LOWER BOUNDS FOR THE EXTRINSIC TOTAL CURVATURES OF A SPACE-LIKE CODIMENSION 2 SURFACE IN MINKOWSKI SPACE

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Abstract. There are three invariant curvature functions defined on any smooth space-like 2-surfaces in four-dimensional Minkowski space. (If the surface lies in a Euclidean hyperplane then the functions agree with $H^2$, $K^2$, and $(H^2 - K^2)^2$.) For each of these functions we show that there exists a space-like immersion of any oriented compact (or noncompact complete) surface with associated total curvature arbitrarily small.

A two dimensional oriented surface $M$ in four dimensional Minkowski space which inherits a positive definite metric is called space-like. Such a surface carries three functions $H$, $K$, and $U$ that are determined by the first and second fundamental forms. (If the surface lies in a Euclidean hyperplane then these functions reduce to $H^2$, $K^2$, and $(H^2 - K^2)$ respectively.) It is natural to ask for the infimum of the respective total curvatures $\int_M |H| \, dA$, $\int_M |K|^{1/2} \, dA$, and $\int_M U^{1/2} \, dA$ over the set of embedded compact space-like surfaces. In this paper we show that there exists an embedded smooth compact orientable surface $M_g$ of genus $g \geq 0$ with $\int_{M_g} |H| \, dA$ arbitrarily small (Theorem 5). In the process we relate $H$ to the split mean curvatures in the manner of reference [2]. We then show that there exist compact (and complete noncompact) embedded orientable surfaces of arbitrary genus with either $K$ or $U$ identically zero, (Theorems 6 and 7). The author would like to thank P. Ehrlich, R. Howard, T. K. Milnor and R. Penrose for inspiration.

Throughout this paper $M^4$ will denote Minkowski space, the real four dimensional vector space equipped with bilinear form $(\ , \ )$ of type $(3, 1)$, (i.e., the normal form has 3 plus signs and 1 minus sign). We will assume that $M^4$ is oriented and time oriented (i.e., a 4-volume form $dV$, and future time-like vector field Fut, have been chosen.) We will write $LC = \{v \in M^4 | (v , v) = 0\}$ to denote the light cone in Minkowski space. A space-like surface in $M^4$ is
a smooth immersion of a smooth compact oriented 2-manifold \( k: M \to \mathbb{M}^4 \), such that the induced metric \( k^*\langle \ , \ \rangle \) is positive definite. It follows that the normal bundle \( M^\perp \to M \) is a trivial rank two bundle with fibres carrying a type \((1, 1)\) metric. Now choose an orthogonal splitting of Minkowski space \( \mathbb{M}^4 = E^3 + E^- \) with projections \( t: E^3 + E^- \to E^- \) and \( \pi_E: E^3 + E^- \to E^3 \) and let \( U^i, \ i = F, P \) denote the two unique sections of \( M^\perp \) which satisfy the following:

\[(a) \quad \langle U^i_s, U^i_s \rangle = 0, \quad i = F, P,\]
\[(b) \quad t_s(U^i_s(x)) \text{ is of unit length for all } x \in M, \quad i = F, P,\]
\[(c) \quad \langle U^F_s, \text{FUT} \rangle < 0,\]
\[(d) \quad dV(-, -, U^P_s, U^F_s) \equiv dA(-, -),\]

(Here \( dA \) is the induced area form on \( M \) and equivalence is in the sense of orientations. The subscript \( s \) refers to the choice of splitting.)

We also have the splitting dependent function \( \frac{1}{\sqrt{2}} (U^F_s, U^P_s): M \to \mathbb{R} \), which we denote by \( \text{TILT}_s \), and which satisfies:

\[(a) \quad 0 < \text{TILT}_s \leq 1\]
\[(b) \quad \text{TILT}_s \text{ is constant iff } \text{TILT}_s = 1 \text{ and } M \text{ lies in a fiber of } t: E^3 + E^- \to E^- .\]

See Lemma 2 in [1].

