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ON THE MAL'CEV CORRESPONDENCE

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ABSTRACT. We generalize Mal'cev's correspondence [5] to other nilpotent groups, namely to certain maximal unipotent subgroups of Chevalley groups. If $\mathscr{L}$ is a root system and K is an infinite field of characteristic different to 2 or 3, then we show that a group G elementarily equivalent to $U_{\mathscr{L}}(K)$ is isomorphic to $U_{\mathscr{L}}(F)$, where F is a field elementarily equivalent to K.

1. INTRODUCTION

One of the most attractive and useful tools of the logician is the method of interpretations. For example, the theory of rational numbers is undecidable because one can interpret integers inside rationals. Groups of matrices with coefficients in a ring provide a natural context where one may try the method of interpretations. Indeed, one wants to know what relations exist between the group theory and ring theory. This study seems to have been started by A. I. Mal'cev in his paper "A correspondence between groups and rings," where the main emphasis is on upper unitriangular matrix groups. Here we will consider other classes of nilpotent groups, including Mal'cev's. However, we only consider matrices over fields. Matrices over rings could be considered and results generalized; that topic will be pursued in a later work.

Our main result is Theorem 5.2. Here we show that a group elementarily equivalent to one of these matrix groups over a field K is a matrix group over a field elementarily equivalent to K.

We will not treat fields of characteristic 2 or 3. These cases, especially for groups of small nilpotency class, are interesting, but so far we have only partial results. It seems possible to find nonmatrix groups in these cases.

The proof goes more or less as follows. Our groups U are built from basic subgroups, called root groups, which can be defined from a certain set X of parameters. These parameters act like a frame, in which the root subgroups fit. An elementarily equivalent group H will have many frames with the corresponding root subgroups.

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The problem consists in showing that no matter what frame one uses, the corresponding subgroups fit correctly.

2. The nilpotent group $U_{\mathscr{L}}(K)$

For any (irreducible) root system $\mathscr{L}$, and any field K, Chevalley constructed a group now known as the Chevalley group of type $\mathscr{L}$ over K. We concentrate on a certain subgroup, namely a maximal unipotent subgroup which we write as $U_{\mathscr{L}}(K)$, or simply U if $\mathscr{L}$ and K have been fixed. This group can be given in terms of generators and relations as follows. Let Σ, Σ^+, and π denote sets of roots, positive roots and simple roots, respectively (of $\mathscr{L}$). We will use the letters $r, s, \dots, \alpha, \beta, \sigma, \dots$, to denote roots. If $\alpha \in \Sigma^+$ then $\sigma = \Sigma_{\alpha\in\pi} n_\alpha \alpha$ with $n_\alpha \in \mathbb{N}$, uniquely. Define the height of σ, $ht(\sigma) = \Sigma n_\alpha$. Σ^+ can be partially ordered as follows: $\beta < \alpha$ iff $\alpha - \beta$ is the sum of simple roots. Note that $ht(\beta) \le ht(\alpha)$ if $\beta < \alpha$. The number of simple roots is called the rank of $\mathscr{L}$. For general information on root systems see [4, Chapter III].

Let $x_\alpha(t)$ be a symbol, for each $\alpha \in \Sigma^+$ and $t \in K$. Chevalley discovered that U is generated as an abstract group by the symbols above and has the following relations.

$$x_\alpha(u) \cdot x_\alpha(t) = x_\alpha(u+t), \tag{2.1}$$

$$[x_\alpha(u), x_\beta(t)] = \begin{cases} 1 & \text{if } \alpha+\beta \notin \Sigma^+ \\ \prod_{i,j} x_{i\beta+j_\alpha}(C_{ij\beta\alpha} \cdot (-t)^i u^j) & \text{otherwise}, \end{cases} \tag{2.2}$$

where the product extends over all pairs (i, j) such that $i\beta + j\alpha \in \Sigma^+$, the roots are taken in increasing order, and where each $C_{ij\beta\alpha}$ is an integer of absolute value ≤ 3. As a consequence of (2.1) and (2.2), we have that every element in U can be written uniquely in the form $x_{\alpha_1}(t_1)x_{\alpha_2}(t_2)\cdots x_{\alpha_N}(t_N)$ where $\alpha_1, \alpha_2, \dots, \alpha_N$ are the positive roots in increasing order. Also $K_\alpha = \{x_\alpha(t) : t \in K\}$ is a subgroup of U isomorphic to $(K, +)$; it is called the root subgroup corresponding to α. Since we will always consider positive roots we will simply call them roots.

