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HIGHER-DIMENSIONAL SHIFT EQUIVALENCE AND STRONG SHIFT EQUIVALENCE ARE THE SAME OVER THE INTEGERS

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ABSTRACT. Let $RS(\Lambda)$ and $S(\Lambda)$ denote, respectively, the spaces of strong shift equivalences and shift equivalences over a subset Λ of a ring which is closed under addition and multiplication. For example, let Λ be the integers Z or the nonnegative integers Z^+. For any principal ideal domain Λ, we prove that the continuous map $RS(\Lambda) \to S(\Lambda)$ is a homotopy equivalence. The methods also show that any inert automorphism, i.e,, an element in the kernel of $\pi_1(RS(Z^+), A) \to \pi_1(S(Z^+), A)$ can be represented by a closed loop in $RS(Z^+)$ which in $RS(Z)$ is spanned by a triangulated 2-disc supporting a positive 1-cocycle. These cocycles are used in work of Kim-Roush that leads to a counterexample to Williams' lifting problem for automorphisms of finite subsystems of subshifts of finite type.

1. INTRODUCTION

R. F. Williams [Wi] introduced shift equivalence SE and strong shift equivalence SSE over the non-negative integers Z^+ in connection with the classification of subshifts of finite type (X_A, σ_A) up to topological conjugacy, and it remains an open question whether SE and SSE are the same over the non-negative integers. He did prove that SE and SSE are the same over the ring of integers Z. Study of the automorphism group $\mathrm{Aut}(\sigma_A)$ of (X_A, σ_A) led in [W1, W2, W3] to the introduction of the CW-complexes $RS(\Lambda)$ of strong shift equivalences and $S(\Lambda)$ of shift equivalences over a subset Λ of a ring which is closed under addition and multiplication. It follows from the definitions of these spaces that the set $\pi_0(RS(\Lambda))$ of path components of $RS(\Lambda)$ is just the set of SSE classes over Λ and that $\pi_0(S(\Lambda))$ is the set of SE classes over Λ. A step in [W3] for obtaining eventual finite-order generation for inert automorphisms is the isomorphism [W3, 1.7] between the group of automorphisms $\mathrm{Aut}(\sigma_A)$

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modulo the simple ones and $\pi_1(RS(Z^+), A))$. As known to Boyle, Fiebig, and Krieger for several years, results of the type obtained by Kim and Roush in their paper [KR] would lead to examples of a shift commuting automorphism of the periodic points of order 6 of the full 2-shift which is not the restriction of a shift commuting automorphism of the 2-shift itself. A key insight in [KR] is how to convert certain statements over Z to other appropriate ones over Z^+. In particular, a fact they use is that $\pi_1(RS(Z), A) \to \pi_1(S(Z), A)$ is a monomorphism. Kim and Roush also introduce the concept of a "positive 1-cocycle" and need to know that any inert automorphism, i.e., an element in the kernel of $\pi_1(RS(Z^+), A) \to \pi_1(S(Z^+), A)$, can be represented by a closed loop in $RS(Z^+)$, which in $RS(Z)$ is null-homotopic by a triangulated 2-disc supporting a positive 1-cocycle. The specific construction of such cocycles given in (2.7), (2.10), and (2.16) of this paper is used in [KR].

(1.1) **Theorem.** *Let Λ be a principal ideal domain. Then $RS(\Lambda) \to S(\Lambda)$ is a homotopy equivalence and $\pi_n(RS(\Lambda), A) = \pi_n(S(\Lambda), A) = 0$ for $n \geq 2$.*

Consider Krieger's dimension group constructed over Λ from the stationary system $A: \Lambda^m \to \Lambda^m$. (See [BLR] and [W1].) Let $\mathrm{Aut}(s_{A/\Lambda})$ denote the group of automorphisms of this dimension group which commute with s_A, but which do not necessarily preserve the order structure. In fact, there generally is not an order structure on the dimension group unless Λ is ordered anyway.

(1.2) **Proposition.** $\pi_1(S(\Lambda), A) = \mathrm{Aut}(s_{A/\Lambda})$.

The proof of (1.2) is entirely similarly to [W1, 4.18]. Simply ignore any considerations of the order structure.

