INVERSE-CLOSED CARLEMAN ALGEBRAS OF INFINITELY DIFFERENTIABLE FUNCTIONS

JAMIL A. SIDDIQI

(Communicated by John B. Conway)

Abstract. We characterize the classes $\mathscr{C}_M(I)$ and $\mathscr{C}_M^*(I)$ of infinitely differentiable functions which are inverse-closed thereby giving a complete solution to a problem first posed by W. Rudin [11] and solved by him and J. Boman and L. Hörmander [2] for classes $\mathscr{C}_M(\mathbb{R})$ alone.

1. Introduction

Given a positive sequence $M = \{M_n\}$ and an interval $I$, let $\mathscr{C}_M(I)$ denote the Carleman class of all infinitely differentiable complex functions $f$ defined on $I$ for which $\sup_n \{\|f^{(n)}\|_\infty / M_n\}^{1/n} < \infty$. A class $\mathscr{C}_M(I)$ is said to be inverse-closed if $f^{-1}$ is in $\mathscr{C}_M(I)$ whenever $f$ is in $\mathscr{C}_M(I)$ and is bounded away from zero. More generally, analytic functions are said to operate on $\mathscr{C}_M(I)$ if for any $f$ in $\mathscr{C}_M(I)$ and for any $g$ analytic in an open set containing the closure of the range of $f$, $g \circ f$ is in $\mathscr{C}_M(I)$. A sequence $M$ is called log-convex if $M_{2n} < M_{n-1}M_{n+1}$ for $n \geq 1$.

In his paper on the symbolic calculus for the algebra of real functions which are Fourier transforms of functions in $L^1(\mathbb{R})$, P. Malliavin [8] (see J.-P. Kahane [7] for this and other related results) proved the following:

Theorem A. If $M$ is a log-convex sequence and the sequence $A \equiv \{A_n = (M_n/n!)^{1/n}\}$ is almost increasing in the sense that there exists a constant $K > 0$ such that $A_m \leq KA_n$ for each $m$ and $n$ with $m < n$, then the class $\mathscr{C}_M(I)$ is inverse-closed.

The problem as to whether the converse of Theorem A holds was first taken up by W. Rudin [11] who proved that it is so if $\mathscr{C}_M(\mathbb{R})$ is a non-quasi-analytic class of $2\pi$-periodic functions, a restriction that was later removed by J. Boman and...
L Hörmander [2]. Thus even when the Carleman classes on \( I \neq \mathbb{R} \) are defined by log-convex sequences, the converse of Theorem A is not known.

It is clear that Malliavin’s sufficiency condition is applicable only to classes \( C_M(I) \) which are defined by log-convex sequences. However there exist classes \( C_M(I) \) which cannot be so defined. In fact, H. Cartan [5] has shown that if \( I \) is closed, then \( C_M(I) \neq C_{M'}(I) \) even if \( C_n(I) \subseteq C_M(I) \), where \( M^c \) denotes the largest log-convex minorant of \( M \). Thus for classes \( C_M(I) \) not defined by log-convex sequences, the problem of finding necessary and sufficient condition on \( M \) in order that they be inverse-closed over arbitrary \( I \), has remained open so far (see, however, [12]). The same holds for local Carleman classes \( C^*_M(I) \) of functions which belong to class \( C_M(J) \) for each compact subinterval \( J \) of \( I \).

In this paper, we give a complete solution of the problem for these classes. Instead of limiting a priori to classes defined by log-convex sequences \( M \), we consider arbitrary classes which, as is well known (see [10]), can also be defined by regularized sequences \( M' \) which vary according to the nature of the interval \( I \). Using either the characteristic functions of these classes or Baire’s category theorem applied to certain Fréchet spaces, we are able to show that they are inverse-closed if and only if \( \{(M'/n^r)!/n^r\} \) is almost increasing. The techniques employed here are different. They are simpler than those used for \( I = \mathbb{R} \) in [11] and [2] and enable us to solve, in particular, the problem of characterization of inverse-closed algebras \( C^{2\pi}_{M}(I) \) of 2\( \pi \)-periodic functions, posed by W. Rudin in [11].

