is revealed. It is shown that for a certain class of initial data functions there is a critical positive time such that the support of the solution is unbounded above at any earlier time, whilst the opposite is the case at any later time.

In this article we shall consider the Cauchy problem

(1) $$u_t = (u^m)_{xx} + (u^n)_x \quad \text{in } \mathbf{R} \times \mathbf{R}^+,$$

(2) $$u(x, 0) = u_0(x) \quad \text{for } x \in \mathbf{R},$$

where m and n are positive real constants and u_0 is a given bounded continuous nonnegative function.

Because, in general, problem (1), (2) does not admit a classical solution [4, 8, 10, 12], it is necessary to introduce the notion of a generalized solution. We shall say that a function $u(x, t)$ is a generalized supersolution of equation (1) in a domain

(3) $$D = (\eta_1, \eta_2) \times (0, T]$$

with

(4) $$-\infty \le \eta_1 < \eta_2 \le \infty \quad \text{and} \quad 0 < T < \infty$$

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if it is defined, real, nonnegative, bounded, and continuous in $\overline{D}$ and satisfies the integral inequality

$$(5) \qquad \int_{t_1}^{t_2} \int_{x_1}^{x_2} \left\{ u\phi_t + u^m \phi_{xx} - u^n \phi_x \right\} dx\, dt$$

$$\leq \int_{x_1}^{x_2} \left\{ u(x, t_2)\phi(x, t_2) - u(x, t_1)\phi(x, t_1) \right\} dx$$

$$+ \int_{t_1}^{t_2} \left\{ u^m(x_2, t)\phi_x(x_2, t) - u^m(x_1, t)\phi_x(x_1, t) \right\} dt$$

for all bounded rectangles $R = (x_1, x_2) \times (t_1, t_2] \subseteq D$ and nonnegative functions $\phi \in C^{2,1}(\overline{R})$ such that $\phi(x_1, t) = \phi(x_2, t) = 0$ for all $t \in [t_1, t_2]$. Similarly, we shall say that a function is a generalized subsolution of equation (1) in D if it satisfies the previous criteria with the inequality sign in (5) reversed. We define a generalized solution of problem (1), (2) as a function $u(x, t)$ which is a generalized supersolution and a generalized subsolution of (1) in all domains of the form (3), (4) and which satisfies (2).

With the above-mentioned notion of a generalized solution, problem (1), (2) has been shown to be uniquely solvable. Furthermore, for any generalized solution of problem (1), (2), $u(x, t)$, the following comparison principle has been proven. If $v(x, t)$ denotes a generalized supersolution of equation (1) in a domain D of the form (3), (4) such that $v(x, t) \geq u(x, t)$ for all $(x, t) \in \overline{D}\backslash D$ then $v(x, t) \geq u(x, t)$ for all $(x, t) \in \overline{D}$. Correspondingly, if $v(x, t)$ denotes a generalized subsolution of equation (1) in a domain D of the type (3), (4) such that $v(x, t) \leq u(x, t)$ for all $(x, t) \in \overline{D}\backslash D$ then $v(x, t) \leq u(x, t)$ for all $(x, t) \in \overline{D}$ [4, 8].

Suppose now that u_0 is not identically zero, and let $u(x, t)$ denote the unique generalized solution of problem (1), (2). Define

$$\zeta^-(t) = \inf \{ x \in \mathbf{R} : u(x, t) > 0 \}$$

and

$$\zeta^+(t) = \sup \{ x \in \mathbf{R} : u(x, t) > 0 \}$$

for all $t \geq 0$, i.e. let $\zeta^{\pm}$ denote the limits of the support of the solution as a function of t. When they are finite these functions are interfaces or free boundaries entering the solution of problem (1), (2), and it is at points on these interfaces that the generalized solution $u(x, t)$ may fail to be a classical solution [4, 7, 8, 12].

As u_0 is not identically zero

$$(6) \qquad\qquad -\infty \leq \zeta^-(t) < \zeta^+(t) \leq \infty \qquad \text{for all } t \geq 0$$

[7]. Moreover, the following can be stated when $n \geq 1$ [7, 10, 12]:

    (i) If $m \leq 1$ or $\pm\zeta^{\pm}(0) = \infty$ then $\pm\zeta^{\pm}(t) = \infty$ for all $t > 0$.
    (ii) If $m > 1$ and $\pm\zeta^{\pm}(0) < \infty$ then $\pm\zeta^{\pm}(t) < \infty$ for all $t > 0$.