2. Proof of the main theorem (1.1) for π_0 and π_1

Step 1. $\pi_0(RS(\Lambda), A)) \to \pi_0(S(\Lambda, A))$ is a bijection of sets.
From the definitions we know immediately that this map is surjective. Williams also showed that this map is injective (at least for $\Lambda = Z$). For completeness, we include a proof by showing

$$\pi_1(S(\Lambda), RS(\Lambda)) = 0 \tag{2.1}$$

In other words, any path from A to B in $S(\Lambda)$ can be deformed back to a path from A to B in $RS(\Lambda)$ keeping endpoints fixed. Notation that will be used in the remainder of the paper is introduced in the proof of (2.1).

Any path from A to B to $S(\Lambda)$ is the concatenation of elementary shift equivalences $R: P \to Q$ or their inverses. So the proof reduces to verifying (2.1) for $R: P \to Q$. We now show how to reduce the argument further to the case where each of P and Q are monomorphisms and therefore are isomorphisms when tensored by $F(\Lambda)$, the field of fractions of Λ. This implies $R: P \to Q$ is also a monomorphism and an isomorphism when tensored with $F(\Lambda)$. By definition, each vertex M of $RS(\Lambda)$ and of $S(\Lambda)$ is a square $m \times m$-matrix over Λ where m can vary from vertex to vertex. View M as an endomorphism of the free Λ-module V of rank m equipped with the standard basis with respect to which the endomorphism M has the original matrix representation M. We write this as $\{V, M\}$. More generally, we let $\{V, M\}$ denote a free Λ-module of finite rank equipped with a basis and an endomorphism M. If M comes from an $m \times m$-matrix, we always take the basis to be the standard one. Now consider the SSE

$$(\pi, M): \{V, M\} \to \{V/\ker M, M\} \tag{2.2}$$

where $\pi: V \to V/\ker M$ is the projection and both $M: V/\ker M \to V$ and $M: V/\ker M \to V/\ker M$ are the homomorphisms induced by M. While V comes with a basis, $V/\ker M$ does not have a canonical one. So we will choose a basis for $V/\ker M$ in such a way that if $\ker M = 0$, then the basis for $V/\ker M = V$ remains the same. This makes (2.2) a SSE in the category of matrices over Λ.

The subgroup of those elements in V which are killed by some power of M is free and finitely generated because Λ is a PID and V is free of finite rank over Λ. So the construction (2.2) may be repeated a finite number of times to produce a chain of integral SSEs to a monomorphism. Moreover, starting with the shift equivalence $R: \{V, P\} \to \{W, Q\}$ over Λ, there is the following commutative diagram in $S(\Lambda)$:

$$\begin{tikzcd}
\{V, P\} \arrow[r, "R"] \arrow[d, "{(\pi, P)}"'] \arrow[dr, "R\pi"] & \{W, Q\} \arrow[d, "{(\pi, Q)}"] \\
\{V/\ker P, P\} \arrow[r, "R"'] & \{W/\ker Q, Q\}.
\end{tikzcd} \tag{2.3}$$

Continuing this procedure a finite number of times deforms $R: \{V, P\} \to \{W, Q\}$ to a monomorphism as required.

Assume that $R: \{V, P\} \to \{W, Q\}$ is a SE where P, Q, and R are monomorphisms. Choose $S: \{W, Q\} \to \{V, P\}$ and $k > 0$ such that $RS = P^k$ and $SR = Q^k$. Let VR be the image of R in W. Then we have the following diagram.

$$\begin{tikzcd}
\{V, P\} \arrow[rrrr, "R"] \arrow[d, "{(R, SP^{-(k-1)})}"'] & & & & \{W, Q\} \\
\{VR, Q\} \arrow[urrrr, "I"'] \arrow[dr, "{(I, Q)}"'] & & & & \\
& \{VR + WQ^{k-1}, Q\} \arrow[uurrr, "I"'] \arrow[dr, "{(I, Q)}"'] & & & \\
& & \{VR + WQ^{k-2}, Q\} \arrow[uuurr, "I"'] \arrow[dr, "{(I, Q)}"'] & & \\
& & & \cdots \arrow[dr, "{(I, Q)}"'] & \\
& & & & \{VR + WQ, Q\} \arrow[uuuuu, "{(I, Q)}"']
\end{tikzcd} \tag{2.4}$$

As above, we arbitrarily choose bases for the $VR + WQ^j$ to get a diagram in $S(\Lambda)$. This completes the proof of Step 1.