The inverse-closed algebras \( C_M(I) \) and \( C^*_M(I) \) are, respectively, inductive limits of Banach and Fréchet spaces. Although, with the usual seminorms, they are not locally convex algebras (see [9] for relevant definitions), we can describe their (algebraic) maximal ideals and complex homomorphisms. Thus every maximal ideal of the inverse-closed algebra \( C_M(I) \) is of the form \( \mathcal{F}_x = \{ f \in C_M(I) : f(x) = 0 \} \) for some \( x \in I \) and every complex homeomorphism is a point evaluation.

We may remark that J. Bruna [3] has proved that if \( M \) is log-convex, then the differentiable Beurling classes and their projective limits are inverse-closed if and only if \( \{(M'/n)!/n^r\} \) is almost increasing.

2. Inverse-closed Carleman classes

A Carleman class \( X = C_M(\mathbb{R}) \) is always an algebra. In fact, if

\[
\liminf_{n \to \infty} M_n^{1/n} = 0, \quad X \equiv \{\text{const}\}
\]

and if

\[
0 < \liminf_{n \to \infty} M_n^{1/n} < \infty, \quad X \equiv C_1(\mathbb{R}).
\]

In both these cases \( X \) is an algebra. Suppose now that \( \lim_{n \to \infty} M_n^{1/n} = \infty \). If
we set
\[ T_M(r) = \sup_{n \geq 1} \frac{r^n}{M_n}, \]
then
\[ M^c_n = \sup_{r \geq 1} \frac{r^n}{T_{sc}(r)}, \quad T_{sc}(r) = \sup_{n \geq 1} \frac{r^n}{M^c_n} = T_{M^c}(r). \]

Since \( X \equiv C_{M^c}(\mathbb{R}) \) (see [10], p. 226), and \( M^c \) is log-convex, using the Leibnitz formula for successive derivatives of a product, it is easily seen that \( X \) is an algebra.

Similarly \( X = C_M(\mathbb{R}^+) \) is always an algebra. In fact, if \( \liminf_{n \to \infty} nM_n^{1/n} = 0 \), \( X \equiv \{ \text{const} \} \) and if \( 0 < \liminf_{n \to \infty} nM_n^{1/n} < \infty \) then \( X \equiv C_{n^{-}}(\mathbb{R}^+) \).

Suppose now that \( \lim_{n \to \infty} nM_n^{1/n} = \infty \). Then \( X \equiv C_{M^d}(\mathbb{R}^+) \) (see [10], p. 226), where \( M^d \equiv \{ M^d_n \} \) is defined by setting \( n^d M_n^d = (n^d M_n^c)^c \) \((n \geq 1)\). If we put
\[ T_{M^d}^*(r) = \sup_{n \geq 1} \frac{r^n}{n^d M_n^d} \quad (r \geq 1), \]
then
\[ n^d M_n^d = \sup_{r \geq 1} \frac{r^n}{T_{sc}^d(r)} \quad \text{and} \quad T_{sc}^d(r) = \sup_{n \geq 1} \frac{r^n}{n^d M_n^d} = T_{M^d}^*(r). \]

Here, as before, in all the above three cases \( C_M(\mathbb{R}^+) \) is an algebra.

A Carleman class \( C_M(I) \) consisting of the constants alone is always inverse-closed. For nontrivial inverse-closed Carleman classes, we have the following characterization.

**Theorem 1.** Let \( X = C_M(\mathbb{R}) \) or \( C_M(\mathbb{R}^+) \). If \( X \) is nontrivial, then the following assertions are equivalent:

(a) \( \lim_{n \to \infty} M_n^{1/n} = \infty \) and the sequence \( A \) is almost increasing.

(b) Analytic functions operate on \( X \).

(c) \( X \) is an inverse-closed algebra.

Here \( A = \{ A_n \} \), where \( A_n = (M_n^c/n!)^{1/n} \) or \( (M_n^d/n!)^{1/n} \) \((n \geq 1)\) according as \( X = C_M(\mathbb{R}) \) or \( C_M(\mathbb{R}^+) \).

