A PROPERTY OF PURELY INFINITE SIMPLE $C^*$-ALGEBRAS

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(Communicated by Palle E. T. Jorgensen)

Abstract. An alternative proof is given for the fact ([13]) that a purely infinite, simple $C^*$-algebra has the FS property: the set of self-adjoint elements with finite spectrum is norm dense in the set of all self-adjoint elements. In particular, the Cuntz algebras $O_n$ ($2 \leq n \leq +\infty$) and the Cuntz-Krieger algebras $O_A$, if $A$ is an irreducible matrix, have the FS property. This answers a question raised in [2, 2.10] concerning the structure of projections in the Cuntz algebras. Moreover, many corona algebras and multiplier algebras have the FS property.

A $C^*$-algebra $A$ is said to be purely infinite if $(xAx)^{-}$ contains an infinite projection for every nonzero positive element $x$ in $A$ ([7, 12]). The author recently proved ([13]) that purely infinite, simple $C^*$-algebras have the FS property; namely, the set of self-adjoint elements with finite spectrum is norm dense in the set of all self-adjoint elements. Actually, many interesting $C^*$-algebras have the FS property. For example, the Bunce-Deddens algebras have FS ([1, 3]); many corona algebras and multiplier algebras have FS ([5, 13]); certain irrational rotation algebras have FS ([6]). Certainly, all AF algebras, von Neumann algebras, and $AW^*$ algebras have FS.

In this short note, we provide another proof for the fact that purely infinite, simple $C^*$-algebras have the FS property. The algebras $O_n$ ($2 \leq n \leq +\infty$) and $O_A$, if $A$ is an irreducible matrix, are purely infinite and simple ([7, 8, 9]), and many corona algebras are purely infinite and simple ([12, 13]). Hence, these $C^*$-algebras have the FS property. In particular, this answers a question of B. Blackadar raised in [2, 2.10] concerning the projection structure of the Cuntz algebras.

1. Theorem. If $A$ is a purely infinite, simple $C^*$-algebra, then $A$ has the FS property, and hence $RR(A) = 0$.

Proof. To prove the conclusion, by [2, 2.7; 10], it is equivalent to prove that every hereditary $C^*$-subalgebra of $A$ has an approximate identity consisting of
projections (the HP property). It suffices to show that for any positive element \( x \) in \( A \) and any positive number \( \delta \), there exists a projection \( p \) in the hereditary \( C^* \)-subalgebra \( A_x \) of \( A \) generated by \( x \) such that

\[
\|(1 - p)x\| < \delta.
\]

We can assume that \( \|x\| = 1 \). If \( 0 \notin \sigma(x) \), then \( A_x = A \) and \( A \) has a unit. Let \( p \) be the unit. If \( 0 \) is an isolated point of \( \sigma(x) \), then let \( p \) be the spectral projection \( E_{(0, \infty)}(x) \) of \( x \) over the interval \((0, \infty)\), which is in \( A_x \). Hence, we can, furthermore, assume that \( 0 \) is a limit point of \( \sigma(x) \). From now on, we will denote the Banach space double dual of \( A \) by \( A^{**} \).

For any positive number \( \varepsilon \), we define a real-valued continuous function \( f_\varepsilon(t) \) on the interval \([0, 1]\) as follows:

\[
f_\varepsilon(t) = \begin{cases} 
  t & \text{if } 2\varepsilon < t, \\
  \text{linear} & \text{if } \varepsilon < t \leq 2\varepsilon, \\
  0 & \text{if } t < \varepsilon.
\end{cases}
\]

Clearly,

\[
\|x - f_\varepsilon(x)\| < 2\varepsilon.
\]

Let \( B_\varepsilon \) be the hereditary \( C^* \)-subalgebra of \( A \) supported by the open projection \( p_\varepsilon = E_{(\varepsilon, \infty)}(x) \), where \( E_{(\varepsilon, \infty)}(x) \) is the spectral projection of \( x \) in \( A^{**} \) over the interval \((\varepsilon, \infty)\). Clearly, \( f_\varepsilon(x) \) is a strictly positive element of \( B_\varepsilon \), and hence \( B_\varepsilon \) is \( \sigma \)-unital. Similarly, let \( A_\varepsilon \) to be the hereditary \( C^* \)-subalgebra of \( A \) supported by the open projection \( E_{(0, \varepsilon)}(x) \). It is obvious that \( A_\varepsilon \) and \( B_\varepsilon \) are mutually orthogonal, purely infinite, and simple.