We will need the following property of roots. Let r_N denote the unique root of largest height in the finite set Σ^+.

Proposition 2.3. *Let $\alpha \in \Sigma^+$, $ht(\alpha) \ge 2$ and $\alpha \ne r_N$. Then there exists a simple root β such that $\alpha - \beta \in \Sigma^+$ but $r_N - \beta \notin \Sigma^+$.*

This property has apparently not been noticed previously. A (not very illuminating) proof consists in checking it by looking at tables; see illustrations I–VII in [2].

We will use the terminology of nodes and Dynkin diagrams. For these one can consult [4]. The subgroup U^ℓ is defined to be the group generated by all $x_r(t)$ with $ht(r) \ge \ell$.

3. Definability in $U_{\mathscr{L}}(K)$

To show that if G is elementarily equivalent to $U_{\mathscr{L}}(K)$ then G is of the form $U_{\mathscr{L}}(F)$ with $F \equiv K$, we need to know that the root groups K_r are definable. Recall that a subset $S \subset G$ is definable (with parameters) if $S = \{s \in G \mid \varphi(s)\}$, where φ is a formula in the language of groups which may use elements from G. If φ has no parameters, then we say that S is 0-definable.

Proposition 3.1. *Let K be a field,* $\mathrm{ch}(K) \neq 2, 3$. *Then each of the subgroups U^ℓ is 0-definable.*

Proof. Let $U = U' \supset U^2 \supset \cdots \supset U^N = C(U)$. Fix $2 \leq j \leq N$. If x belongs to U^j then it is congruent to $x_{r_{j_1}}(t_1)x_{r_{j_2}}(t_2)\cdots x_{r_{j_{n_j}}}(t_{n_j}) \bmod U^{j+1}$. Here the r_{j_i} are the n_j roots of height j. For each term of level j, say $x_{r_{j_1}}(t_1)$, it is easy to see that there exists a j-fold commutator $c_1 = x_{r_{j_1}}(t_1)w$, where w may include terms of height j but for roots $\neq r_{j_1}$.

Let $x' = x^{-1}c_1$. Then x' does not contain a term with root r_{j_1}. In a similar way we find $(n_j - j)$-fold commutators $c_1c_2\cdots c_{n_j}$ such that $x' = x^{-1}c_1c_2 \cdots c_{n_j} \in U^{j+1}$. Repeat the above to get $(n_{j+1} - (j+1))$-fold commutators $d_1, \ldots, d_{n_{j+1}}$ such that $x'' = x'^{-1}d_1d_2\cdots d_{n_{j+1}} \in U^{j+2}$.

This process terminates since U is nilpotent. It is clear that the above can be written down as a first order statement.

Proposition 3.2. *Let K be a field with* $\mathrm{ch}(K) \neq 2, 3$. *In $U_{\mathscr{L}}(K)$ each of the subgroups k_r, for r not simple, and $K_r \cdot K_{r_N}$ for r simple is definable from the set $\{x_\alpha(1) \colon \alpha \in \pi\}$.*

Proof. For simplicity, if $\alpha, \beta \in \Sigma^+$ we will assume that the span $\langle \alpha, \beta \rangle$ is of type A_2. After we explain our procedure it will be clear how to modify it to cover the other possibilities; hence we omit the details.

Let $r \in \Sigma^+$ with $ht(r) \geq 2$. Our definition is inductive, starting with the highest root r_N. K_{r_N} is the center of U (remember that $\mathrm{ch}(K) \neq 2, 3$). Assume that $r_0 \in \Sigma^+$, $ht(r_0) = \ell \geq 2$. The definition of K_{r_0} will involve K_s for s with $ht(s) > \ell$ and the parameters $x_\alpha(1)$, $\alpha \in \pi$.

Let $x \in U^\ell$, $x = \prod_{ht(r) \geq \ell} x_r(t_r)$. To eliminate all terms $x_r(t_r)$ with $ht(r) = \ell, r \neq r_0$ appearing in x, we divide the set of simple roots into two sets, A and B: A is the set of simple roots α such that $\alpha + r_0 \in \Sigma^+$, and B is the complement of A. For all $\alpha \in B$ we put the condition $x \in C(x_\alpha(1))$, where $C(g)$ is the centralizer of g.