**Proof.** (i) Let \( X = C_M(\mathbb{R}) \). Suppose that (a) holds. Then \( X \equiv C_{M^c}(\mathbb{R}) \). That analytic functions operate on \( X \) now follows since \( A \) is almost increasing, if we use Faà di Bruno's formula viz.,
\[ (g \circ f)^{(n)}(x) = \sum_{k_1! \cdots k_n!} g^{(k)}(f(x)) \left( \frac{f'(x)}{1!} \right)^{k_1} \cdots \left( \frac{f^{(n)}(x)}{n!} \right)^{k_n}, \]
where
\[ 2^{n-1} = \sum_{k_1! \cdots k_n!} \frac{k_1! \cdots k_n!}{k_1! \cdots k_n!}, \]
and the summation in the two cases is over all \( n \)-tuples \((k_1, \ldots, k_n)\) such that \( k_1 + \cdots + k_n = k \) and \( k_1 + 2k_2 + \cdots + nk_n = n \) \((0 \leq k \leq n)\). Thus (b) holds.
Trivially (b) implies (c). It remains to be shown that (c) implies (a). Let (c) hold. Since X is nontrivial, we cannot have \( \liminf_{n \to \infty} M_{n}^{1/n} = 0 \), for otherwise, we would have \( \mathcal{E}_{M}(\mathbb{R}) \equiv \{\text{const}\} \). We cannot have \( 0 < \liminf_{n \to \infty} M_{n}^{1/n} < \infty \) either, for then we would have \( \mathcal{E}_{M}(\mathbb{R}) \equiv \mathcal{E}_{1}(\mathbb{R}) \) which is not inverse-closed contradicting the hypothesis, since \( f \), where \( f(x) = 2 + \sin x \) is in \( \mathcal{E}_{1}(\mathbb{R}) \) but not its inverse (see [1], p. 25). Thus \( \lim_{n \to \infty} M_{n}^{1/n} = \infty \). It follows from (1) that there exists a positive sequence \( \{r_{n}\} \) such that \( r_{n}^{n} = M_{n}^{c} T_{M_{n}^{c}}(r_{n}) \) \( (n \geq 1) \).

The function

\[
(4) \quad f(x) = \sum_{j=1}^{\infty} \frac{e^{jx}}{2^{j} T_{M_{j}^{c}}(r_{j})},
\]
is a characteristic function of \( X \) since it is easily seen that

1. \( |f^{(n)}(x)| \leq M_{n}^{c} \) \((n \geq 0; x \in \mathbb{R})\),
2. \( f^{(n)}(0) = i^{n} s_{n} \) \((n \geq 0)\), where \( s_{n} \geq 2^{-n} M_{n}^{c} \).

If \( \lambda > 1 + \|f\|_{\infty} \), then \( \lambda - f \in X \) and since it does not vanish on \( \mathbb{R} \) and \( X \) is inverse-closed, \( (\lambda - f)^{-1} \) belongs to \( X \). From (3), we get

\[
\sum_{k_{1} \cdots k_{n}} \frac{k!}{k_{1}! \cdots k_{n}!} (\lambda - f(0))^{-k} \left( \frac{M_{1}^{c}}{112} \right)^{k_{1}} \cdots \left( \frac{M_{n}^{c}}{n!2^{n}} \right)^{k_{n}} \leq AB \frac{M_{n}^{c}}{n!}.
\]

Let \( p > 1 \) be a fixed integer and suppose that \( n = pk \). The term in the above inequality that corresponds to the choice \( k_{p} = k \) and \( k_{q} = 0 \) for \( q \neq p \) does not exceed that on the right so that

\[
\left( \frac{M_{p}^{c}}{p!} \right)^{1/p} \leq K_{1} \left( \frac{M_{n}^{c}}{n!} \right)^{1/n}.
\]

If \( n \) is not a multiple of \( p \), let \( pm \leq n \leq p(m+1) \). Since \( \{(M_{n}^{c})^{1/n}\} \) is increasing, we get

\[
\left( \frac{M_{n}^{c}}{n!} \right)^{1/n} \geq \left( \frac{M_{pm}^{c}}{(pm)!} \right)^{1/pm} \left( \frac{(pm)!}{(n)!} \right)^{1/n} \geq \frac{1}{K_{2}} \left( \frac{M_{p}^{c}}{p!} \right)^{1/p}
\]
so that (a) holds.