In this respect, equation (1) resembles the well-known *porous media equation* which may be obtained by deleting the term $(u^n)_x$ in (1), and which, as a prototype for nonlinear diffusion equations, has been the subject of considerable study in recent years [1, 2, 12]. In contrast, when $n < 1$, equation (1) gives rise to behavior which is quite different from that exhibited by solutions of the porous media equation. The following is known to be the case when $n < 1$ [3, 7]:

    (i) If $m \leq n$, then irrespective of the value of $\zeta^{\pm}(0)$, $\pm\zeta^+(t) = \infty$ for all $t > 0$.

    (ii) If $m > n$, then irrespective of the value of $\zeta^-(0)$, $-\zeta^-(t) = \infty$ for all $t > 0$.

    (iii) If $m > n$ and $\zeta^+(0) < \infty$ then $\zeta^+(t) < \infty$ for all $t > 0$.

In this article we shall shed light on the remaining possibility when $n < 1$, namely $m > n$ and $\zeta^+(0) = \infty$, by proving the following theorem.

Theorem 1. *Suppose that* $n < 1$, $m > n$ *and*

$$(7) \qquad u_0(x) \sim Cx^{-1/(1-n)} \qquad \text{as } x \to \infty$$

for some constant $C > 0$. *Then*

$$\zeta^+(t) = \infty \qquad \text{for all } t \in (0, C^{1-n}/n)$$

and

$$\zeta^+(t) < \infty \qquad \text{for all } t \in (C^{1-n}/n, \infty).$$

In view of the comparison principle for solutions of problem (1), (2), this theorem infers that if

$$u_0(x) = o(x^{-1/(1-n)}) \qquad \text{as } x \to \infty$$

and u_0 is not identically zero for large x, then there is *instantaneous shrinking* of the support of the solution of problem (1), (2). This is to say, despite the fact that $\zeta^+(0) = \infty$, $\zeta^+(t) < \infty$ for any $t > 0$.

This remarkable behavior has been proved for the Cauchy problem for the equation

$$(8) \qquad u_t = (a(u))_{xx} + (b(u))_x - c(u),$$

where $a, b \in C^1([0, \infty))$ and $C \in C([0, \infty))$ satisfy

$$a(0) = 0, \quad a'(s) > 0 \qquad \text{for all } s > 0,$$

$$b'(s)/a'(s) \text{ is bounded on compact subsets of } [0, \infty),$$

$$c(0) = 0, \quad c(s) > 0 \qquad \text{for all } s > 0,$$

and

$$a'(s)\{a(s)c(s)\}^{-1/2}, \quad \{c(s)\}^{-1} \in L^1(0, 1),$$

together with certain other conditions, by Evans and Knerr [5]. For the particular case

$$(9) \qquad u_t = (u^m)_{xx} + (u^n)_x - u^p,$$

with m, n and p real constants, these conditions are equivalent to

$$m \geq 1,$$
$$n = 0 \quad \text{or} \quad n \geq m,$$

and

$$0 < p < 1.$$

Instantaneous shrinking has also been established for the equation

$$(10) \qquad u_t = u_{xx} - (1 + x^2)^{-1/2} c(u),$$

where

$$c(u) = u^p \qquad \text{with } 0 < p < 1,$$

by Kalashnikov [11]. However, to the best of the authors' knowledge, this is the first time that this phenomenon has been shown to occur for an equation of the form (8) (or (10)) with the term c absent.

Equations of the general type (8) arise in a number of physical situations. In these contexts, t denotes time and x a spatial coordinate. Furthermore, the term $(a(u))_{xx}$ is commonly referred to as the diffusion term, $(b(u))_x$ as the convection term, and $c(u)$ as the absorption or sink term. Apparently, for equation (1) with $n < 1$ and $n < m$, the convection mechanism is sufficiently powerful in comparison to the diffusive mechanism that in some way the equation emulates nonlinear diffusion with strong absorption.

Theorem 1 though brings to light a previously unreported phenomenon which is even more remarkable than instantaneous shrinking alone. One could speak of *deferred instantaneous shrinking*. When (7) holds, there is a critical time $\tau^* = C^{1-n}/n$, $0 < \tau^* < \infty$, such that the support of the solution is unbounded above for any $t < \tau^*$, whilst the opposite is the case for any $t > \tau^*$. What actually happens at the critical time τ^* is a riddle.

To prove Theorem 1 we shall utilize the following result which is of some inherent interest.