Then the conditions imposed eliminate all terms $x_r(t_r)$ with $r + \alpha \in \Sigma^+$ (and $\alpha \in B$) in x. Suppose $x_r(t_r)$ $r \neq r_0$ remains in the expression for x. It follows that the set of simple roots α such that $r + \alpha \in \Sigma^+$ is contained in A. Take any such root α and put the condition $[x, x_\alpha(1)] \in K_{r_0+\alpha}$. This eliminates the

term $x_r(t_r)$ in x. We conclude that $x = x_{r_0}(t_{r_0})g$, where $g \in U^{\ell+1}$. We can continue as before to eliminate all terms of height j with $\ell + 1 \leq j < N$. This yields $x = x_{r_0}(t_{r_0})x_{r_N}(t_{r_N})$. Let α_0 be a simple root such that $r_0 - \alpha_0 \in \Sigma^+$ but $r_N - \alpha_0 \notin \Sigma^+$. The final condition on x is that there is a $y \in U^{\ell-1}$ such that $[y, x_{\alpha_0}(1)] = x$.

Next, we define the subgroups $K_\alpha K_{\alpha_N}$ with $\alpha \in \pi$. We consider two cases.

Case 1. α is connected to one other simple root only. For $\mathscr{L} = A_\ell, B_\ell, C_\ell, E_6, E_7, E_8, F_4$, and G_2 there are two such roots, label them α_1 and α_ℓ (where for types B_ℓ, C_ℓ, and G_2 we make the convention that α_1 is shorter than α_ℓ). Here are our definitions:

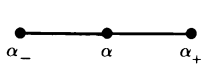
For types B_ℓ, C_ℓ, and G_2 we have

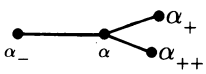
$$x \in K_{\alpha_\ell} K_{\alpha_N} \leftrightarrow x \in \bigcap_{i \neq \ell - 1} C(x_{\alpha_i}(1)) \quad \text{and} \quad [x, x_{\alpha_{\ell-1}}(1)] \in K_{\alpha_{\ell-1}+\alpha_\ell} \cdot K_{\alpha_{\ell-1}+2\alpha_\ell}.$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_\ell$ are the simple roots. For any type $\mathscr{L}$ (except D_ℓ), $x \in K_{\alpha_1} K_{\alpha_N} \leftrightarrow x \in \bigcap_{i \neq 2} C(x_{\alpha_i}(1))$ and $[x, x_{\alpha_2}(1)] \in K_{\alpha_1+\alpha_2}$, where α_2 is the unique root connected to α_1. Finally, another modification is needed for D_ℓ, but we omit it.

Case 2. α is connected to two roots.

In terms of the Dynkin diagram of $\mathscr{L}$, the situation reduces to

(A) 

or 

(B)

In (A) we use

$$x \in K_\alpha K_{\alpha_N} \Leftrightarrow x \in \bigcap_{\substack{\beta \in \pi \\ \beta \neq \alpha_-, \alpha_+}} C(x_\beta(1))$$

and in (B),

$$x \in K_\alpha K_{\alpha_N} \Leftrightarrow x \in \bigcap_{\substack{\beta \in \pi \\ \beta \neq \alpha_-, \alpha_+, \alpha_{++}}} C(x_\beta(1))$$

and

$$[x, x_{\alpha_-}(1)] \in K_{\alpha+\alpha_-}, \; [x, x_{\alpha+}(1)] \in K_{\alpha+}\alpha_+,$$
$$[x, x_{\alpha_{++}}(1)] \in K_{\alpha+}\alpha_{++}.$$

In the style of Mal′cev we interpret the field K in the group.