(ii) Let \( X = \mathcal{E}_{M}(\mathbb{R}_{+}) \). Suppose that (a) holds. Since \( \lim_{n \to \infty} nM_{n}^{1/n} = \infty \), \( X \equiv \mathcal{E}_{M_{n}}(\mathbb{R}_{+}) \). As in (i), (a) implies (b) and (b) implies (c). Suppose that (c) holds. Since \( X \) is nontrivial, we cannot have \( \liminf_{n \to \infty} nM_{n}^{1/n} = 0 \). We cannot have \( 0 < \liminf_{n \to \infty} nM_{n}^{1/n} < \infty \). For then \( X \equiv \mathcal{E}_{n^{-n}}(\mathbb{R}_{+}) \) which is not inverse-closed. In fact, let

\[
h(x) = \sum_{k=0}^{\infty} \frac{(-x)^{k}}{(2k+3)!}, \quad (x \in \mathbb{R}_{+}).
\]

Using the properties of Mittag–Leffler function, we prove that

\[
|h^{(n)}(x)| \leq 2e^{n} n^{-n} \quad (n \geq 1), \quad (x \in \mathbb{R}_{+}),
\]
so that \( h \in X \). Choosing \( \lambda > 1 + \| h \|_\infty \) and applying (3), we get

\[
[(\lambda - h(x))^{-1}]_{x=0}^{(2m)} \geq \frac{(2m)!}{(7!)^m m! (\lambda - h(0))^{m+1}}.
\]

It follows that

\[
\limsup_{n \to \infty} n \left( \max_{x \in \mathbb{R}_+} |(\lambda - h(x))^{-1}|^{(n)} \right)^{1/n} = \infty.
\]

Hence \( (\lambda - h)^{-1} \notin X \), i.e. \( X \) is not inverse-closed contradicting the hypothesis. Thus \( \lim_{n \to \infty} n M_n^{1/n} = \infty \) and \( X = \mathcal{C}_{M^d}^\ast (\mathbb{R}_+) \). It follows from (2) that there exists a positive sequence \( \{ r_n \} \) such that \( r_n^n = n^n M_n^d T_{M^d}^* (r_n) \) \((n \geq 1)\). The function

\[(6) \quad f(x) = \sum_{j=1}^{\infty} \frac{h(r_j x)}{2^j T_{M^d}^* (r_j)} \]

is a characteristic function of \( X \) since, by (5),

1°. \( |f^{(n)}(x)| \leq 2 e^n M_n^d \) \((n \geq 0; x \in \mathbb{R}_+)\),

2°. \( f^{(n)}(0) = (-1)^n s_n \), where \( s_n \geq \mu^n M_n^d \) \((\mu > 0)\).

If we choose \( \lambda > 1 + \| f \|_\infty \), then reasoning as in (i), we conclude that if \( p > 1 \) is a fixed integer and \( n \) is a multiple of \( p \), say, \( n = pk \), we get

\[
\frac{1}{(\lambda - f(0))^{k+1}} \left( \frac{p^p p! M_p^d}{(2p+3)! 2^p} \right)^k \leq AB^n M_n^d \frac{n!}{n^!}
\]

and consequently

\[
\left( \frac{M_p^d}{p!} \right)^{1/p} \leq K \left( \frac{M_n^d}{n!} \right)^{1/n}.
\]

The result for arbitrary \( n \) now follows as in (i) if we remember that \( \{(nM_n^d)^{1/n}\} \) is increasing.

3. Inverse-closed local Carleman classes

We now proceed to characterize the inverse-closed local Carleman classes. If \( I \) is a finite or infinite open interval, then \( \mathcal{C}_{M^d}^\ast (I) \equiv \mathcal{C}_{M^d}^\ast (I) \) (see [10], p. 223), where \( M^0 = \{ M^0_n \} \) is defined by setting

\[
S_M(r) = \max_{n \leq r} \frac{r^n}{M_n} , \quad M^0_n = \sup_{r \geq n} \frac{r^n}{S_M(r)}.
\]

Then \( S_M(r) = S_{M^0}(r) \). But if \( I \) is an arbitrary interval then \( \mathcal{C}_{M^d}^\ast (I) \equiv \mathcal{C}_{M^d}^\ast (I) \) (see [10], p. 223 and [4], p. 718), where \( M^f = \{ M^f_n \} \) is defined by setting

\[
U_{sc}(r) = \max_{n \leq r} \frac{r^{2n}}{n M_n} \quad \text{and} \quad n^n M^f_n = \sup_{r \geq n} \frac{r^{2n}}{U_M(r)}.
\]

Then \( U_M(r) = U_{M^f}(r) \).