Step 2. $\pi_1(RS(\Lambda), A) \to \pi_1(S(\Lambda), A)$ is an isomorphism.

From (2.1), we know this is surjective. Thus, we must prove it is injective. This will be done in a way which produces the "positive 1-cocycles" used by Kim-Roush in [KR].

Consider an elementary SSE $(R, S)\colon \{V, P\} \to \{W, Q\}$. We have the diagram

(2.5)
$$\begin{tikzcd}
\{V, P\} \arrow[rr, "{(R,S)}"] \arrow[dr, "{(\pi,P)}"] \arrow[dd, "{(\pi,P)}"'] & & \{W, Q\} \arrow[dd, "{(\pi,Q)}"] \\
& \{V/\ker R, P\} \arrow[ur, "{(R,S\pi)}"] \arrow[dl, "{(\pi,P\pi)}"] \arrow[dr, "{(R\pi,S\pi)}"] & \\
\{V/\ker P, P\} \arrow[rr, "{(R,S)}"'] & & \{W/\ker Q, Q\}
\end{tikzcd}$$

in $RS(\Lambda)$. Recall from [W2, W3] that we often let $\gamma(R, S)$ denote the path in $RS(\Lambda)$ corresponding to the elementary strong shift equivalence $(R, S)\colon P \to Q$. Moreover, we have the identities

$$\gamma(R, S)\gamma(S, R) = \gamma(P, 1),$$
$$\gamma(S, R)\gamma(R, S) = \gamma(Q, 1),$$

and

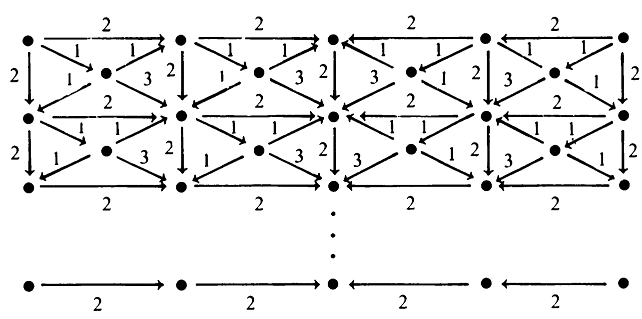
$$\gamma(1, P) = \gamma(1, Q) = 1.$$

Using these, we can represent any element of $\pi_1(RS(\Lambda), A)$ as a loop

(2.6) $$V = V_0 \to V_1 \to \cdots \to V_n \leftarrow \cdots \leftarrow V_{2n-1} \leftarrow V_{2n} = V,$$

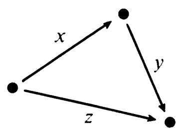
where the number of forward and backward arrows is the *same*. Each free Λ-module V_i comes equipped with an endomorphism A_i and a basis giving a matrix representation for A_i. For example, if $\Lambda = Z^+$ and the loop represents an element of $\mathrm{Aut}(\sigma_A)$, then the matrices are non-negative. For simplicity we have omitted the notation for the A_i, as well as for the R and S matrices giving the SSEs between the vertices in the path. Applying the procedure in (2.5) a finite number of times produces a homotopy of the loop (2.6) like the following $(n = 2)$:

(2.7)



where the vertex endomorphisms and the homomorphisms in the elementary SSEs are all *monomorphisms* along the *bottom* loop. Clearly, the construction applies to all path lengths $2n$. The right and left vertical paths are identified to obtain a homotopy of closed loops. The numbers along each edge give a specific positive 1-cocycle on the homotopy.

This notion was introduced by Kim and Roush. It consists of a function f from the oriented edges of the homotopy to the *positive* integers satisfying the cocycle condition $f(z) = f(x) + f(y)$ whenever x, y, and z form the boundary of a oriented triangle as in the diagram



Now we assume the vertex A matrices and the edge (R, S) matrix pairs in the loop (2.6) are monomorphisms and we must show that when (2.6) is inert, it is possible to deform it to a point by a homotopy supporting a positive 1-cocycle fitting together with the one in (2.7). For the remainder of this section all endomorphisms and homomorphisms will be assumed to be monomorphisms.

