LIFTING HOMEOMORPHISMS OF A COMPLEX SURFACE TO ITS NORMALIZATION

WILLIAM A. ADKINS

(Communicated by Clifford J. Earle, Jr.)

Abstract. It is proved that any homeomorphism of a complex analytic surface lifts to a homeomorphism of the normalization.

The purpose of the present note is to provide a proof of the following result which was mentioned in the paper [3] of Steenbrink and Stevens without proof.

Theorem. Any homeomorphism of a complex analytic surface lifts to a homeomorphism of the normalization.

Before giving the proof of this theorem we recall the basic notations and results which are needed. By a complex analytic surface we mean a reduced, second countable, complex space of pure dimension 2. If \((X, \mathcal{O}_X)\) is a complex space with structure sheaf \(\mathcal{O}_X\) and \(a \in X\), then \((X, a)\) will denote the germ of \(X\) at \(a\). A theorem of Mumford [2] states that if the germ \((X, a)\) is homeomorphic to the germ \((\mathbb{C}^2, 0)\), then \(a\) is a nonsingular point of \(X\).

If \(X\) is a complex surface, then \(S(X)\) will denote the singular subspace of \(X\) with the reduced structure. We say that a point \(a \in S(X)\) is a topological multicross if the germ \((X, a)\) can be written as a union of surface germs \((X_i, a)\)

\[
(X, a) = (X_1, a) \cup \cdots \cup (X_r, a)
\]

where

1. \((X_i, a)\) is analytically isomorphic to \((\mathbb{C}^2, 0)\) for \(1 \leq i \leq r\).
2. \((X_i, a) \cap (X_j, a) = (S(X), a)\) for all \(i \neq j\).
3. \((S(X), a)\) is homeomorphic to \((\mathbb{C}, 0)\).

A point \(a \in S(X)\) is said to be an analytic multicross, or just a multicross, if the germ \((X, a)\) satisfies conditions (1), (2), and the following condition \((3')\), instead of (3).

\((3')\) \((S(X), a)\) is analytically isomorphic to the germ \((\mathbb{C}, 0)\).

The multicross singularities of complex surfaces were introduced in [1] in order to provide a structure theorem for weakly normal surface singularities.
Recall that a complex space is weakly normal if the sheaf \( \mathcal{O}^c_X \) of germs of \( c \)-holomorphic functions on \( X \) is equal to the sheaf \( \mathcal{O}_X \) of germs of holomorphic functions on \( X \).

We now introduce notation for certain subsets of \( X \): \( X_{\text{reg}} \) denotes the set of all regular (smooth) points of \( X \), \( X_{\text{tm}} \) denotes the set of all topological multicross points of \( X \), and \( X_{\text{am}} \) denotes the set of all analytic multicross points of \( X \). Then we let \( X_s = X \backslash (X_{\text{reg}} \cup X_{\text{tm}}) \) and \( X_u = X \backslash (X_{\text{reg}} \cup X_{\text{am}}) \). Thus \( X \) can be written as a disjoint union in two different ways: \( X = X_{\text{reg}} \cup X_{\text{tm}} \cup X_s = X_{\text{reg}} \cup X_{\text{am}} \cup X_u \). Since \( X_{\text{am}} \subseteq X_{\text{tm}} \), it follows that \( X_s \subseteq X_u \). If \( X \) is a weakly normal complex surface, then the Oka theorem for weakly normal singularities (Theorem 2.4 of [1]) shows that \( X_u \) (and hence \( X_s \subseteq X_u \)) is a discrete subset of \( X \).

**Lemma.** Let \( X \) be a complex surface and let \( h : X \to X \) be a homeomorphism. Then \( h(X_{\text{reg}}) = X_{\text{reg}} \), \( h(X_{\text{tm}}) = X_{\text{tm}} \), and \( h(X_s) = X_s \).

**Proof.** If \( a \) is a regular point of \( X \), then \( h(a) \) is a regular point of \( X \) by Mumford's theorem. Thus \( h(X_{\text{reg}}) = X_{\text{reg}} \). Suppose \( a \in X_{\text{tm}} \). Then \( (X, a) = (X_1, a) \cup \cdots \cup (X_r(a), a) \) where \( (X_i, a) \) is analytically isomorphic to \( (C^2, 0) \) and \( (X_i, a) \cap (X_j, a) = (S(X), a) \) for all \( i \neq j \) and with \( (S(X), a) \) homeomorphic to \( (C, 0) \). Now \( h : (X, a) \to (X, h(a)) \) is a homeomorphism of germs and \( h(S(X)) = S(X) \) so that \( h : (X \backslash S, a) \to (X \backslash S, h(a)) \) is also a homeomorphism of germs, where \( S(X) = S \). Therefore, \( (X \backslash S, h(a)) \) has the same number of connected components as \( (X \backslash S, a) \), so that \( (X, h(a)) \) can be written as a union \( (X_1', h(a)) \cup \cdots \cup (X_r(a), h(a)) \) and \( h : (X, a) \to (X, h(a)) \) decomposes (after reindexing, if necessary) into a union of homeomorphisms \( h_i : (X_i, a) \to (X_i', h(a)) \) with the property that \( h_i|_{X_i \cap X_j = S(X)} = h_j|_{X_i \cap X_j = S(X)} \). By Mumford's theorem, \( (X_i', h(a)) \) is nonsingular for all \( i \), and \( (S(X), h(a)) \) is homeomorphic to \( (S(X), a) \), which is homeomorphic to \( (C, 0) \). Therefore, \( h(a) \in X_{\text{tm}} \). Finally, \( h(X_s) = X_s \) since \( X \) is a disjoint union of \( X_{\text{reg}} \), \( X_{\text{tm}} \), and \( X_s \).