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
A local Carleman class $X = C_M^*(I)$ is not always an algebra (see [6], p. 337). However, it is so if it contains the local analytic class. In fact, if $I$ is open, then, as shown by H. Cartan (see [5], p. 7), $X = C_M^*(I)$ so that $X = C_M^{*0}(I) = C_M^*(I)$ is an algebra. The same is true for local Carleman class $C_M^*(I)$ defined on a closed or a semi-closed interval $I$. However, in this case the following analogue of Cartan’s result holds.

**Lemma I.** If $X = C_M^*(I) \supseteq C_n^*(I)$, then $X = C_M^*(I) = C_M^d(I)$ for any interval $I$. Consequently $X$ is an algebra.

**Proof.** Since $X = C_M^*(I) \supseteq C_n^*(I)$, it follows that $M_n \geq k^n M_0 n!$ $(n \geq 0)$. Choose $f \in C_M^*(I)$. Let $J$ be a compact subinterval of $I$ and let $J'$ be a compact subinterval of $I$ such that each point $x$ of $J$ is in a subinterval of fixed length $\lambda$ $(< 1)$ contained in $J'$, where $|f^{(n)}(x)| \leq K \sigma^n M_n$ $(n \geq 0)$.

Choose $\sigma$ so large that $\lambda \sigma \geq K^{-1}$. Clearly $n! k M_0 \leq K(\lambda \sigma)^n M_n$ $(n \geq 0)$. Let $\{n_i\}$ be the sequence of principal indices for $M' = \{K\sigma^n M_n\}$ and let $n_i < n < n_i + 1$. Since $(K\sigma^n n^n M_n^d)$ is log-convex, applying Cartan-Gorny inequalities (see [10], p. 219), we conclude that for any $x$ in $J$

$$|f^{(n)}(x)| \leq 2(e^2 r p^{-1})^n (K(\lambda \sigma)^n M_n) q/r (K(\lambda \sigma)^{n_i+1} M_{n_i+1})^p/r \leq 2K(e^2 r p^{-1}) n^{-n/q} r^{-n/p} (\lambda \sigma)^n M_n^d,$$

where $p = n - n_i$, $q = n_i + 1 - n$, $r = n_i + 1 - n_i$. Set $u = n_i + 1/n_i$. Then

$$n^{-n/q} r^{-n/p} = \frac{u^{n/(u-1)}(u-1))^{n-n_i} \leq 1.$$

Hence

$$|f^{(n)}(x)| \leq \mu^n M_n^d \quad (x \in J).$$

Since $M_n = M_n^d$, this inequality holds for all $n \geq 0$. Thus $f \in C_M^d(I)$. Clearly $C_M^*(I)$ is an algebra.

Thus a local Carleman algebra $X = C_M^*(I) \supseteq C_n^*(I)$ has two regularizations viz. $X = C_M^{*0}(I) \equiv C_M^*(I)$ or $X = C_M^d(I) \equiv C_M^*(I)$ according as $I$ is open or arbitrary.

Although, in general, it is not true, the equivalence of classes in these two cases does imply that the sequences $M^0$ and $M^c$ and the sequences $M^f$ and $M^d$ are equivalent in the sense that for some constants $\alpha > 0$ and $\beta > 0$

$$\sum \begin{align*}
& \beta^n M_n^0 \leq M_n^c \leq \alpha^n M_n^0, & (a) \\
& \beta^n M_n^f \leq M_n^d \leq \alpha^n M_n^f \quad (n \geq 1).
\end{align*}$$

The second halves of (7)(a) and (b) are obviously true with $\alpha = 1$. For $I$ finite, (7) follows from the inclusion theorems of H. Cartan and S. Mandelbrojt (see [10], p. 238). The arguments used by these authors fail when $I$ is infinite. However, using Baire’s category theorem, we get the same result valid in all cases.

**Lemma II.** For any finite or infinite open interval $I$, $C_M^*(I) \subseteq C_n^*(I)$ if and only if $(M_n^0)^{1/n} = O((N_n^0)^{1/n})$ or $(M_n^0)^{1/n} = O((N_n)^{1/n})$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Proof. Since \( \mathcal{C}_M^*(I) \equiv \mathcal{C}_M^{*0}(I) \) and \( \mathcal{C}_N^*(I) \equiv \mathcal{C}_N^{*0}(I) \), the sufficiency of the conditions is obvious. So we prove that they are necessary. We may suppose, without loss of generality, that \( I = ]-1, 1[ \) or \( ]-1, \infty[ \) or \( \mathbb{R} \). For each \( n \), choose an integer \( h_n \geq n \) such that

\[
\frac{h_n^n}{S_{M^0}(r)} \geq \frac{M^0_n}{e}.
\]