Theorem 3.3. *Let $U_{\mathscr{L}}(K)$ be given. Then we can define K in $U_{\mathscr{L}}(K)$, using $x_\alpha(1)$, $\alpha \in \pi$.*

Proof. As the underlying set for K we take the center of the group, that is K_{α_N}. To define multiplication, we proceed as follows. Let $\alpha_0 \in \pi$ be such

that $\alpha_N - \alpha_0 \in \Sigma^+$. Given $x \in K_{\alpha_N}$ we can define $g \in K_{\alpha_N - \alpha_0}$ such that $[x_{\alpha_0}(1), g] = x$. Note that either $x_{\alpha_N - \alpha_0}(1)$ or $x_{\alpha_N - \alpha_0}(-1)$ is definable from the set $\{x_\alpha(1) \colon \alpha \in \pi\}$. Hence, given $y \in K_{\alpha_N}$ we can define $w \in K_{\alpha_0} K_{\alpha_N}$ with $[w, x_{\alpha_N - \alpha_0}(\pm 1)] = y$. The product of x and y is then defined to be $[w, g]$. Furthermore, we have the following: given a formula $\varphi(\overline{x})$ in the language of rings we can find recursively another formula $\overline{\varphi}(\overline{x}, \overline{c})$ in the language of groups together with constants $\overline{c}$ naming parameters $x_\alpha(1)$, $\alpha \in \pi$ such that

$$K_{\alpha_N} \models \varphi(\overline{a}) \Leftrightarrow U \models \overline{\varphi}(\overline{a}, \overline{c}).$$

4. Frames and automorphisms of $U_{\mathscr{L}}(K)$

We begin with a definition.

Definition 4.1. A frame $\mathscr{F}$ in $U_{\mathscr{L}}(K)$ is a set of $\ell = \operatorname{rank} \mathscr{L}$ elements of $U_{\mathscr{L}}(K)$ such that the commutation relations between elements of $\mathscr{F}$ are the same as those relations between elements of $\{x_\alpha(1) \colon \alpha \in \pi\}$.

What this means is that to each $g \in \mathscr{F}$ we can make correspond an $x_\alpha(1)$ in such a way that commutators are preserved; that is, iterated commutators of elements in $\mathscr{F}$ are trivial if and only if the corresponding iteration of elements in $\{x_\alpha(1) \colon \alpha \in \pi\}$ is trivial. We call $\{x_\alpha(1) \colon \alpha \in \pi\}$ the standard frame. Clearly, the image of the standard frame under an automorphism of the group is a frame.

We have to look at the "shape" of a frame. The next theorem describes this. We use the results of Gibbs in [3]. Roughly speaking, his results imply that from the graph structure of the Dynkin diagram associated with our group and from the commutator relations, one can show that a frame $\mathscr{F}$ must look like the standard one at least modulo U^2.

Theorem 4.2. *Let $\{g_1, \dots, g_\ell\}$ be a frame in $U_\ell(K)$ where $\ell = \operatorname{rank} \mathscr{L}$. Let $g_j \equiv \prod_{i=1}^{\ell} x_{r_i}(t_{ij}) \pmod{U^2}$. Then the matrix (t_{ij}) is diagonal for all root systems except A_ℓ, D_ℓ $(\ell \geq 4)$, E_6, and B_2. For types A_ℓ, D_ℓ $(\ell \geq 4)$, E_6, and B_2, the matrix is diagonal after applying an automorphism of the group.*

Proof. We will use (the idea of) Lemma 6.3 ([3], p. 211). For completeness we state the results we need.

(G1). Suppose $t_{i\alpha} \neq 0$ and j is a node such that there are two or more nodes between i and j. Then if $t_{j\alpha} \neq 0$ and β is a node connected to α, it follows that there is $\lambda \in K^*$ such that $t_{k\alpha} = \lambda t_{k\beta}$, $k = 1, \dots, \ell$.

In other words, column α is a multiple of column β.

(G2). Suppose $t_{i\alpha} \neq 0$ and α is connected to at least two other nodes β, γ. Assume j is a node such that there is only one node between i and j. If $t_{j\alpha} \neq 0$ then there is $\lambda \in K^*$ such that $(t_{j\alpha}, t_{j\beta}, t_{j\gamma}) = \lambda(t_{i\alpha}, t_{i\beta}, t_{i\gamma})$.

(G3). Suppose $t_{i\alpha} \neq 0$ and that nodes i and j are connected. Assume also that α is connected to at least two other nodes. Then all nonzero entries in columns α, β, γ are in rows i and j if $t_{j\alpha} \neq 0$.

(G4). Suppose α is a node connected to at least two nodes, and that $t_{k\alpha}$ is the only nonzero entry in column α. Then node k is also joined to more than one node; or if it is joined only to node j, and λ, μ are joined to α, then all nonzero entries of columns α, λ, μ are in rows i and j.

These four properties are straightforward to prove, using only the commutator formula.

