CENTRALIZERS OF IMMERSIONS OF THE CIRCLE

CARLOS ARTEAGA

(Communicated by Kenneth R. Meyer)

Abstract. We prove here that for every element \( f \) of an open and dense subset of immersions of the circle \( S^1 \), either the centralizer \( Z(f) \) of \( f \) is trivial (i.e. \( f \) only commutes with its own powers) or \( f \) is topologically conjugate to a map \( f_n: S^1 \to S^1 \) given by \( f_n(z) = z^n \) and, in this case, if \( h \) is the conjugacy between \( f \) and \( f_n \) then \( Z(f) \) is a subgroup of \( \{ h^{-1} \circ \omega f_m \circ h ; m \in \mathbb{N} \text{ and } \omega^{m-1} = 1 \} \).

1. Introduction

Let \( \text{Imm}(S^1) \) be the group of \( C^\infty \) immersions of the circle \( S^1 \) (i.e. maps of \( S^1 \) onto itself without critical points). For \( f \in \text{Imm}(S^1) \), \( Z(f) \) denotes the centralizer group of \( f \), i.e. the set of elements in \( \text{Imm}(S^1) \) that commute with \( f \). We say that \( f \) has trivial centralizer if \( Z(f) \) is reduced to the iterates \( \{ f^n , n \in \mathbb{N} \} \) of \( f \).

We denote by \( f_n \) the immersion of \( S^1 \) given by \( f_n(z) = z^n \).

The purpose of this paper is to prove the following result.

Theorem. Let \( \text{Imm}(S^1) \) be given the \( C^r \) topology, \( r \in \mathbb{N} \). Then there is an open and dense subset \( \mathcal{U} \) of \( \text{Imm}(S^1) \) such that for \( f \in \mathcal{U} \), either \( Z(f) \) is trivial or \( f \) is topologically conjugate to a map \( f_n \) and in this case, if \( h: S^1 \to S^1 \) is the conjugacy between \( f \) and \( f_n \) then \( Z(f) \) is a subgroup of \( \{ h^{-1} \circ \omega f_m \circ h ; m \in \mathbb{N} \text{ and } \omega^{m-1} = 1 \} \).

This result is an extension to immersions of a theorem of Kopell [2, Theorem 3], who showed the triviality of the centralizer for an open and dense subset of diffeomorphisms of the circle.

The proof of the theorem is based on a result of Mañe [3] that states that structural stability is \( C^r \) generic in \( \text{Imm}(S^1) \).

From the theorem it follows that,

\[
Z(f_n) = \{ \omega f_m ; m \in \mathbb{N} , \omega^{m-1} = 1 \}.
\]
Now observe that if an immersion $f: S^1 \rightarrow S^1$ is topologically conjugate to a map $f_n$ by a conjugacy $h$, then $f$ commutes with $h^{-1} \circ \omega f_m \circ h$ for all $m \in N$ and all $\omega \in S^1$ satisfying $\omega^{n-1} = 1$. Since $h^{-1} \circ \omega f_m \circ h$ is not necessarily $C^\infty$, we were not able to prove that

$$Z(f) = \{h^{-1} \circ \omega f_m \circ h; m \in N, \omega^{n-1} = 1\}.$$ 

2. Proof of the theorem

We begin by recalling some basic concepts and establishing preliminary results.

Let $f: S^1 \rightarrow S^1$ be a immersion of $S^1$. As usual we say that $z \in S^1$ is a periodic point of $f$ if $f^n(x) = x$ for some $n \in N$. In this case we say that $x$ is a sink if $|f^n'(x)| < 1$ and a source if $|f^n'(x)| > 1$. The basin of a sink is defined as the set of points $y$ such that $\lim_{n \to \infty} |f^n(x) - f^n(y)| = 0$. It is an open set containing $x$. We denote by $\Sigma(f)$ the complement of the union of the basins of the sinks of $f$. $\Sigma(f)$ is invariant under both $f$ and $f^{-1}$. We say that $\Sigma(f)$ is hyperbolic if there exist constants $k > 0$ and $k > 1$ satisfying

$$|f^n'(x)| > k\lambda^n \quad \text{for all } x \in \Sigma(f) \text{ and } n > 0.$$ 

We note that if $h \in Z(f)$ then the set of sinks of $f$ (sources of $f$) and the basins of the sinks are invariant under $h$.