For \( n \geq 2 \), set \( I_n = ]-1 + (1/h_n), 1 - (1/h_n)[ \) or \( ]-1 + (1/h_n), h_n[ \) or \([-h_n, h_n]\) according as \( I = ]-1, 1[ \) or \( ]-1, \infty[ \) or \( ]-\infty, \infty[ \). Let \( \mathcal{F} \) denote the class of functions \( f \in \mathcal{C}_M^{*0}(I) \) such that

\[
p_k(f) = \sup_{n \geq 0} \max_{x \in I_n} \frac{|f^{(n)}(x)|}{M^0_n}, \quad k = 2, 3, \ldots.
\]

\( \mathcal{F} \) is a Fréchet space with seminorms \( \{p_k\} \) and \( \mathcal{F} \subseteq \mathcal{C}_M^{*0}(I) \). Set

\[
V_j = \{ f \in \mathcal{F} : |f^{(n)}(0)| \leq j^{n+1}N^0_n (n \geq 0), \quad j = 1, 2, \ldots \}
\]

If \( f \in \mathcal{F} \), \( f \in \mathcal{C}_M^{*0}(I) \subseteq \mathcal{C}_M^{*0}(I) \). Hence \( f \in V_j \) for some \( j \). Thus \( \mathcal{F} = \bigcup_{j=1}^{\infty} V_j \). Clearly, \( V_j \)'s are closed in \( \mathcal{F} \). Hence, by Baire's category theorem, there exists a seminorm \( p_r \), a \( \delta > 0 \) and a \( V_s \) such that \( p_r(f) \leq \delta \) implies that \( f \in V_s \). Set \( \alpha = 1/6h_r \) and let

\[
f(x) = \frac{\delta Z_{h_n}(\alpha x)}{2S_{M^0}(h_n)},
\]

where

\[
Z_{n}(x) = (-1)^{[n/2]}T_n(x) + (-1)^{[(n-1)/2]}T_{n-1}(x),
\]

\( T_n(x) \) denoting the Chebyshev polynomial of degree \( n \) and \( [t] \), the integral part of \( t \). For \( x \in I_q \), \( f^{(k)}(x) \equiv 0 \) if \( k > h_n \), and

\[
|f^{(k)}(x)| = \frac{1}{2} \delta \alpha^k \frac{|Z_{h_n}(\alpha x)|}{S_{M^0}(h_n)} \leq A_q M^0_k
\]

if \( 1 \leq k \leq h_n \), since

\[
|T_{n}^{(k)}(x)| \leq (4n)^{k} e^{4n e|x|}, \quad x \in \mathbb{R}.
\]

Thus \( f \in \mathcal{F} \). If \( x \in I_r \), then

\[
|f^{(k)}(x)| \leq \delta \frac{h_n^k}{S_{M^0}(h_n)} \frac{3^k \alpha^k}{(1 - \alpha^2 h_r^2)^k} \leq \delta M^0_k
\]

since (see [10], p. 206) for \( -1 < x < 1 \):

\[
|Z_{n}(x)| \leq \frac{2.3^k n^k}{(1 - x^2)^k} \quad (0 \leq k \leq n).
\]
Hence $p_r(f) \leq \delta$. Then $f \in V_s$ and so for every $k \geq 1$,
\[
\frac{1}{2} \delta \alpha^k \frac{|Z_{k_n}(0)|}{S_{M^0} h_n} \leq s^{k+1} N_k^0.
\]
Since
\[
(n/e)^k \leq |Z_{n}(0)| \leq n^k,
\]
it follows that
\[
\frac{1}{2} \delta \alpha^k \frac{h^k}{S_{M^0} h_n} \leq s^{k+1} N_k^0.
\]
Choosing $k = n$ and using (8), we see that, for some $\beta > 0$,
\[
\beta^n M_n^0 \leq N_n^0 \quad (n \geq 1).
\]
Thus the first condition is necessary and so also the second.

If we apply this lemma with $N_n = M_n^0$, we get the first half of (7)(a). To prove (7)(b), we need the following:

**Lemma III.** If $I$ is not an open interval, then $C^*_M(I) \subseteq C^*_N(I)$ if and only if $(M^f_n)^{1/n} = O((N^f_n)^{1/n})$ or $(M^f_n)^{1/n} = O((N^f_n)^{1/n})$.

