A PROPERTY OF PURELY INFINITE SIMPLE C*-ALGEBRAS

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Abstract. An alternative proof is given for the fact ([13]) that a purely infinite, simple C*-algebra has the FS property: the set of self-adjoint elements with finite spectrum is norm dense in the set of all self-adjoint elements. In particular, the Cuntz algebras $O_n$ ($2 \leq n \leq +\infty$) and the Cuntz–Krieger algebras $O_A$, if $A$ is an irreducible matrix, have the FS property. This answers a question raised in [2, 2.10] concerning the structure of projections in the Cuntz algebras. Moreover, many corona algebras and multiplier algebras have the FS property.

A C*-algebra $A$ is said to be purely infinite if $(\pi Ax)^{-}$ contains an infinite projection for every nonzero positive element $x$ in $A$ ([7, 12]). The author recently proved ([13]) that purely infinite, simple C*-algebras have the FS property; namely, the set of self-adjoint elements with finite spectrum is norm dense in the set of all self-adjoint elements. Actually, many interesting C*-algebras have the FS property. For example, the Bunce–Deddens algebras have FS ([1, 3]); many corona algebras and multiplier algebras have FS ([5, 13]); certain irrational rotation algebras have FS ([6]). Certainly, all AF algebras, von Neumann algebras, and AW* algebras have FS.

In this short note, we provide another proof for the fact that purely infinite, simple C*-algebras have the FS property. The algebras $O_n$ ($2 \leq n \leq +\infty$) and $O_A$, if $A$ is an irreducible matrix, are purely infinite and simple ([7, 8, 9]), and many corona algebras are purely infinite and simple ([12, 13]). Hence, these C*-algebras have the FS property. In particular, this answers a question of B. Blackadar raised in [2, 2.10] concerning the projection structure of the Cuntz algebras.

1. Theorem. If $A$ is a purely infinite, simple C*-algebra, then $A$ has the FS property, and hence $RR(A) = 0$.

Proof. To prove the conclusion, by [2, 2.7; 10], it is equivalent to prove that every hereditary C*-subalgebra of $A$ has an approximate identity consisting of
projections (the HP property). It suffices to show that for any positive element \(x\) in \(A\) and any positive number \(\delta\), there exists a projection \(p\) in the hereditary \(C^*\)-subalgebra \(A_x\) of \(A\) generated by \(x\) such that

\[\|(1 - p)x\| < \delta.\]

We can assume that \(\|x\| = 1\). If \(0 \notin \sigma(x)\), then \(A_x = A\) and \(A\) has a unit. Let \(p\) be the unit. If \(0\) is an isolated point of \(\sigma(x)\), then let \(p\) be the spectral projection \(E_{(0, \infty)}(x)\) of \(x\) over the interval \((0, \infty)\), which is in \(A_x\). Hence, we can, furthermore, assume that \(0\) is a limit point of \(\sigma(x)\). From now on, we will denote the Banach space double dual of \(A\) by \(A^{**}\).

For any positive number \(\varepsilon\), we define a real-valued continuous function \(f_\varepsilon(t)\) on the interval \([0, 1]\) as follows:

\[f_\varepsilon(t) = \begin{cases} 
  t & \text{if } 2\varepsilon < t, \\
  \text{linear} & \text{if } \varepsilon < t \leq 2\varepsilon, \\
  0 & \text{if } t < \varepsilon.
\end{cases}\]

Clearly,

\[\|x - f_\varepsilon(x)\| < 2\varepsilon.\]

Let \(B_\varepsilon\) be the hereditary \(C^*\)-subalgebra of \(A\) supported by the open projection \(p_\varepsilon = E_{(\varepsilon, \infty)}(x)\), where \(E_{(\varepsilon, \infty)}(x)\) is the spectral projection of \(x\) in \(A^{**}\) over the interval \((\varepsilon, \infty)\). Clearly, \(f_\varepsilon(x)\) is a strictly positive element of \(B_\varepsilon\), and hence \(B_\varepsilon\) is \(\sigma\)-unital. Similarly, let \(A_\varepsilon\) to be the hereditary \(C^*\)-subalgebra of \(A\) supported by the open projection \(E_{(0, \varepsilon)}(x)\). It is obvious that \(A_\varepsilon\) and \(B_\varepsilon\) are mutually orthogonal, purely infinite, and simple.