The following lemma is basic in the proof of the theorem.

Lemma 2.1. Let $g: S^1 \rightarrow S^1$ be a strictly monotone continuous map of $S^1$. If $g \circ f_k = f_k \circ g$ for some $k > 1$ then $g = \omega f_m$ where $m = \text{degree } g$ and $\omega$ is a $(k-1)$th root of unity.

Proof. By hypothesis, $g$ satisfies

$$g(z^k) = g(z)^k \quad \text{for all } z \in S^1. \tag{*}$$

Then, if $\omega = f(1)$, we have $\omega^{k-1} = 1$ and this implies that $\omega^{-1} g$ commutes with $f_k$. Hence we can assume without loss of generality that $g(1) = 1$, so we have to show that

$$g(z) = z^m \quad \text{for all } z \in S^1.$$ 

First we claim that $g$ is order preserving. Otherwise, by hypothesis, there exists a point $z \in S^1$ sufficiently close to 1 and such that

$$1 < z < z^k \quad \text{and} \quad g(z^k) < g(z) < g(z)^k < 1.$$ 

This contradicts $(*)$ and proves the claim.

We now consider the inverse image of 1 under $f_k$ and $g$. The points mapping to 1 under $f_k$ are exactly the $k$th roots of unity, which can be represented by the points

$$1 < z_1 < z_1^2 < \cdots < z_1^{k-1} \quad \text{where } z_1^k = 1.$$
Similarly, \( g \) being of degree \( m \) and strictly monotone, maps exactly \( m \) points

\[
l = p_0 < \cdots < p_{m-1}
\]
to 1. We claim that

\[
g(z_i^l) = (z_i^l)^m \quad \text{for all } i = 0, 1, \ldots k-1.
\]

In fact, if \( i = 0 \), the claim is obvious because \( g(1) = 1 \). Suppose now, inducting on \( i \) that \( g(z_i^l) = (z_i^l)^m \). Since \([z_i^l, z_i^{l+1}]\) maps under \( f_k \) bijectively to \( S^1 \), there exist exactly \( m+1 \) points

\[
z_i^l = x_0 < x_1 < \cdots < x_m = z_i^{l+1}
\]
satisfying \( f_k(x_j), \ j = 0, \ldots m-1 \) and \( f(x_m) = 1 \). This and (*) imply that \( g(x_j)^k = 1 \) and so \( g(x_j) = z_i^l \) for some \( l = 0, \ldots k-1 \). We note that between two successive elements \( x_j, x_{j+1} \) there is no other inverse image of \( z_i^l \) under \( g \), because if \( g(x) = z_i^l \) for some \( l \), then by (*)

\[
g(x^k) = g(x)^k = (z_i^l)^k = 1,
\]

so \( x^k = p_j \) for some \( 0 \leq j \leq m-1 \) and this implies that \( x = x_i \) for some \( 0 \leq l \leq m \). These properties together with the facts that \( g \) is order preserving and \( g(z_i^l) = (z_i^l)^m \), imply that \( g(x_j) = (z_i^l)^m z_i^l \). Hence

\[
g(z_i^{l+1}) = (z_i^l)^m z_i^l = (z_i^{l+1})^m.
\]

Similarly, since \( g \circ f_k^n = f_k^n \circ g \) and \( f_k^n = f_k^* \) for \( n \in N \), we have by the arguments above that \( g(z) = z^m \) for all inverse images of 1 under \( f_k^n \). Since \( \bigcup_{n \in N} f_k^n(1) \) is dense in \( S^1 \), we have that \( g(z) = z^m \) for all \( z \in S^1 \) and the lemma is proved.