**Proof.** Since $C^*_M(I) \equiv C^*_M(I)$ and $C^*_N(I) \equiv C^*_N(I)$, we need prove only the necessity part of the lemma. Suppose $I = R_+$. Let $I_h = [0, h]$ ($h \geq 2$) and let $\mathcal{F}$ denote the class of functions $f \in C^\infty(I)$ such that for each $h \geq 2$,
\[
p_h(f) = \sup_{n \geq 0} \left( \max_{x \in I_h} |f^{(n)}(x)|/M^f_n \right) < \infty.
\]
{p_h} is a family of seminorms on $\mathcal{F}$ making $\mathcal{F}$ a Fréchet space. Let
\[
V_j = \{ f \in \mathcal{F} : |f^{(n)}(1)| \leq j^{n+1} N_n^f \quad (n \geq 1), \quad j = 1, 2, \ldots \}
\]
Clearly $\mathcal{F} = \bigcup_1^\infty V_j$ and $V_j$'s are closed. Applying Baire's category theorem, we get a seminorm $p_r$, a $\delta > 0$ and a $V_s$ such that $p_r(f) \leq \delta$ implies that $f \in V_s$. Let
\[
f(x) = \frac{\delta T_{k_n}(x/r)}{U_{M^f}(k_n)},
\]
where $T_n$ denotes the Chebyshev polynomial of degree $n$ and $\{k_n\}$ a sequence of integers chosen such that $k_n \geq n$ and
\[
\frac{k_n^{2n}}{U_{M^f}(k_n)} \geq \frac{n M^f_n}{e^2}.
\]
f $\in \mathcal{F}$ since $f^{(m)}(x) = 0$ for $x \in I_h$ and $m > k_n$ and, by (9),
\[
|f^{(m)}(x)| \leq A N^f_m, \quad x \in I_h, \quad m \leq k_n.
\]
Since, for \( x \in [-1, 1] \), \( |T_n^{(j)}(x)| \leq (en/2)^j \), we have

\[
p_r(f) = \frac{\delta}{U_{M^f}(k_n)} \sup_{m \leq k_n} \frac{k_n^{2m}}{M_n^{m^2}} \leq \delta,
\]

Therefore \( f \in V_s \) and so for each \( m \geq 1 \).

\[
\delta r^{-m} \left( \frac{2}{em} \right)^m \frac{k_n^{2m}}{U_{M^f}(k_n)} \leq s^{m+1} N_m.
\]

Since \( T_n^{(j)}(1) \geq (2n^2/ej)^j \). Choosing \( m = n \), we get from (11)

\[
\delta r^{-n} \left( \frac{2}{en} \right)^n e^{-2n^2} M_n^{f^2} \leq s^{n+1} N_n^{f^2}.
\]

Thus \( (M_n^{f^2})^{1/n} = O[(N_n^{f^2})^{1/n}] \) and so also \( (M_n^{f^2})^{1/n} = O[(N_n^{f^2})^{1/n}] \).

If \( I \) is finite, we may take it to be \([-1, 1]\) or \([-1, 1] \). Let \( B \) denote the Banach space of functions \( f \in \mathcal{C}^\infty(I) \) such that

\[
\|f\|_B = \sup_{n \geq 0} \left( \max_{0 \leq x \leq 1} |f^{(n)}(x)|/M_n^{f^2} \right) > \infty.
\]

Clearly \( B \) is a union of the closed sets \( V_j \), where

\[
V_j = \{ f \in B : |f^{(n)}(1)| \leq j^{n+1} N_n^{f^2} \ (n \geq 1), \quad j = 1, 2, \ldots \}.
\]

Arguing as before with \( r = (e/2) \) in (10), we complete the proof.

If \( X = \mathcal{C}_M^*(I) \supseteq \mathcal{C}_n^*(I) \), we choose \( N_n = M_n^{d^2} \) in Lemma III, and get the first half of (7)(b).

The following theorem characterizes the inverse-closed local Carleman classes:

**Theorem 2.** Let \( X = \mathcal{C}_M^*(I) \). The following assertions are equivalent:

(a) The sequence \( A = \{ A_n \} \) is almost increasing.
(b) Analytic functions operate on \( X \).
(c) \( X \) is an inverse-closed algebra.