Now for any choice of splitting \( \mathbb{M}^4 = E^3 + E^- \) we get two split second fundamental forms \( \Pi^i_s: TM \times TM \to \mathbb{R}, \ i = F, P \) defined by \( \Pi^i_s(X, Y) = \langle \nabla_X U^i_s, Y \rangle \) where \( X, Y \) are vector fields on \( M \) and \( \nabla \) is the ambient connection. We define the split curvatures by \( K_s^i = \frac{\text{DETI'}_{\Pi^i_s}}{\text{DETI}}, \ i = F, P \) and split mean curvatures by \( H^i_s = \frac{1}{2} \frac{\text{TRI'}_{\Pi^i_s}}{\text{DETI}}, \ i = F, P \) where \( I \) is the first fundamental form on \( M \).

For any normal vector \( u_x \in M_x \), we have an endomorphism of \( T_x M \) defined by \( v_x \to (\text{tangential projection of } \nabla_x u \text{ at } x) \). There are unique global sections \( R^F, R^P \) of \( M \) that satisfy \( \langle R^F, R^P \rangle = \sqrt{2} \) and \( a, c, d \) of (E1) above. For such a pair we have the \textit{quadratic mean curvature} \( H = \frac{1}{4} \text{TR}(\nabla R^F) \cdot \text{TR}(\nabla R^P) \) and the \textit{quartic curvature} \( K = \text{DETI}(\nabla R^F) \cdot \text{DETI}(\nabla R^P) \). Observe that if we fix a splitting \( \mathbb{M}^4 = E^3 + E^- \), then

\[ H = \frac{H^F_s H^P_s}{\text{TILT}_s}, \]
\[ K = \frac{K^F_s K^P_s}{(\text{TILT}_s)^2}, \]

From these splitting dependent descriptions of \( H \) and \( K \) it is apparent that

\[ U = \frac{(K^F_s - (H^F_s)^2)(K^P_s - (H^P_s)^2)}{(\text{TILT}_s)^2} \geq 0 \]
is also an invariant, which we will call the *umbilic curvature*. All three functions $H$, $K$, $U : M \to \mathbb{R}$ are congruence invariants for $(M, k)$.

**Proposition 1.** Given $(M, k)$, then for all splittings $M^4 = E^3 + E^-$,

$$
\int_M |H| dA \leq \left[ \sup_{\text{TILT}_s} \frac{1}{2} \right] \int_M (H^F)^2 + (H^P)^2 dA.
$$

Further if equality holds for some splitting then $(M, k)$ is an immersion into a space-like hyperplane $E^3 \subset M^4$. (The supremum is taken over all splittings $M^4 = E^3 + E^-$).

**Proof.** Choose a splitting. Then (E3) implies that

$$2|H| \leq \frac{1}{\text{TILT}_s} \left[ (H^F)^2 + (H^P)^2 \right],$$

from which follows the inequality

$$2 \int_M |H| dA \leq \int_M \frac{(H^F)^2 + (H^P)^2}{\text{TILT}_s} dA \leq \left[ \sup_{\text{TILT}_s} \frac{1}{\text{TILT}_s} \right] \int_M (H^F)^2 + (H^P)^2 dA.$$ 

Since equality implies (TILT$_s$) is constant (E2b) implies our claim.

**Theorem 2.** Given $(M, k)$ then for all splittings $M^4 = E^3 + E^-$,

$$8\pi \leq \frac{1}{\text{TILT}_s} \int_M (H^F)^2 + (H^P)^2 dA.$$ 

Further if equality holds for some splitting then $M = S^2$ and $k$ embeds $S^2$ as a round sphere in a space-like hyperplane $E^3 \subset M^4$.

**Proof.** Recall that for any splitting we have:

$$K^i_s \leq (H^i_s)^2 \quad i = F, P$$

with equality at $x \in M$ if and only if $x$ is an umbilic point for $\Pi_s^i$, $i = F, P$. Now with (E7) of [1] we have that

$$8\pi \leq \int_{\{K^F_s > 0\}} K^F_s dA + \int_{\{K^P_s > 0\}} K^P_s dA$$

$$\leq \int_{\{K^F_s > 0\}} (H^F_s)^2 dA + \int_{\{K^P_s > 0\}} (H^P_s)^2 dA$$

(5)

$$\leq \int_M (H^F_s)^2 + (H^P_s)^2 dA$$

Finally if we have equality, then TILT$_s = 1$, hence $(M, k)$ is an immersion into an $E^3$ factor of the splitting with $E^3$-mean curvature given by $H^F_s = H^P_s$ and Gaussian curvature given by $K^F_s = K^P_s$. Thus $(M, k)$ is a totally umbilic surface in a $E^3 \subset M^4$. 