Now we prove the theorem. Let

\[
\beta = \{ f \in \text{Imm}(S^1); \Sigma(f) \text{ is hyperbolic}\}.
\]

It follows from [3] that \( \beta \) is open and dense in \( \text{Imm}(S^1) \). Let

\[
\mathscr{U}_1 = \{ f \in \beta; \Sigma(f) \neq S^1 \text{ and } Z(f) \text{ is trivial}\}
\]

\[
\mathscr{U}_2 = \{ f \in \beta; \Sigma(f) = S^1 \}.
\]

By [1, Theorem A], \( \mathscr{U}_2 \) is open in \( \text{Imm}(S^1) \) and if \( f \in \mathscr{U}_2 \) then \( f \) is topologically conjugate to a map \( f_k \) for some \( k > 1 \). Let \( h: S^1 \to S^1 \) be a conjugacy between \( f \) and \( f_k \). Then, for \( g \in Z(f) \) we have that \( h \circ g \circ h^{-1} \) is strictly monotone and

\[
(h \circ g \circ h^{-1}) \circ f_k = h \circ g \circ f \circ h^{-1} = h \circ f \circ g \circ h^{-1} = (h \circ f \circ h^{-1}) \circ (h \circ g \circ h^{-1}) = f_k \circ (h \circ g \circ h^{-1}).
\]
Thus by Lemma 2.1 \( h \circ g \circ h^{-1} = \omega f_m \) where \( m = \text{degree } g \) and \( \omega \) is a \((k - 1)\)th root of unity. Therefore to show the theorem, it is sufficient to show that \( \mathcal{Z}_1 \) is open in \( \beta \) and dense in \( \beta - \mathcal{Z}_2 \).

We shall prove that \( \mathcal{Z}_1 \) is dense in \( \beta - \mathcal{Z}_2 \) by adapting a technique due to Kopell [2]. Let \( f \in \beta - \mathcal{Z}_2 \). If degree \( f = 1 \) then by [2, Theorem 3], \( f \) can be arbitrarily approximated by some \( \hat{f} \in \mathcal{Z}_1 \), so we assume degree \( f > 1 \). Since \( \Sigma(f) \) is hyperbolic and \( S^1 - \Sigma(f) \) is nonempty, the set of sinks of \( f \) is finite and nonempty. Let \( p_0^0, p_1^0, \ldots, p_m^0 \) be the sinks of \( f \) and let \( (p_i^1, p_i^2) \) be the component of the basin of \( p_i^0 \) containing \( p_i^0 \), \( i = 0, \ldots, m \). Let \( F = f^n \) be the first iterate of \( f \) such that \( F(p_i^k) = p_i^k \), \( k = 0, 1, 2 \) and \( i = 0, \ldots, m \). By making a small perturbation in \( f \), we may assume that \( F'(p_i^k) = F'(p_i^l) \), unless \( \exists s \in N \) satisfying \( f^s(p_i^k) = p_i^l \). By [2, Lemma 5], one can choose a diffeomorphism \( \tilde{F}_0 : [p_0^0, p_0^2] \to [p_0^0, p_0^2] \) arbitrarily close to \( F / [p_0^0, p_0^2] \) such that \( \tilde{F}_0(p_0^0) = F'(p_0^0) \), \( i = 0, 2 \) and \( Z(\tilde{F}_0) = \{ \tilde{F}_0^n, n \in \mathbb{Z} \} \). Let \( f / [p_0^0, p_0^2] = f / [p_0^0, p_0^2] \) and

\[
\hat{f}/[p_0^0, p_0^2] = \varphi \circ \tilde{F}_0
\]

where \( \varphi \) is the inverse of \( f^{n-1} / f / [p_0^0, p_0^2] \). Let \( \tilde{F} = \hat{f}^n \). We will show that \( Z(\hat{f}) \) is trivial.