Fix a monomorphism $A: V \to V$ of the free Λ-module V. As noted earlier, let $F(\Lambda)$ denote the field of fractions of Λ. Let $\mathrm{Lat}(A)$ denote the set of lattices in $V \otimes F(\Lambda)$ which are invariant under A. Make $\mathrm{Lat}(A)$ into a simplicial complex by letting an n-simplex be an $(n+1)$-tuple $[L_0, \dots, L_n]$ where each L_i is a lattice in $V \otimes F(\Lambda)$ such that $L_i \subset L_j$ and $A(L_j) \subset L_i$ whenever $i < j$.

(2.8) **Proposition.** *Each connected component of* $\mathrm{Lat}(A)$ *is contractible.*

First, we prove that each component is simply connected. We use the notation $K \to L$ to mean $[K, L]$ is an edge in $\mathrm{Lat}(A)$. Similarly for triangles. We give the argument for the connected component containing V. The argument for other components is the same. So start with a loop

$$V = L_0 \to L_1 \to \cdots \to L_n \leftarrow \cdots \leftarrow L_{2n-1} \leftarrow L_{2n} = V \tag{2.9}$$

in $\mathrm{Lat}(A)$. The well-known formula for the product of a matrix and its classical adjoint implies that the intersection of two lattices in $V \otimes F(\Lambda)$ is again a lattice. Apply this to produce the null-homotopy of (2.9) in $\mathrm{Lat}(A)$ when $n = 2$ as in diagram (2.10) below.

The numbers along the edges give a positive 1-cocycle. The construction clearly generalizes to any path length $2n$. The two 45-degree paths arising from the lowest vertex are identified to obtain a homotopy of closed loops. The diagram in (2.10) resembles the one in the proof of [BFK, 2.12].

(2.10)

$$\begin{tikzcd}[column sep=small]
 & & & & L_2 & & & & \\
 & & L_1 \arrow[urr, "2"] & & & & L_3 \arrow[ull, "2"'] & & \\
L_0 \arrow[urr, "2"] & & & & L_1 \cap L_3 \arrow[ull, "2"'] \arrow[urr, "2"] \arrow[uu, "4"] & & & & L_4 \arrow[ull, "2"'] \\
 & & L_0 \cap L_1 \cap L_3 \arrow[ull, "2"] \arrow[uu, "4"] \arrow[urr, "2"] & & & & L_1 \cap L_3 \cap L_4 \arrow[ull, "2"] \arrow[uu, "4"] \arrow[urr, "2"'] & & \\
 & & & & L_0 \cap L_1 \cap L_3 \cap L_4 \arrow[ull, "2"] \arrow[uu, "4"] \arrow[urr, "2"'] & & & &
\end{tikzcd}$$

To finish the proof of (2.8), we must show that the higher homotopy groups of each component of $\mathrm{Lat}(A)$ vanish. By the Whitehead Theorem [Sp], it suffices to show that the higher homology groups are zero, because we have just shown that the components are simply connected. The proof for this is essentially a verbatim copy of the proof for Step III in [W1, 2.12]. The notation $U \to V$ for Markov partitions is replaced by the notation $K \to L$ for lattices.

Finally, we can finish the proof of Step 2 by fitting (2.7) and (2.10) together via (2.16) below. Represent an inert element α of $\pi_1(RS(\Lambda), A)$ by a loop

$$\alpha = \prod_{i=1}^{2n} \gamma(R_i, S_i)^{\varepsilon_i} \tag{2.11}$$

where $(R_i, S_i)\colon (V_{i-1}, A_{i-1}) \to (V_i, A_i)$ and $\varepsilon_i = +1$ for $1 \le i \le n$, and $(R_i, S_i)\colon (V_i, A_i) \to (V_{i-1}, A_{i-1})$ and $\varepsilon_i = -1$ for $n+1 \le i \le 2n$. For $1 \le i \le 2n$, let B_i denote the isomorphism from $V \otimes F(\Lambda) = V_0 \otimes F(\Lambda)$ to $V_i \otimes F(\Lambda)$ given by the formula

$$B_i = \prod_{p=1}^{i} (R_p \otimes 1)^{\varepsilon_p} . \tag{2.12}$$

Let $B_0 = I$. Observe that since α is *inert*, we also have $B_{2n} = I$. Moreover, we have $A = B_i A_i B_i^{-1}$ for each i. Let W_i be the lattice in $V \otimes F(\Lambda)$ defined by the equation

$$W_i B_i = V_i . \tag{2.13}$$

Then we have the following two diagrams in $RS(\Lambda)$ for the two cases $\varepsilon_i = +1$ and $\varepsilon_i = -1$, respectively.