For each type, we show that one column has the desired form and we then use (G1)–(G4) to show that the others also have the required form.

Type $A_\ell (\ell \geq 4)$. There are two simple roots α_1 and α_ℓ such that $r_N - \alpha_i \in \Sigma^+$. From our frame we have two elements $g_{r_N-\alpha_1}$ and $g_{r_N-\alpha_\ell}$, (obtained by commutation as $x_{r_N-\alpha_1}(1)$ and $x_{r_N-\alpha_1}(1)$ are from $\{x_\alpha(1) \colon \alpha \in \pi\}$), such that $[g_1, g_{r_N-\alpha_1}] \neq \mathrm{id}$, $[g_1, g_{r_N-\alpha_\ell}] = \mathrm{id}$, $[g_1, g_{r_N-\alpha_1}] = \mathrm{id}$, and $[g_\ell, g_{r_N-\alpha_\ell}] \neq \mathrm{id}$. If we set $g_{r_N-\alpha_1} \equiv x_{r_N-\alpha_1}(t_1) x_{r_N-\alpha_\ell}(t_\ell) \pmod{C(U)}$ and $g_{r_N-\alpha_\ell} \equiv x_{r_N-\alpha_1}(t_1') x_{r_N-\alpha_\ell}(t_1') \pmod{C(U)}$, then the above relations give

$$\begin{array}{ll} t_{11}t_1 - t_{1\ell}t_\ell \neq 0 & t_{\ell 1}t_1 = t_\ell t_{\ell\ell} \\ t_{11}t_1' = t_{1\ell}t_\ell' & t_{\ell 1}t_1' - t_{\ell\ell}t_\ell' \neq 0 \end{array}$$

These relations imply that $t_\ell t_1' - t_\ell' t_1 \neq 0$.

Let j be any mode $\neq i, \ell$. From the definition of a frame we get that $[g_j, g_{r_N-\alpha_i}] = \mathrm{id}$ for $i = 1, \ell$.

Hence $t_{j1}t_1 = t_\ell t_{j\ell}$, $t_{j1}t_1' = t_\ell' t_{j\ell}$. It follows that $t_{j1}(t_1' t_\ell - t_1 t_\ell') = 0$. Hence $t_{j1} = 0$ and so $t_{j\ell} = 0$.

Next we show that $t_{11}t_{1\ell} = 0$ and $t_{\ell 1}t_{\ell\ell} = 0$. Note that both $t_{11} = 0$ and $t_{1\ell} = 0$ (similarly for ℓ) is not possible.

If $t_{i1}t_{i\ell} \neq 0$ (for $i = 1, \ell$) then (G1) implies that all rows j with $1 < j < \ell$ are zero. This is clearly impossible. Using (G2) and (G3) we conclude that each column can have only one nonzero entry. Without loss of generality we may assume that $t_{11} \neq 0$ (in view of the existence of a graph automorphism interchanging α_1 with α_ℓ and leaving α_j fixed). It follows that $t_{\ell\ell} \neq 0$. Now it is easy to see that $t_{jj} \neq 0$, for otherwise g_{r_N} (the element corresponding to $x_{r_N}(1)$) would be trivial.

Type $B_\ell (\ell \geq 4)$. If the simple roots are $\alpha_1, \alpha_2, \ldots, \alpha_\ell$, then there is only one simple root, namely α_2, such that $r_N - \alpha_2 \in \Sigma^+$. Let $g_{r_N-\alpha_2}$ be the element defined from our frame corresponding to $x_{r_N-\alpha_2}(1)$. Let $g_{r_N-\alpha_2} \equiv x_{r_N-\alpha_2}(t) \pmod{U^N}$.

From $[g_j, g_{r_N-\alpha_2}] = \mathrm{id}$ for all nodes $j \neq 2$ one concludes that $t_{j2} = 0$.

From $[g_{\alpha_2}, g_{r_N-\alpha_2}] \neq \mathrm{id}$ one gets $t_{22} \neq 0$. Now, using column 2 and (G1), (G2), (G3) it is easy to see that column j for $1 < j < \ell - 1$ has only one

nonzero entry. This nonzero entry must be connected to more than one node, by (G4). It follows that $t_{ii} \neq 0$, $i = 1, \ldots, \ell$, as otherwise the element g_{r_N} would not correspond to $x_{r_N}(1)$. It remains to show that $t_{21} = 0$ and $t_{31} = 0$ (a similar argument applies to the last column). Here again, an argument in Gibbs [3, p. 213] gets our result.