We now proceed to the proof of the theorem. Thus let \( X \) be a complex analytic surface, let \( \eta : \tilde{X} \to X \) be the normalization of \( X \), and let \( \tilde{X}_{\text{reg}} = \eta^{-1}(X_{\text{reg}}) \), \( \tilde{X}_{\text{tm}} = \eta^{-1}(X_{\text{tm}}) \), and \( \tilde{X}_s = \eta^{-1}(X_s) \). Assume that \( h : X \to X \) is a homeomorphism. Then we claim that there is a homeomorphism \( \tilde{h} : \tilde{X} \to \tilde{X} \) such that the diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{h}} & \tilde{X} \\
\downarrow \eta & & \downarrow \eta \\
X & \xrightarrow{h} & X
\end{array}
\]

is commutative. The proof proceeds by constructing the homeomorphism \( \tilde{h} : \tilde{X} \to \tilde{X} \) successively on \( \tilde{X}_{\text{reg}} \), \( \tilde{X}_{\text{tm}} \), and \( \tilde{X}_s \). Without loss of generality,
we may assume that $X$ is weakly normal (since the weak normalization $X^w$ of $X$ is homeomorphic to $X$ [1]). As observed above, $X_s$ is a discrete subset of $X$ and since $\eta: \tilde{X} \to X$ is a finite map, $\tilde{X}_s$ is a discrete subset of $\tilde{X}$. Also note that $\tilde{X}_s = \eta^{-1}(h(X_s))$ since, according to the lemma, $h(X_s) = X_s$.

Define $\tilde{h}': \tilde{X}_{\text{reg}} \to \tilde{X}_{\text{reg}}$ by $\tilde{h}' = \eta^{-1} \circ h \circ \tilde{\eta}$ where $\tilde{\eta} = \eta|_{\tilde{X}_{\text{reg}}} : \tilde{X}_{\text{reg}} \to \tilde{X}_{\text{reg}}$ is biholomorphic. It remains to extend $\tilde{h}'$ homeomorphically across $\tilde{X}_t \cup \tilde{X}_s$. Suppose that $a \in X_t$. According to the lemma $(X, a) = (X_1, a) \cup \ldots \cup (X_r(a), a)$ where each $(X_i, a)$ is analytically isomorphic to $(\mathbb{C}^2, 0)$ and $(X_i, a) \cap (X_j, a)$ is homeomorphic to $(\mathbb{C}, 0)$ whenever $i \neq j$, with a similar expression for $h(X, a) = (X', h(a)) = (X'_1, h(a)) \cup \ldots \cup (X'_r(a), h(a))$. Let $h_i = h|_{(X_i, a)}$. Then $h_i$ is a homeomorphism from $(X_i, a)$ to $(X'_i, h(a))$. Since $\eta^{-1}(a) = \{b_1, \ldots, b_r(a)\}$ with $\eta_i = \eta|_{(\tilde{X}, b_i)} : (\tilde{X}, b_i) \to (X_i, a)$ biholomorphic and $\eta^{-1}(h(a)) = \{c_1, \ldots, c_r(a)\}$ with $\eta_i = \eta|_{(\tilde{X}, c_i)} : (\tilde{X}, c_i) \to (X'_i, h(a))$ biholomorphic, we can define $\tilde{h}: (\tilde{X}, b_i) \to (\tilde{X}, c_i)$ by the formula $\tilde{h} = \eta_i^{-1} \circ h \circ \eta_i$. Therefore, we have extended $\tilde{h}'$ to a homeomorphism $\tilde{h}: \tilde{X} \setminus \tilde{X}_s \to \tilde{X} \setminus \tilde{X}_s$ where $\tilde{X}_s$ is a discrete subset of $\tilde{X}$. Thus we need to extend $\tilde{h}$ across $\tilde{X}_s$. If $b \in \tilde{X}_s$, then $\eta(\tilde{h}(\tilde{X}_{\text{reg}}, b))$ is a connected component of $(X \setminus S(X), h(\eta(b)))$ so the closure is an irreducible component of the germ $(X, h(\eta(b)))$. Thus the points of $\eta^{-1}(h(\eta(b)))$ are in one-to-one correspondence with the irreducible components of $(X, h(\eta(b)))$. Let $\tilde{h}(b)$ be the point corresponding to $\eta(\tilde{h}(\tilde{X}_{\text{reg}}, b))$. This completes the construction of $\tilde{h}$ and from the method of construction it is clear that $\tilde{h}: \tilde{X} \to \tilde{X}$ is a homeomorphism, and the theorem is proved.

REFERENCES


DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA 70803