Since \(0\) is a limit point of \(\sigma(x)\), we can choose a nonzero projection \(r\) in \(A_\varepsilon\). Since \(A\) is purely infinite and simple, by a routine argument (for example, see the proof of [2, 3.12]), we can obtain a sequence of nonzero subprojections of \(r\), say \(\{q_i\}\), such that

\[q_i \sim q_j \quad \text{if } i, j \geq 1, \quad \text{and} \quad q_j q_i = 0 \quad \text{if } i \neq j,\]

where \(\sim\) means the Murray–von Neumann equivalence of projections in \(A\).

Set \(p_0 = \sum_{i=1}^\infty q_i\). It is easily verified that \(p_0\) is an open projection in \(A^{**}\) and the hereditary \(C^*\)-subalgebra \(B_0\) of \(A\) supported by \(p_0\) is simple and stable; actually, \(B_0 \cong q_1 A_\varepsilon \otimes K\), where \(K\) is the \(C^*\)-algebra consisting of all compact operators on a separable Hilbert space. Since both \(q_1 A_\varepsilon\) and \(B_\varepsilon\) are \(\sigma\)-unital full hereditary \(C^*\)-subalgebras of \(A_x\) and \(A_x\) is \(\sigma\)-unital also, by [4, 2.8], we have that \(B_0 \cong B_\varepsilon \otimes K\). Obviously, \(B_0\) is a hereditary \(C^*\)-subalgebra of \(A_x\), and of course is orthogonal to \(B_\varepsilon\). Let \(q_0 = p_\varepsilon + p_0\). Then \(q_0\) is an open projection in \(A^{**}\) and the hereditary \(C^*\)-subalgebra \(B_1\) of \(A_x\) supported by \(q_0\) is \(*\)-isomorphic to \(B_\varepsilon \otimes K\). Hence \(B_1\) is \(*\)-isomorphic to \(q_1 A_\varepsilon \otimes K\). It follows that \(B_1\) has an approximate identity consisting of projections. Thus, we can find a projection \(p\) in \(B_1 \subset A_x\) such that

\[\|(1 - p)f_\varepsilon(x)\| < \varepsilon.\]
Therefore,
\[\|(1 - p)x\| \leq \|(1 - p)(x - f_\varepsilon(x))\| + \|(1 - p)f_\varepsilon(x)\| < 3\varepsilon.\]
Since \(\varepsilon\) can be arbitrarily small, this completes the proof. \(\Box\)

2. **Corollaries.** (i). The Cuntz algebras \(O_n\) (2 \(\leq n \leq \infty\)) and the Cuntz–Krieger algebra \(O_A\), if \(A\) is an irreducible matrix, have the FS property.

(ii). If \(A\) is a \(\sigma\)-unital, nonunital, simple \(C^*\)-algebra with the FS property, then \(M(A)/A\) has the FS property provided \(M(A)/A\) is simple. If, in addition, every projection in \(M(A)/A\) lifts to a projection in \(M(A)\), then \(M(A)\) has the FS property.

**Proof.** (i) follows from Theorem 1 and [7, 1.6]. (ii) follows from Theorem 1, [12, 1.3] and [13] or [5]. \(\Box\)

3. **Remark.** The author has pointed out in [11, 3.1], under the assumption that \(A\) is a \(\sigma\)-unital \(C^*\)-algebra with the FS property, that \(M(A)\) has the FS property if and only if every self-adjoint element of \(M(A)\) can be written in the following form:

\[
\sum_{i=1}^{\infty} \lambda_i p_i + a,
\]
where \(\{\lambda_i\}\) is a bounded real sequence, \(\{p_i\}\) is a sequence of mutually orthogonal projections of \(A\), and \(a\) is a self-adjoint element of \(A\). In other words, the general Weyl–von Neumann theorem holds in \(M(A)\), if and only if \(M(A)\) has FS. The reader is referred to [5, 11, 13, 14] for more examples of \(C^*\)-algebras with the FS property and related results.

**References**


13. _____, *$C^*$-algebras with real rank zero and the internal structure of their corona and multiplier algebras*, Part I, Part II, Part IV, preprints.


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