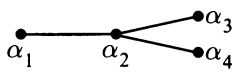
Types $C_\ell (\ell \geq 4)$, $D_\ell (\ell \geq 5)$, E_6, E_7, **and** E_8. The proofs are similar: for C_ℓ there is only one simple root α such that $r_N - \alpha \in \Sigma^+$. This root is joined to only one other root. One proceeds as in A_1 but deals with the last column as in type B_ℓ. Type D_ℓ has only one simple root α such that $r_N - \alpha \in \Sigma^+$, and it is joined to two other roots. One proceeds as with type B_ℓ. The other types are dealt with in a similar fashion.

To finish, we will show how to do D_4 and G_2 explicitly. The others, A_3, A_2, B_3, B_2 and F_4, require some modifications, but the proof is basically the same (taking into account the different types of automorphisms that exist).

Type G_2. Let α and β be the simple roots. The positive roots are $\alpha + \beta$, $2\alpha + \beta$, $3\alpha + \beta$, and $3\alpha + 2\beta$. As before, consider the matrix (t_{ij}). Since $[g_\alpha, g_{r_N - \beta}] = \mathrm{id}$, it follows that $t_{12} = 0$.

Therefore $t_{11}t_{22} \neq 0$. It remains to show that $t_{21} = 0$. Now, as can be seen, $[[x_\alpha(1), x_\beta(1)], x_\beta(1)]$ belongs to the center. Therefore the same should happen with $[[g_\alpha, g_\beta], g_\beta]$. This commutator is equal to $x_{2\alpha+\beta}(2t_{11}t_{22}t_{21})$ mod U^3. Hence $t_{21} = 0$.

Type D_4. The Dynkin diagram is



The positive roots are

$$\begin{array}{ccccccc} \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 \\ & \alpha_1 + \alpha_2 & & \alpha_2 + \alpha_3 & & \alpha_2 + \alpha_4 & \\ \alpha_1 + \alpha_2 + \alpha_3 & & & \alpha_1 + \alpha_2 + \alpha_4 & & & \alpha_3 + \alpha_2 + \alpha_4 \\ & & & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & & & \\ & & & \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 & & & \end{array}.$$

There are six automorphisms of this diagram; all leave α_2 fixed.

Let g_{α_1}, g_{α_2}, g_{α_3}, and g_{α_4} be a frame. As before, we immediately conclude that $t_{22} \neq 0$ and $t_{j2} = 0$, $j = 1, 3, 4$. Since (G1) applies, we also have $t_{1i}t_{4i} = 0$ for $i = 1, 3, 4$.

The next step is to show that $t_{2i} = 0$ for $i = 1, 3, 4$.

We obtain this as follows: Since we have a frame, the relations $[[g_1 g_2]g_2] = [[g_3 g_2]g_2] = [[g_4 g_2]g_2] = \mathrm{id}$ hold.

Calculating the terms of height 3 and setting them equal to zero, one gets the following equations:

$$\begin{array}{lll} t_{31}t_{23} = c_1 t_{21}t_{33} & t_{23}t_{41} = c_6 t_{43}t_{21} & t_{11}t_{24} = c_7 t_{21}t_{14} \\ t_{34}t_{23} = c_2 t_{24}t_{33} & t_{41}t_{24} = c_5 t_{44}t_{21} & t_{13}t_{24} = c_8 t_{23}t_{14} \\ t_{34}t_{21} = c_3 t_{31}t_{24} & t_{43}t_{24} = c_6 t_{44}t_{23} & t_{11}t_{23} = c_9 t_{21}t_{13} \end{array}$$

The constants c_i are all nonzero and come from the structure constants. Now, either $t_{11} = 0$ or $t_{41} = 0$. In both cases one shows that $t_{21} = 0$ for $i = 1, 3, 4$. Each column can only have one nonzero entry. Suppose, for example, column α_1 has two nonzero entries. Then either $t_{11}t_{31} \neq 0$ or $t_{31}t_{41} \neq 0$. Leaving α_1 and α_2 fixed and interchanging α_3 with α_4 yields a frame. Hence $(g_{\alpha_1}, g_{\alpha_2}, g_{\alpha_4}, g_{\alpha_3})$ is a frame, but its matrix (t'_{ij}) has $t'_{11}t'_{41} \neq 0$, a contradiction.