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Next we examine how the split mean curvatures behave individually.

**Theorem 3.**

(a) Given \((M, k)\) with \(M = S^2\), then for all splittings \(M^4 = E^3 + E^-\)

\[
4\pi \leq \int_{S^2}(H^i_f)^2 dA \quad i = F, P
\]

Further equality for \(i = F\) or \(P\) implies that \(k\) embeds \(S^2\) in a light cone \(LC \subset M^4\). Equality in both \(i = F\) and \(P\) implies that \((M, k)\) is congruent to a round \(S^2\) in a space-like hypersurface \(E^3 \subset M^4\).

(b) Given \(\epsilon > 0\) and an integer \(g > 0\), then there exists an orientable compact space-like embedded surface of genus \(g\) and a splitting \(M^5 = E^3 + E^-\) such that for \(i = F\) or \(P\)

\[
0 \leq \int_{M_g}(H^i_f)^2 dA \leq \epsilon.
\]

(Note that because of (E5) above this holds for only one of the split mean curvatures.)

**Proof.** (a) Referring to Theorem 1 of [1] we have that

\[
4\pi \leq \int_{\{K^i_f > 0\}} K^i_f dA, \quad i = F, P
\]

where the integration is over the subset of \(M\) where \(K^i_f > 0\). As a consequence we have that for \(i = F, P\),

\[
4\pi \leq \int_{\{K^i_f > 0\}} K^i_f dA \\
\leq \int_{\{H^i_f > 0\}} (H^i_f)^2 dA \leq \int_{S^2}(H^i_f)^2 dA.
\]

Now suppose there is equality for \(i = F\) or \(P\). From (E6) of [2] it follows that **as conformal structures** on \(M\), \(\Pi^i_f\) for \((M, k)\) agrees with \(\overline{\Pi}\) for the light-like hypersurface \(LL^i(M, k)\). Thus if we choose any (affine) space-like hyperplane \(E^3 \subset M^4\), at any immersed point in \(\{E^3 \cap LL^i(M, k)\}\) this intersection is locally an umbilic surface in \(E^3\), and hence is a piece of a round sphere or of a plane. Since we are assuming \(M\) is compact, we will have our result if we show that there exists an \(E^3 \subset M^4\) with intersection \(\{E^3 \cap LL^i(M, k)\}\) consisting of only immersive points (i.e., \(LL^i(M, k)\) is an icon). To prove this choose a hyperplane \(\tilde{E}^3 \subset M^4\) so that the intersection \(\{\tilde{E}^3 \cap LL^i(M, k)\}\) contains an immersive point. Recall that \(LL^i(M, k) \xrightarrow{\pi} M\) is fibered by null lines and each fiber intersects the chosen \(\tilde{E}^3\) in a unique point. (See [1] for details.) Since \(k: M \to M^4\) has image, which is a section of this fibering of \(LL^i(M, k)\),
we now have smooth maps
\[ S^2 = M \xrightarrow{k} \mathbb{L}^i(M, k) \subset M^4 \]
\[ \tilde{E}^3 \subset M^4 \]
where \( p \circ k(x) \) is intersection point of the \( \pi^i \)-fibered through \( k(x) \) and the chosen \( \tilde{E}^3 \). We know the immersive points of \( p \circ k \) are open in \( S^2 = M \), we need only show that they are closed. Let \( x_n \to x \) be a convergent sequence of points in \( S^2 \) with \( p \circ k \) immersive at \( x_n \). Now consider the \( E^3 \) parallel to the above chosen \( \tilde{E}^3 \) that contains \( k(x) \). We know that \( \{E^3 \cap \mathbb{L}^i(M, k)\} = V \) is a piece of a sphere or a plane. Hence \( (\pi^i)^{-1}(\pi^i(V)) \) is a subset of a light cone or a null 3-plane. But for any affine \( E^3 \subset M^4 \), the intersection of \( E^3 \) with a light cone or a null 3-plane consists entirely of immersive points or entirely of nonimmersive points. Thus the intersection of \( (\pi^i)^{-1}(\pi^i(V)) \cap \mathbb{L}^i(M, k) \) with the chosen \( \tilde{E}^3 \) contains \( p \circ k(x_n) \) (for \( n \) sufficiently large). These are immersive points for \( p \circ k \) and hence immersive points for the intersection with \( \tilde{E}^3 \). Since \( p \circ k(x) \) lies in this intersection it is also an immersive point. We conclude \( \mathbb{L}^i(M, k) \) is congruent to a light cone. For the last claim in Theorem 3 (a) we now know that both \( \mathbb{L}^i(M, k) \) are congruent to light cones. Hence \( (M, k) \) is space-like, compact, and lies in their intersection. It must be a round \( S^2 \subset E^3 \subset M^4 \).