First we note that since degree \( \hat{f} > 1 \) and \( f \) is monotone, \( p_0^2 \) is either a limit point of \( \Sigma(\hat{f}) \) or an endpoint of some interval \( (p_i^1, p_i^2) \) with \( p_i^0 \neq p_0^0 \). Hence by adjoining these intervals to \( (p_i^0, p_i^2) \) if necessary, we obtain an interval \( I = (p_i^1, p_i^2) \) such that \( (p_i^1, p_i^2) \subseteq I, \hat{f}^l(I) = I \) for some \( l \in N \), and the endpoints of \( I \) are limit points of \( \Sigma(\hat{f}) \). It follows from the hyperbolicity of \( \Sigma(\hat{f}) \) that for every neighborhood \( U \) of \( \tilde{I} \), \( f^j(U) = S^1 \) for some \( j \in N \).

Now we claim that \( g \in Z(\hat{f}) \) is completely determined by \( g(p_0^0) \) and \( g'(p_0^0) \). In fact, let \( g_1, g_2 \in Z(\hat{f}) \), \( g_1(p_0^0) = g_2(p_0^0) \), and \( g_1'(p_0^0) = g_2'(p_0^0) \). Since the set of sinks of \( \hat{f} \) (sources of \( \hat{f} \)) and the basins of the sinks are \( g \)-invariant, \( g_1(I) = g_2(I) \neq S^1 \). Hence we can choose neighborhoods \( U_1 \) and \( U \) of \( \tilde{I} \) with \( U_1 \subset U \) and such that \( \tilde{F} / U \) and \( g_i / U \) are injective and

\[
(g_2 / U)^{-1} \circ g_1 \circ \tilde{F}(x) = \tilde{F} \circ (g_2 / U)^{-1} \circ g_1(x) \quad \text{for all } x \in U_1.
\]

Since \( (g_2 / U)^{-1} \circ g_1 \circ (g_2 / U)^{-1} \circ g_1'(p_0^0) = 1 \), we have by [2, Lemma 1(b)] that

\[
g_1(x) = g_2(x) \quad \text{for all } x \in U_1.
\]

Therefore, using the facts that \( \tilde{F}^l(U_1) = S^1 \) for some \( l \in N \), and \( \tilde{F}^l \circ g_i = g_i \circ \tilde{F}^l \), \( i = 1, 2 \), we conclude that \( g_1 = g_2 \) and the claim is proved.
Now let $g \in Z(\tilde{f})$. It is easily checked that $\tilde{F}'(p_0^0) = \tilde{F}'(g(p_0^0))$. This together with the fact that $g(p_0^0) = p_i^0$ for some $0 \leq i \leq m$, implies $g(p_0^0) = \tilde{f}^k(p_0^0)$ for some $1 \leq k \leq n$. Since $\tilde{F}(p_0^0) = p_0^0$, $g(p_0^0) = \tilde{f}^k \circ \tilde{F}^j(p_0^0)$ for all $j \in N$. Hence, to show that $g$ is a power of $\tilde{f}$, it is sufficient to show that $g'(p_0^0) = (\tilde{f}^k)'(p_0^0)(\tilde{F}^j)'(p_0^0)$ for some $j \in N$. But, if $\varphi_1$ is the inverse of $\tilde{f}^k/[p_0^0, p_0^2]$ then $\varphi_1 \circ g/[p_0^0, p_0^2] \in Z(\tilde{F}_0)$, so $(\varphi_1 \circ g)'(p_0^0) = (\tilde{F}^j)'(p_0^0)$ for some $j \in N$. Hence we are done.

The proof that $\mathcal{Z}_1$ is open in $\beta$ uses similar arguments and the Lemma 4 of [2].

References


Departamento de Matemática, Universidade Federal de São Carlos, 13560, São Carlos-S.P., Brazil