Theorem 2. *Suppose that $n < 1$, $m > n$ and*

$$u_0(x) \leq C x^{-\gamma} \qquad \text{for all } x \geq x_0$$

for some constants $C > 0$,

$$\gamma > 1/(1 - n)$$

and $x_0 > 0$. Set

$$\sigma = 1/\{(1 - n)\gamma - 1\}$$

and

$$\mu = (n\omega\gamma)^{-\sigma} \qquad \text{where} \quad \omega = \min\{m, 1\} - n.$$

Then, given any $\varepsilon > 0$ there exists a $T > 0$ such that

$$\zeta^+(t) \leq (\mu C^{(1-n)\sigma} + \varepsilon)t^{-\sigma} \qquad \text{for all } t \in (0, T].$$

Theorem 2 provides a lower estimate of the speed of shrinking under conditions appropriate to instantaneous shrinking.

The remainder of this article is devoted to the proof of Theorems 1 and 2. First, we shall prove Theorem 1 assuming that Theorem 2 is true. Thereafter, we prove Theorem 2.

We refer the interested reader to [7, 9, 12, 15] for additional information on the qualitative behavior of solutions of problem (1), (2) and their associated interfaces $\zeta^\pm$, and to [1, 2, 12] for surveys of work on these and related topics. Data on diverse characteristics of nonnegative solutions of equation (9) such as the occurrence of similar interfaces in generalized solutions of this equation, the localization and extinction of solutions, and on the role in which the exponents in the diffusion term, the convection term and the absorption term in (9) play in determining these characteristics may be found in [6, 10, 13–15].

THE PROOF OF THEOREM 1

To prove the theorem we shall use the following two lemmas.

Lemma 1. *Suppose that*

$$u_0(x) \geq Cx^{-1/(1-n)} \qquad \text{for all } x \geq x_0$$

for some $C > 0$ and $x_0 > 0$. Then $\zeta^+(t) = \infty$ for all $t \in [0, C^{1-n}/n)$.

Proof. Let $P_0 = \{x \in \mathbf{R} : u_0(x) > 0\}$. In view of the comparison principle for generalized solutions of problem (1), (2), without loss of generality we may suppose that P_0 is connected. Set $T = C^{1-n}/n$.

Recalling (6), $\sup\{u(x, T) : x \in \mathbf{R}\} > 0$. Hence, utilizing an argument in [7] there exists a function $\xi \in C([0, T])$ such that $u(\xi(t), t) > 0$ for all $t \in [0, T]$. Furthermore, since P_0 is connected, $u_0(x) > 0$ for all $x \in [\xi(0), x_0]$. Consequently, if we define

$$\Omega = \{(x, t) : \xi(t) < x < \infty, 0 < t \leq T\}$$

and

$$v(x, t) = C(x + \rho)^{-1/(1-n)}(1 - t/T)^{1/(1-n)},$$

we can pick a ρ sufficiently large that $x + \rho > 0$ and

(11) $$v(x, t) \leq u(x, t) \qquad \text{for all } (x, t) \in \overline{\Omega} \backslash \Omega.$$

Letting N denote the nonlinear parabolic differential operator

(12) $$N(z) = (z^m)_{xx} + (z^n)_x - z_t,$$

by direct computation

$$N(v) = m(m + 1 - n)(1 - n)^{-2}(x + \rho)^{-2}v^m \geq 0 \qquad \text{in } \Omega.$$

So v satisfies (11) and is a classical subsolution of equation (1) in Ω.

From the last remark we would like to be able to conclude that

$$(13) \qquad\qquad v(x, t) \leq u(x, t) \qquad \text{for all } (x, t) \in \overline{\Omega}.$$

Unfortunately the quoted comparison principle cannot deliver this conclusion since the domain Ω is not of the form (3), (4). However [8], u can be constructed on the domain $(-\infty, \infty) \times (0, T]$ as the pointwise limit of a decreasing sequence of positive classical solutions of equation (1), $\{u_k\}_{k=1}^{\infty}$, each one of which is defined on the corresponding member of an expanding sequence of bounded domains $\{\overline{Q}_k\}_{k=1}^{\infty}$, where

$$Q_k = (-k, k) \times (0, T],$$
$$u_k(x, 0) \geq u_0(x) \qquad \text{for all } x \in [-k, k],$$
$$u_k(\pm k, t) \geq M \qquad \text{for all } t \in [0, T],$$

and

$$M = \sup \{u(x, t): -\infty < x < \infty, \ 0 \leq t \leq T\}.$$

Thus, by applying the classical maximum principle to compare v with each solution u_k in $\Omega \cap Q_k$ and then taking limits, we can still draw the conclusion (13).