The above results imply that the matrix (t_{ij}) can only have one of six possible configurations.

The frame $\mathscr{F}$ can be further reduced.

Proposition 4.3. *Let $\mathscr{F}$ be a frame in $U_{\mathscr{F}}(K)$. Then, if $\mathscr{L} \neq C_\ell$ there is an inner automorphism i such that $i(\mathscr{F}) \equiv \{x_\alpha(t_\alpha) \colon \alpha \in \pi\} \mod U^{N-1}$. If $\mathscr{F} = C_\ell$ then there are inner and extremal automorphisms i, e such that $\varepsilon \cdot i(\mathscr{F}) \equiv \{x_\alpha(t_\alpha) \colon \alpha \in \pi\} \mod U^{N-1}$.*

Proof. The proofs of Lemmas 6.7 and 6.8 in [3] can be adapted for our purposes by substituting our $g_\alpha \in \mathscr{F}$ for Gibbs' $\theta(x_\alpha(t))$, $\alpha \in \pi$. For the definition of extremal automorphisms see Gibbs [3].

We need one more reduction.

Proposition 4.4. *Suppose $\mathscr{F}$ is a frame as above; then there are inner and extremal automorphisms i and e such that $e \cdot i(\mathscr{F}) \equiv \{x_\alpha(t_\alpha) \colon \alpha \in \pi\} \mod U^N$.*

Proof. We exclude type A_ℓ. A proof similar to the one below works, but we omit the details. There is one simple root β such that $r_N - \beta \in \Sigma^+$. If $\alpha \in \pi$ is not connected to β then $[g_\alpha, g_\beta] = \mathrm{id}$. This implies that $g_\alpha \equiv x_\alpha(t_\alpha) \mod U^N$ for such α. Suppose now that α is connected to β.

Let $g_\alpha \equiv x_\alpha(t_\alpha) x_{r_N - \beta}(U_\alpha) \mod U^N$.

It follows that $(r_N - \beta) - \alpha$ is a root (see [3, (3.2)]).

We can conjugate g_α with an element in $K_{(r_N-\beta)-\alpha}$ into $x_\alpha(\lambda)$. This conjugation leaves K_r $(r \in \pi, r \neq \alpha)$ invariant. Hence, all g_α, $\alpha \neq \beta$ have the correct form. Finally, with an extremal automorphism we can fix g_β without changing the g_α's.

Remark 4.5. If we have a frame $\mathscr{F}$ as in Proposition 4.4 and if we replace the standard frame by $\mathscr{F}$ in the definitions of Proposition 3.2, the sets defined are the same as those defined by the standard frame. The above propositions imply that the orbit of the standard frame under the action of the automorphism group of $U_{\mathscr{L}}(K)$ is 0-definable.

5. The isomorphism theorem

In this section we establish a criterion on a group G that will make it isomorphic to $U_{\mathscr{L}}(K)$, for some field K. This is completely analogous to Mal'cev's "reciprocity theorem" (see [5]). We impose two types of conditions on G: the first type are elementary; the second are not.

Conditions A. Let G be a group. We require that G possess a frame $\mathscr{F}$. Replacing the elements of $\mathscr{F}$ by the parameters $\{x_\alpha(1): \alpha \in \pi\}$ in the definitions in Proposition 3.2, we require that the subsets G_r so defined be abelian subgroups of G. Furthermore, if r, s are simple then $G_r \cap G_s = Z(G)$, (the center of G).

We require next that the ring interpreted in G via Theorem 3.3 be a field.

Conditions B. We need some preliminaries before stating our conditions. First, given a root r, different from r_N, there always exists a simple root α such that $r + \alpha$ is a root. Continuing in this way one obtains a sequence of simple roots $\alpha_1, \dots, \alpha_{r_j}$ such that $r + \alpha_1 + \cdots + \alpha_{r_i}$ is a root for $1 \le i \le r_j$ and $r + \alpha_1 + \cdots + \alpha_{r_j} = r_N$. Second, given α, β positive roots, if $\alpha + \beta$ is a root then the combinations $i\alpha + j\beta$ $(i, j \ge 1)$ which are roots form a system of type A_2, B_2, or G_2.