(b) The construction of \( (M_n, k) \) is but a slight modification of the construction presented in Appendix 2 of [1]. Consider a fixed splitting \( M^4 = E^3 + E^- \) with projection \( t: E^3 + E^- \to E^- \) and the torus of revolution \((T^2, k)\) obtained by rotating an “embedded Figure 8” about a \( M^2 \) in \( M^4 \) so that a linear segment in the figure 8 generates a cylinder in \( E^3 \); and the \( S^2 \) valued Gauss maps are not surjective. For this torus and splitting \( H^i_s = \frac{1}{2} \frac{\text{TR}(\nabla U^U_i)}{\text{DET} I} \), \( i = F, P \). Now because the Gauss maps are not surjective, there exist isometries \( b_\theta: M^4 \to M^4 \) such that for all positive integers \( n > 0 \), there exist \( \theta_n \), so that \( n < \inf |t_s(b_{\theta_n} \circ U^F_s)| \) where the infimum is over points of the torus. Since \( b_{\theta_n} \) is an isometry we have that \((T^2, b_{\theta_n} \circ k)\) has \( U^F_{\theta_n} = \lambda(b_{\theta_n} \circ U^F_s) \) where \( \lambda > \frac{1}{n} \) is a function on \( T^2 \) and \( U^F_{\theta_n} \) is the vector field satisfying (E1) above relative to the fixed splitting. We have that,
\[
H^F_{\theta_n} = \frac{\text{TR}(\nabla U^F_{\theta_n}, -)}{\text{DET} I} < \frac{1}{n} \frac{\text{TR}(\nabla U^F_s, -)}{\text{DET} I} < \frac{H^F_s}{n}
\]
and hence a torus with \( \int_{T^2} (H^F_s)^2 \, dA \) arbitrarily small. Note that the key step of constructing \( b_{\theta_n} \) requires only that the Gauss map be nonsurjective. Thus to construct higher genus examples take two copies of \((T^2, k)\) and shrink one
copy so that the two copies do not intersect, and the cylinders in $E^3$ mentioned at the outset are concentric. We can connect these two cylinders in the $E^3$ with catanoidal necks in such a way that removing the $E^3$-minimal parts of the necks results in two punctured tori with $M^3$-Gauss maps nonsurjective. If we denote this connected sum by $(M_g, k)$ then for any splitting $M^4 = E^3 + E^{-}$, both the $H^i$, $i = F, P$ will vanish on the $E^3$-minimal part of the necks. Thus we may repeat the above argument on $(M_g - \{H^F = 0\}, b_{\theta_g} \circ k)$ and we have surfaces which satisfy our claim for $i = F$. The same argument adapts to construct an example for $i = P$.

By a space-like graph in $M^4$ we mean a smooth space-like hypersurface which, relative to some (hence any) splitting of $M^4 = E^3 + E^{-}$ is globally the graph of a function $f: E^3 \to E^{-}$.

Theorem 4. If $(M, k)$ is an embedding into a space-like graph, then for all splittings of $M^4 = E^3 + E^{-}$

$$4\pi \leq \int_M (H^F)^2 dA,$$

Further equality for $i = F$ or $P$ implies that $M = S^2$ so that Theorem 3, Part (a) applies.