(2.14)

$$\begin{tikzcd}
V_{i-1} \arrow[r, "{(R_i, S_i)}"] \arrow[d, "{(B_{i-1}^{-1}, B_{i-1}A_{i-1})}"'] \arrow[dr, "{(R_i B_i^{-1}, B_i S_i)}" description] & V_i \arrow[d, "{(B_i^{-1}, B_i A_i)}"] \\
W_{i-1} \arrow[r, "{(I, A)}"'] & W_i
\end{tikzcd}$$

(2.15)
$$\begin{tikzcd}[column sep=huge, row sep=large]
V_{i-1} \arrow[d, "{(B_{i-1}^{-1}, B_{i-1}A_{i-1})}"'] & V_i \arrow[l, "{(R_i, S_i)}"'] \arrow[d, "{(B_i^{-1}, B_iA_i)}"] \arrow[dl, "{(R_iB_{i-1}^{-1}, B_{i-1}S_i)}"] \\
W_{i-1} & W_i \arrow[l, dash, "{(I, A)}"]
\end{tikzcd}$$

Diagrams (2.14) and (2.15) piece together to form the homotopy in (2.16) below from the bottom line in (2.7) to the top line in (2.10). Again, we take $2n = 4$ for simplicity. The construction clearly generalizes to all lengths $2n$.

(2.16)
$$\begin{tikzcd}
\bullet \arrow[r, "2"] \arrow[d, "2"'] \arrow[dr, "4"'] & \bullet \arrow[r, "2"] \arrow[d] \arrow[dr, "4"'] & \bullet \arrow[d] & \bullet \arrow[l, "2"'] \arrow[d] \arrow[dl, "4"] & \bullet \arrow[l, "2"'] \arrow[d, "2"] \arrow[dl, "4"] \\
\bullet \arrow[r, "2"'] & \bullet \arrow[r, "2"'] & \bullet & \bullet \arrow[l, "2"] & \bullet \arrow[l, "2"]
\end{tikzcd}$$

The positive 1-cocycle compatible with those in (2.7) and (2.10) is marked on the diagram. This completes the proof of Step 2.

3. Proof of theorem (1.1) for π_n when $n \geq 2$

Let $MRS(\Lambda)$ and $MS(\Lambda)$ denote the subspaces of $RS(\Lambda)$ and $S(\Lambda)$, respectively, where each SSE or SE is a monomorphism. Using generalizations of (2.3) and (2.5), we first show

(3.1) **Theorem.** *Assume Λ is a principal ideal domain. Then the inclusions $MS(\Lambda) \subset S(\Lambda)$ and $MRS(\Lambda) \subset RS(\Lambda)$ are homotopy equivalences.*

Proof that $MS(\Lambda) \subset S(\Lambda)$ is a homotopy equivalence.

Let $\Delta = [V_0, \dots, V_n]$ be an n-simplex in $S(\Lambda)$, where each V_i is a free, based Λ-module of finite rank with an endomorphism A_i such that whenever $i < j$, we are also given a shift equivalence $R_{ij}\colon V_i \to V_j$ between A_i and A_j satisfying $R_{ij}R_{jk} = R_{ik}$ for $i < j < k$. As in (2.2) and (2.3), let $\Delta' = [V_0/\ker A_0, \dots, V_n/\ker A_n]$ with the corresponding vertex endomorphisms A_i and edge shift equivalences R_{ij} induced from those in Δ. Let I denote the unit interval. Then, as in [S], we can triangulate $\Delta \times I$ as follows: The vertices consist of the vertices V_i of Δ together with the vertices $V_i/\ker A_i$ of Δ'. The directed edges are of the form $R_{ij}\colon V_i \to V_j$, $R_{ij}\colon V_i/\ker A_i \to V_j/\ker A_j$, $R_{ij}\colon V_i \to V_j/\ker A_j$ when $i < j$, and $V_i \to V_i/\ker A_i$. These satisfy the triangle relation $R_{ij}R_{jk} = R_{ik}$ for $i < j < k$, and the procedure gives

(3.2) a deformation of Δ to Δ' compatible with the face and degeneracy maps in the complex $S(\Lambda)$.