Let G be a group satisfying conditions A. The final conditions can now be stated: first we require that there exist abelian group homomorphisms f_r: $C(G) \to G_r$ (for all roots) such that $[\dots[[f_r(z), g_{\alpha_1}]\dots], g_{\alpha_j}] = z$, and second [Chevalley's formulas] that for each pair α, β such that $\alpha + \beta \in \Sigma^+$

$$[f_\alpha(z), f_\beta(t)] = \prod_{i,j} f_{i\alpha+j\beta}(C_{ij,\beta\alpha}(-t)^j z^i).$$

Here, the product $(-t)^j z^i$ is the one defined in the center of G via the frame.

Theorem 5.1. *Let G be a group satisfying conditions* A *and* B. *Then there exists an $\mathscr{L}$ and a field F such that $G \cong U_{\mathscr{L}}(F)$.*

Proof. The field is the center of G equipped with multiplication.

Define $\theta: U_{\mathscr{L}}(F) \to G$ by

$$x_{r_1}(t_1)x_{r_2}(t_2)\cdots x_{r_N}(t_n) \to f_{r_1}(t_1)f_{r_2}(t_2)\cdots t_N.$$

It is a tedious calculation to show that θ is an isomorphism, but it should be clear that all we need is contained in A and B.

We need a few more observations. First, recall that for abelian groups $A \subset B$, A is n-pure in B if whenever the equation $nx = a$, $a \in A$ is solvable in B it is solvable in A. A is pure in B if it is n-pure for all n. Note that K_{r_N} is a direct summand of $K_r \cdot K_{r_N}$, $r \in \pi$. The best we can say is that $C(G)$ is n-pure in G_r $(r \in \pi)$ for each n. This follows from the fact that given $\mathscr{F}$ a frame and $n \in N$, K_{r_N} is n-pure in the subgroup corresponding to r defined using $\mathscr{F}$ (see Proposition 4.3). Second, if $K \models \theta$ then $U_{\mathscr{L}}(K) \models \tilde{\theta}(g_1, \dots, g_\ell)$ where $\{g_1, \dots, g_\ell\}$ is a frame, since the multiplications defined using different frames are all "conjugate."

We are now ready for our main result.

Theorem 5.2. *Suppose* $G \equiv U_{\mathscr{L}}(K)\,\mathrm{ch}(K) \neq 2, 3$. *Then, there exists a field* $F \equiv K$ *such that* $G \cong U_{\mathscr{L}}(F)$.

Proof. We embed G elementarily in a saturated model G^*. In view of Theorem 3.3 and general model theory we may assume that $G^* = U_{\mathscr{L}}(K^*)$, $K^* \equiv K$. Let $\mathscr{F}$ be a frame in G (it is also a frame in G^*). Without loss of generality, assume that $\mathscr{F}$ is the standard frame $\bmod G^{*N}$ (by Theorem 4.2 and the existence of a diagonal automorphisms, see [3 p. 207]). We know that $G_r \subset K_r \cdot K_{r_N}$, for $r \in \pi$. The abelian group $(C(G), *)$ is pure in G_r, and since it supports a field structure it is a direct summand of G_r. Let $G_r = H_r \oplus C(G)$. If $\alpha_1, \ldots, \alpha_{r_j}$ is a sequence of simple roots as in Conditions B, then the map $h_r\colon K_r \cdot K_{r_N} \to C(G)$

$$g \to [g\colon g_{\alpha_1}; g_{\alpha_2} \cdots ; g_\alpha]$$

is an abelian group homomorphism. It is surjective with kernel K_{r_N}.

Hence, we obtain a surjective homomorphism between G_r and $C(G)$ with kernel $C(G)$. It follows that H_r is isomorphic to $C(G)$. In this way we obtain the homomorphisms necessary to apply Theorem 5.1. Note that the commutator formula is satisfied, since $G \subset G^*$.

Corollary 5.3. *Let* $I(\lambda, T)$ *be the number of models of* T *of size* λ. *If* K *is a field with* $\mathrm{ch}(K) \neq 2, 3$ *then*

$$I(\lambda, Th(K)) = I(\lambda, Th(U_{\mathscr{L}}(K))).$$

There are several problems left. Perhaps the most interesting (returning to Mal′cev) is:

Problem: Is $I(\lambda, Th(R)) = I(\lambda, Th(UT(n, R)))$? Here $UT(n, R)$ is the group of upper unitriangular matrices over R, where R is an associative ring with unit.

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