Since \( 0 \) is a limit point of \( \sigma(x) \), we can choose a nonzero projection \( r \) in \( A_\varepsilon \). Since \( A \) is purely infinite and simple, by a routine argument (for example, see the proof of [2, 3.12]), we can obtain a sequence of nonzero subprojections of \( r \), say \( \{q_i\} \), such that

\[
q_i \sim q_j \quad \text{if } i, j \geq 1, \quad \text{and} \quad q_j q_i = 0 \quad \text{if } i \neq j,
\]

where \( \sim \) means the Murray–von Neumann equivalence of projections in \( A \).

Set \( p_0 = \sum_{i=1}^{\infty} q_i \). It is easily verified that \( p_0 \) is an open projection in \( A^{**} \) and the hereditary \( C^* \)-subalgebra \( B_0 \) of \( A \) supported by \( p_0 \) is simple and stable; actually, \( B_0 \cong q_1 Aq_1 \otimes K \), where \( K \) is the \( C^* \)-algebra consisting of all compact operators on a separable Hilbert space. Since both \( q_1 Aq_1 \) and \( B_\varepsilon \) are \( \sigma \)-unital full hereditary \( C^* \)-subalgebras of \( A_x \) and \( A_x \) is \( \sigma \)-unital also, by [4, 2.8], we have that \( B_0 \cong B_\varepsilon \otimes K \). Obviously, \( B_0 \) is a hereditary \( C^* \)-subalgebra of \( A_x \), and of course is orthogonal to \( B_\varepsilon \). Let \( q_0 = p_\varepsilon + p_0 \). Then \( q_0 \) is an open projection in \( A^{**} \) and the hereditary \( C^* \)-subalgebra \( B_1 \) of \( A_x \) supported by \( q_0 \) is \( * \)-isomorphic to \( B_\varepsilon \otimes K \). Hence \( B_1 \) is \( * \)-isomorphic to \( q_1 Aq_1 \otimes K \). It follows that \( B_1 \) has an approximate identity consisting of projections. Thus, we can find a projection \( p \) in \( B_1 \subset A_x \) such that

\[
\|(1 - p)f_\varepsilon(x)\| < \varepsilon.
\]
Therefore,
\[ \| (1 - p)x \| \leq \| (1 - p)(x - f_\epsilon(x)) \| + \| (1 - p)f_\epsilon(x) \| < 3\epsilon. \]
Since \( \epsilon \) can be arbitrarily small, this completes the proof. \( \Box \)

2. **Corollaries.** (i). The Cuntz algebras \( O_n \) (\( 2 \leq n \leq \infty \)) and the Cuntz–Krieger algebra \( O_A \), if \( A \) is an irreducible matrix, have the FS property.

(ii). If \( A \) is a \( \sigma \)-unital, nonunital, simple \( C^* \)-algebra with the FS property, then \( M(A)/A \) has the FS property provided \( M(A)/A \) is simple. If, in addition, every projection in \( M(A)/A \) lifts to a projection in \( M(A) \), then \( M(A) \) has the FS property.

**Proof.** (i) follows from Theorem 1 and [7, 1.6]. (ii) follows from Theorem 1, [12, 1.3] and [13] or [5]. \( \Box \)

3. **Remark.** The author has pointed out in [11, 3.1], under the assumption that \( A \) is a \( \sigma \)-unital \( C^* \)-algebra with the FS property, that \( M(A) \) has the FS property if and only if every self-adjoint element of \( M(A) \) can be written in the following form:
\[ \sum_{i=1}^{\infty} \lambda_i p_i + a, \]
where \( \{\lambda_i\} \) is a bounded real sequence, \( \{p_i\} \) is a sequence of mutually orthogonal projections of \( A \), and \( a \) is a self-adjoint element of \( A \). In other words, the general Weyl–von Neumann theorem holds in \( M(A) \), if and only if \( M(A) \) has FS. The reader is referred to [5, 11, 13, 14] for more examples of \( C^* \)-algebras with the FS property and related results.

**References**


13. _____, *C*-algebras with real rank zero and the internal structure of their corona and multiplier algebras, Part I, Part II, Part IV, preprints.


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