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THE OPERATOR INEQUALITY $P \leq A^*PA$

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ABSTRACT. A short proof of the result that if P is a positive compact operator and A is a contraction such that $P \leq A^*PA$, then $P = A^*PA$, $\overline{\text{ran}}P$ reduces A and $A|\overline{\text{ran}}P$ is unitary is given.

We consider operators, i.e. elements of the algebra $B(H)$ of bounded linear transformations on a complex Hilbert space H into itself. In an often referred to paper (by R. G. Douglas) it has been shown that if P is a positive compact operator and A is a contraction such that $P \leq A^*PA$, then $P = A^*PA$, $\overline{\text{ran}}P$ reduces A and $A|\overline{\text{ran}}P$ is unitary (see [1, Theorem 8 and Corollary 6.5]). The purpose of this short note is to give a proof of this result requiring little more than the well-known fact that a compact hyponormal operator is normal [3, Problem 206].

Theorem 1. *2 Let A be a contraction and P be a positive operator such that $P \leq A^*PA$. If $P^{1/2}A$ is compact, then $P = A^*PA$, $\overline{\text{ran}}P$ reduces A and $A|\overline{\text{ran}}P$ is unitary.*

Proof. Set $P^{1/2}A = M$. Then $MM^* = P^{1/2}AA^*P^{1/2} \leq P$. Hence

$$0 \leq A^*PA - P \leq A^*PA - P^{1/2}AA^*P^{1/2} = M^*M - MM^*,$$

i.e. the operator M is hyponormal. Since M is compact, M is normal. The normality of M implies that $P = A^*PA = P^{1/2}AA^*P^{1/2}$, and hence that A^* is an isometry on $\overline{\text{ran}}P$ and P commutes with A (and so also with A^*). Consequently $A^*AP = A^*PA = P = PAA^*$. Hence $\overline{\text{ran}}P$ reduces A and $A|\overline{\text{ran}}P$ is unitary.

A more general version of the result of Douglas holds. Let $R(A^*, A)\colon B(H) \to B(H)$ be defined by $R(A^*, A)X = A^*XA - X$, and let $R^n(A^*, A)$ denote an n-times application of $R(A^*, A)$.

Theorem 2. *Let A be contraction and X be a compact operator such that $R^{n-1}(A^*, A)X \geq 0$ and $R^n(A^*, A)X \geq 0$ for some natural number $n \geq 2$. Then $A^*XA = X$, $\overline{\text{ran}}X$ reduces A and $A|\overline{\text{ran}}X$ is unitary.*

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Proof. Set $R^{n-1}(A^*, A)X = P$. Then the proof of Theorem 1 implies that $A^*PA - P = 0$ (i.e. $R^n(A^*, A)X = 0$), $\overline{\text{ran}}P$ reduces A and $A|\overline{\text{ran}}P$ is unitary. Proceeding now as in the latter part of the proof of [2, Theorem 2(a)] (see also [4, Proof of Theorem 1′]) the assertion of the statement follows.

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