Consider a map of a pair (X, Y) of finite CW-complexes into the pair $(S(\Lambda), MS(\Lambda))$. Deform this to a cellular map so that the image of X is contained in a finite subcomplex of $S(\Lambda)$. Since Λ is a principal ideal domain, we can use the deformation (3.2) a finite number of times to deform the map on X down into $MS(\Lambda)$ keeping it fixed on Y.

Proof that $MRS(\Lambda) \subset RS(\Lambda)$ *is a homotopy equivalence.*

This proof proceeds exactly like the argument for the previous proof except that, just as (2.5) is more complicated than (2.3), so is the triangulation of $\Delta \times I$.

Let $\Delta = [V_0, \dots, V_n]$ be an n-simplex in $RS(\Lambda)$ where each V_i is a free, based Λ-module of finite rank equipped with an endomorphism A_i and such that whenever $i < j$, we are also given a strong shift equivalence (R_{ij}, S_{ji}): $V_i \to V_j$ between A_i and A_j satisfying the Triangle Identities in [W3] for $i < j < k$. As in (2.2) and (2.5), let $\Delta' = [V_0/\ker A_0, \dots, V_n/\ker A_n]$ with the corresponding vertex endomorphisms A_i and edge strong shift equivalences (R_{ij}, S_{ji}) induced from those in Δ. We will define a triangulation of $\Delta \times I$ which will be isomorphic to the cone from an interior "center" vertex C of $\Delta \times I$ to an inductively defined triangulation of $\partial(\Delta \times I)$. For any simplex Δ, let $C = V_0/\ker R_{01}$ equipped with the induced endomorphism A_0. The vertices of $\Delta \times I$ will be of four types:

(a) the V_i of Δ,
(b) the $V_i/\ker A_i$ of Δ',
(c) the centers C_i of the faces $(\partial_i\Delta) \times I$ for $0 \le i \le n$, and
(d) the center C of $\Delta \times I$.

From the definition of the centers, we see that C_i is given by

$$C_i = \begin{cases} V_1/\ker R_{12}, & \text{for } i = 0 \\ V_0/\ker R_{02}, & \text{for } i = 1 \\ V_0/\ker R_{01}, & \text{for } i \ge 2 \end{cases}$$

The edges are

(ea) $(R_{ij}, S_{ji})\colon V_i \to V_j$ in Δ,
(eb) $(R_{ij}, S_{ji})\colon V_i/\ker A_i \to V_j/\ker A_j$ in Δ',
(ec) the edges in $(\partial_i\Delta) \times I$, and
(ed) a directed edge connecting C and V for each vertex V of the three types (a), (b), and (c).

More specifically, in the situation (ed) we have the following cases.

V is of Type (a):

If $V = V_0$ in Δ, then $V \to C$ is

$$(\pi, A_0)\colon V_0 \to V_0/\ker R_{01}.$$

If $V = V_i$ in Δ for $i \ge 1$, then $C \to V$ is

$$(R_{0i}, S_{i0}\pi)\colon V_0/\ker R_{01} \to V_i.$$

V is of Type (b):

If $V = V_i/\ker A_i$ in Δ' for $i \ge 0$, then $C \to V$ is

$$(\pi, A_0)\colon V_0/\ker R_{01} \to V_0/\ker A_0 \qquad \text{for } i = 0, \text{ and}$$
$$(R_{0i}, S_{i0})\colon V_0/\ker R_{01} \to V_i/\ker A_i \qquad \text{for } i \ge 1.$$

V is of Type (c):
If $V = C_i$, then $C \to V$ is

$$\begin{aligned}(R_{01}, S_{10})&: V_0/\ker R_{01} \to V_1/\ker R_{12}, \text{ for } i = 0, \\ (\pi, A_0)&: V_0/\ker R_{01} \to V_0/\ker R_{02}, \text{ for } i = 1, \text{ and} \\ (I, A_0)&: V_0/\ker R_{01} \to V_0/\ker R_{01}, \text{ for } i \geq 2.\end{aligned}$$

The Triangle Identities are satisfied by any triple of edges which could possibly form a triangle, and hence we get a triangulation of $\Delta \times I$ in $RS(\Lambda)$. This gives

(3.3) a deformation of Δ to Δ' compatible with the face and degeneracy maps in the complex $RS(\Lambda)$.