From (13) we deduce that $\zeta^+(t) = \infty$ for all $t \in [0, T)$. $\quad\square$

Lemma 2. *Suppose that*

$$u_0(x) \leq C x^{-1/(1-n)} \qquad \text{for all } x \geq x_0$$

for some $C > 0$ and $x_0 > 0$. Then given any $T > C^{1-n}/n$ and $\alpha > 0$ there exists a $K > 0$ and an $x_1 \geq x_0$ such that

$$u(x, T) \leq K x^{-(1+\alpha)/(1-n)} \qquad \text{for all } x \geq x_1.$$

Proof. Fix $T > C^{1-n}/n$ and $\alpha > 0$. Let

$$M = \sup \{u_0(x): x \in \mathbf{R}\}$$

and

$$\gamma = 1/(1 - n),$$

and for $x_1 \geq x_0$ consider the function

$$v(x, t) = C x^{-\gamma} \left(1 - t/T + M^{1-n} C^{n-1} x_1^{1+\alpha} x^{-\alpha} \right)^{\gamma}.$$

Set

$$\Omega = \{(x, t) \in (x_1, \infty) \times (0, T]: v(x, t) < M\}.$$

We assert that we can choose x_1 sufficiently large that v is a classical supersolution of equation (1) in Ω.

Assuming for the moment that v is indeed a classical supersolution of equation (1) in Ω, it can be verified that the function w defined in the closure of the domain

$$D = (-\infty, \infty) \times (0, T]$$

by

$$w(x, t) = \begin{cases} v(x, t) & \text{for } (x, t) \in \overline{\Omega} \\ M & \text{for } (x, t) \in \overline{D} \backslash \overline{\Omega} \end{cases}$$

is a generalized supersolution of equation (1) in this new domain. Furthermore, if $x^* = \sup\{x \in \mathbf{R} : w(x, 0) = M\}$, we have

$$w(x, 0) = M \geq u_0(x) \qquad \text{for all } x \leq x^*$$

and

$$w(x, 0) = v(x, 0) > Cx^{-\gamma} \geq u_0(x) \qquad \text{for all } x \geq x^*.$$

Hence, by the comparison principle for generalized solutions of problem (1), (2),

$$w(x, T) \geq u(x, T) \qquad \text{for all } x \in \mathbf{R}.$$

However, since

$$w(x, T) = v(x, T) = M(x_1/x)^{\gamma(1+\alpha)} \qquad \text{for all } x \geq x_1,$$

this proves the lemma.

To complete the proof of the lemma it therefore remains to demonstrate that we can choose an x_1 sufficiently large that v is a classical supersolution of equation (1) in Ω. With N given by (12), we calculate

$$N(v) = \alpha\gamma M^{1-n} x_1^{1+\alpha} x^{-\alpha-2} v^{2n-1} I_1 + \gamma x^{-1} v^n I_2$$

in Ω, where

(14) $\quad I_1 = -n + m(1 + \alpha + 2m\gamma)x^{-1}v^{m-n} + m\alpha(m\gamma - 1)M^{1-n}x_1^{1+\alpha}x^{-\alpha-2}v^{m-1}$

and

$$I_2 = -n + C^{1-n}/T + m(1 + m\gamma)x^{-1}v^{m-n}.$$

Observing that

$$M \geq v(x, t) \geq M(x_1/x)^{\gamma(1+\alpha)} \qquad \text{for all } (x, t) \in \Omega,$$

we can estimate

$$v^{m-1} \leq M^{m-1} \qquad \text{if } m \geq 1$$

and

$$v^{m-1} \leq M^{m-1}(x_1/x)^{\gamma(1+\alpha)(m-1)} \qquad \text{if } m < 1$$

in Ω. Subsequently, substituting these expressions in (14) and then using the datum that $x \geq x_1$ in Ω, we deduce

(15) $\quad I_1 \leq -n + m(1 + \alpha + 2m\gamma + \alpha|m\gamma - 1)M^{m-n}x_1^{-1} \qquad \text{in } \Omega.$

Analogously, we may also derive

$$(16) \qquad I_2 \leq -n + C^{1-n}/T + m(1 + m\gamma)M^{m-n}x_1^{-1} \qquad \text{in } \Omega.$$