Here \( A_n = (M_n^{0/n!})^{1/n} \) or \( (M_n^{d/n!})^{1/n} \) according as \( I \) is open or not.
\begin{proof}
\setcounter{equation}{0}
(i) Let \( X \equiv \mathbb{C}_M^*(I) \), where \( I \) is an open interval which we may suppose, without loss of generality, to be \([-1, 1]\) or \([-1, \infty]\) or \(\mathbb{R}\). Let (a) hold. Since \( A \) is almost increasing, from (2), we conclude that analytic functions operate on \( X \). This trivially implies that \( X \) is an algebra. Thus (b) holds.

Suppose that (c) holds. If we choose \( f(x) = 1 + x^2 \), then \( f \) and consequently \( f^{-1} \) belongs to \( X \) so that \( n! \leq AB^n M_n^c \) \((n \geq 1)\). Thus \( \mathbb{C}_n^*(I) \subseteq X \) and \( X \equiv \mathbb{C}_M^*(I) \). Moreover, reasoning as in (i) of the proof of Theorem 1 and using the function \( f \) constructed there, we see that \( \{(M_n^c/n!)^{1/n}\} \) is almost increasing. But then, by (7), \( A \) is also almost increasing. Thus (c) holds.

(ii) Let \( X \equiv \mathbb{C}_M^*(I) \), where \( I \) is not open. Then \( X \equiv \mathbb{C}_M(I) \). Here we take \( A_n = (M_n^c/n!)^{1/n} \) \((n \geq 1)\). As in (i), (a) implies (b) and (b) implies (c). We only need show that (c) implies (a).

Without loss of generality, we may suppose that \( I = [0, 1] \) or \([0, 1] \) or \(\mathbb{R}_+ \). Since \( X \) is inverse-closed and \( f \in X \), where \( f(x) = 1 + x \), it follows that \( f^{-1} \in X \) so that \( X \supseteq \mathbb{C}_n^*(I) \). But then, by Lemma 1, \( X \equiv \mathbb{C}_M^*(I) \). Hence the function \( f \) constructed in (ii) of the proof of Theorem 1 or its restriction is in \( X \) and so also its inverse \( f^{-1} \). From this point on the same proof with obvious modifications goes through and we conclude that \( \{(M_n^c/n!)^{1/n}\} \) is almost increasing. But then, by (7), \( A \) is almost increasing.

\end{proof}

4. Remarks

We make a few concluding remarks.

1°. The following theorem which completes Theorem A characterizes inverse-closed Carleman and local Carleman algebras defined by log-convex sequences \( M \).

Theorem 3. Let \( X = \mathbb{C}_M(I) \) or \( \mathbb{C}_M^*(I) \), where \( M \) is log-convex. The following assertions are equivalent.

(a) \( \{(M_n/n!)^{1/n}\} \) is almost increasing.

(b) Analytic functions operate on \( X \).

(c) \( X \) is an inverse-closed algebra.

Proof. It suffices to note that (c) implies (a) as in (i) of the proof of Theorem 1 since, by trivial modifications, the characteristic function constructed there becomes such a function for any class \( \mathbb{C}_M(I) \) or \( \mathbb{C}_M^*(I) \) with the desired properties.

2°. Theorems 1 and 2 characterize all inverse-closed Carleman and local Carleman algebras if we note that \( \mathbb{C}_M(I) \equiv \mathbb{C}_M^*(I) \) and that for finite \( I \), \( \mathbb{C}_M(I) \equiv \mathbb{C}_M(\bar{I}) \equiv \mathbb{C}_M^*(I) \).

3°. If we repeat the proof of Theorem 1 for the class \( X \equiv \mathbb{C}_M^{2\pi}([0, 2\pi]) \), replacing the function \( f \) used there by the function

\[ f(x) = \sum_{j=1}^{\infty} \frac{e^{ir_jx}}{2^j T_{M^c}^*(r_j)}, \]
where \[ t \] denotes, as usual, the integral part of \( t \), we conclude in response to the question raised by W. Rudin [11] that \( X \) is inverse-closed if and only if \( \{(M_n^C/n!)^{1/n}\} \) is almost increasing.

**References**

5. _______, *Sur les classes de fonctions définies par des inégalités portant sur les dérivées successives*, Herman, Paris, 1940.

**Département de Mathématiques et de Statistique, Université Laval, Cité Universitaire, Québec, P.Q. G1K 7P4 Canada**