Now we will proceed in the same manner to complete the proof of (3.1).

Proof that $\pi_n(MRS(\Lambda), A) = 0$ *when* $n \geq 2$.
This is very similar to §4 of [W2]. Let $MRS(\Lambda)_A$ denote the component of $MRS(\Lambda)$ containing A. The universal cover $\widetilde{MRS}(\Lambda)_A$ of $MRS(\Lambda)_A$ is the realization of the following simplicial set. The n-simplices are pairs (γ, Δ) where

(3.4) Δ is an n-simplex of $\widetilde{MRS}(\Lambda)$ given by the data $[V_0, \dots, V_n]$ and $(R_{ij}, S_{ji}): V_i \to V_j$, and γ is a homotopy class of paths from V to V_0.

The ith face operator acts on Δ just as it does in $MRS(\Lambda)_A$. For $1 \leq i \leq n$, it leaves γ unchanged, and for $i = 0$, it changes γ to $\gamma * \gamma(R_{01}, S_{10})$. The covering map

$$\widetilde{MRS}(\Lambda)_A \to MRS(\Lambda)_A$$

is induced by the map of simplicial sets taking (γ, Δ) to Δ.

Let γ be a path from V to V' in $MRS(\Lambda)_A$ written in the form

$$\gamma = \prod_{i=1}^{n} \gamma(R_i, S_i)^{\varepsilon_i}, \tag{3.5}$$

where $(R_i, S_i): \{V_{i-1}, A_{i-1}\} \to \{V_i, A_i\}$ when $\varepsilon_i = +1$ and $(R_i, S_i): \{V_i, A_i\} \to \{V_{i-1}, A_{i-1}\}$ when $\varepsilon_i = -1$. Let $\Theta(\gamma)$ denote the isomorphism from $V \otimes F(\Lambda)$ to $V' \otimes F(\Lambda)$ given by the formula

$$\Theta(\gamma) = \prod_{i=1}^{n} (R_i \otimes 1)^{\varepsilon_i}. \tag{3.6}$$

For each n-simplex (γ, Δ) of $\widetilde{MRS}(\Lambda)_A$, let L_i be the lattice in $V \otimes F(\Lambda)$ that is the image of V_i under $\Theta(\gamma * \gamma(R_{0i}, S_{i0}))^{-1}$. Then $[L_0, \dots, L_n]$ is an n-simplex in $\mathrm{Lat}(A)$, and the correspondence taking (γ, Δ) to $[L_0, \dots, L_n]$ is a map of simplicial sets. This gives a continuous map

$$\widetilde{MRS}(\Lambda)_A \to \mathrm{Lat}(A).$$

Let K and L be A-invariant lattices in $V \otimes F(\Lambda)$ such that $A(L) \subset K$. This gives rise to a path $(I, A): K \to L$ in $MRS(\Lambda)_A$, and the correspondence preserves the Triangle Identities. Therefore, there is a continuous map

$$\mathrm{Lat}(A) \to MRS(\Lambda)_A .$$

Moreover, it follows from [S] and from conjugation squares like (2.14) and (2.15) that there is a homotopy commutative diagram

$$\begin{tikzcd} \widetilde{MRS}(\Lambda)_A \arrow[rr] \arrow[dr] & & \mathrm{Lat}(A) \arrow[dl] \\ & MRS(\Lambda)_A & \end{tikzcd} \tag{3.7}$$

Since the components of $\mathrm{Lat}(A)$ are contractible, we see that the homomorphism from $\pi_n(\widetilde{MRS}(\Lambda)_A, A)$ to $\pi_n(MRS(\Lambda)_A, A)$ is zero. On the other hand, it is an isomorphism for $n \geq 2$, because $\widetilde{MRS}(\Lambda)_A$ is the universal cover.

Proof that $\pi_n(MS(\Lambda), A) = 0$ *when* $n \geq 2$.

This proof follows just like the argument for $MRS(\Lambda)$, except that the definition of $\mathrm{Lat}(A)$ is changed slightly. Namely, an n-simplex is an $(n+1)$-tuple $[L_0, \ldots, L_n]$ where each L_i is an A-invariant lattice in $V \otimes F(\Lambda)$ such that $L_i \subset L_j$, and there is a positive integer k so that $A^k(L_j) \subset L_i$ whenever $i < j$.

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