Since though $T > C^{1-n}/n$, it follows from (15) and (16) that we can choose x_1 so large that $I_1 \leq 0$ and $I_2 \leq 0$ in Ω. Whence, we have shown that we can indeed select an $x_1 \geq x_0$ such that v is a classical supersolution of equation (1) in Ω. □

Armed with Lemmas 1 and 2 and with Theorem 2, we can easily prove Theorem 1. By Lemma 1,

$$\zeta^+(t) = \infty \qquad \text{for all } t \in [0, C^{1-n}/n).$$

On the other hand, by Lemma 2,

$$u(x, T) = \mathrm{O}\left(x^{-(1+\alpha)/(1-n)}\right) \qquad \text{as } x \to \infty$$

for any $T > C^{1-n}/n$ and $\alpha > 0$. But then according to Theorem 2, there exists a $\Delta > 0$ such that

$$\zeta^+(t) < \infty \qquad \text{for all } t \in (T, T + \Delta].$$

In turn this implies that

$$\zeta^+(t) < \infty \qquad \text{for all } t \in (T, \infty)$$

[8]. So, since $T > C^{1-n}/n$ was arbitrary,

$$\zeta^+(t) < \infty \qquad \text{for all } t \in (C^{1-n}/n, \infty).$$

THE PROOF OF THEOREM 2

The proof of Theorem 2 is similar in spirit to the proof of Lemma 2. Pick $A > C$ and $\nu > \mu$. Let

$$\delta = 1/\sigma = (1 - n)\gamma - 1$$

and consider the function $v(x, t)$ defined in the closure of the domain

$$D = (\eta, \infty) \times (0, T]$$

where $x_0 \leq \eta < \infty$ and $0 < T < \infty$ by

$$v(x, t) = Ax^{-\gamma}\left[1 - A^{n-1}\nu^{-\delta}x^{\delta}t\right]_+^{1/\omega}.$$

Here

$$[z]_+ = \max\{0, z\}.$$

With N given by (12), we compute

$$N(v) = x^{-1}v^n I_1 + \delta\omega^{-1}\nu^{-\delta}A^{n-1+\omega}x^{\delta-\gamma\omega-1}v^{n-\omega}tI_2$$

with

$$I_1 = -n\gamma + \omega^{-1}\nu^{-\delta}A^{n-1+\omega}x^{\delta-\gamma\omega+1}v^{1-n-\omega} + m\gamma(m\gamma+1)x^{-1}v^{m-n}$$

and

$$I_2 = -n + m(2m\gamma-\delta+1)x^{-1}v^{m-n} + m\delta(m\omega^{-1}-1)\nu^{-\delta}A^{n-1+\omega}x^{\delta-\gamma\omega-1}v^{m-n-\omega}t$$

at all points $(x, t) \in \Omega$ where

$$\Omega = \{(x, t) \in D \colon v(x, t) > 0\}.$$

Observing that $v(x, t) \le Ax^{-\gamma}$ and $t \le \nu^{\delta}A^{1-n}x^{-\delta}$ for all $(x, t) \in \Omega$, we can estimate

$$I_1 \le -n\gamma + \omega^{-1}\nu^{-\delta} + m\gamma(m\gamma+1)A^{m-n}\eta^{-(m-n)\gamma-1}$$

and

$$I_2 \le -n + m\left\{|2m\gamma-\delta+1| + \delta(m\omega^{-1}-1)\right\}A^{m-n}\eta^{-(m-n)\gamma-1}$$

in Ω. Hence, since $\omega^{-1}\nu^{-\delta} < n\gamma$, we can always choose an η so large that

$$I_1 \le 0 \quad \text{and} \quad I_2 \le 0$$

in Ω. The function v is consequently a classical supersolution of equation (1) at all points $(x, t) \in \Omega$. Noting though that v^m is continuously differentiable in D, it can be verified that v is subsequently a generalized supersolution of (1) in D.

With η fixed, we let T be so small that

$$v(\eta, t) > u(\eta, t) \quad \text{for all } t \in [0, T].$$

From here it follows that

$$v(x, t) \ge u(x, t) \quad \text{for all } (x, t) \in \overline{D}\backslash D.$$

Thus, by the comparison principle,

$$v(x, t) \ge u(x, t) \quad \text{for all } (x, t) \in \overline{D}.$$

However, this means that

$$\zeta^+(t) \le \nu A^{(1-n)/\delta}t^{-1/\delta} \quad \text{for all } t \in (0, T].$$

Since $\nu > \mu$ and $A > C$ were arbitrary, this validates the theorem.

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