Proof. Recall from the proof of Theorem 9 in [1] that for $(M, k)$ a surface embedded in a graph we must have

$$4\pi \leq \int_{\{K^i > 0\}} K^F dA, \quad i = F, P.$$

(This is a restatement of the first line in the proof of Theorem 9.) Now for the case of equality we must have by Theorem 1 of [1] that $M = S^2$.

Theorem 5. Given $\varepsilon > 0$ and an integer $g \geq 0$, there exists an orientable compact embedded space-like surface of genus $g$ with,

$$0 < \int_{M_g} |H| dA < \varepsilon.$$

Proof. We first construct an example with $g = 0$. Take two round spheres $S^2_1$ and $S^2_2$ in $E^3 \subset M^4$ and smoothly connect them with a “Hopf neck” $HN \subset E^3$ of small total squared $E^3$-mean curvature to yield an embedded sphere, $S^2 \# S^2$. This can be done as in Figure 1 so that the transition region between round sphere and Hopf neck consist of a pair of planar parallel anuli in $E^3$. (For details concerning the Hopf neck see Lemma 6b in [3].) Let $P_1$ and $P_2$ be the parallel planes in which these annuli lie. Now extend the $S^2 \# S^2 \subset E^3$ to a Light-like hypersurface $LL(S^2 \# S^2)$ in $M^4$ so that the vertex of round spherical parts lie in the “future” of the $E^3$ fixed at the outset. Similarly extend the round spheres $S^2_1$ and $S^2_2$ to light cones $LC_1$ and $LC_2$ so that their vertices are in the future of the $E^3$. Next extend the 2-planes $P_1$ and $P_2$ to null 3-planes $N^3_1$. 

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and \( N_2 \) so that \( N_2 \) intersects the future of \( \text{LC}_\alpha \) in a bounded set. Let \( \overline{N}_3^F \), \( \alpha = 1, 2 \) denote the closure of the subsets of \( N_3^\alpha \) that lie in the future of \( E^3 \) and let \( D_\alpha = \text{LL}(S^2 \# S^2) \cap \overline{N}_3^F \), \( \alpha = 1, 2 \). These \( D_\alpha \) are embedded space-like 2-disks. Thus by construction \( \{D_1 \cup HN \cup D_2\} \) is an embedded space-like two-sphere that intersects the fixed \( E^3 \) in the Hopf neck along with its pair of planar annuli. Now since each \( D_\alpha \) lies in a null 3-plane \( H \) vanishes on both \( D_\alpha \). Thus the support of \( H \) lies entirely in the Hopf neck where it agrees with the \( E^3 \)-mean curvature squared. We have our sphere. To construct higher genus examples we need only connect the parallel annuli with additional Hopf necks with small total squared \( E^3 \)-mean curvature and we are finished.

We now construct space-like surfaces with \( K \) and \( U \) identically zero. We remark that the existence of \( K \)-flat surfaces is relevant to the conjecture discussed in Appendix 3 of [1] in that these surfaces indicate that there need not exist a point \( x \in M \) where both split curvatures are positive \( K(x) > 0 \) \( i = F, P \).

**Theorem 6.** Given \( g \geq 0 \), there exist orientable embedded compact (or noncompact complete) space-like surfaces of genus \( g \) with \( K = 0 \).

**Proof.** Consider any two \( E^3 \)-flat surfaces \( \bar{k}^i : \text{FLT}^i \to E^3 \subset M^4 \), \( i = F, P \) that intersect transversely on an immersed circle \( \bar{S}^1 \to \text{FLT}^F \cap \text{FLT}^P \). (The reason for indexing these surfaces by \( F \) and \( P \) will be apparent below.) Next choose \( E^3 \)-normals for each surface \( n^i : \text{FLT}^i \to S^2 \subset E^3 \), \( i = F, P \). As in the proof of Proposition 6 of [2] we use these \( E^3 \)-normals to construct two light-like hypersurfaces \( j^i : \text{LL}^i(\text{FLT}^i) \to M^4 \), \( i = F, P \). Now two null 3-planes in \( M^4 \) intersect in an \( E^2 \subset M^4 \) or coincide. Since \( \text{FLT}^i \) intersect transversely in \( E^3 \) we have that \( \text{LL}^F(\text{FLT}^F) \) intersects \( \text{LL}^P(\text{FLT}^P) \) transversely in \( M^4 \). Thus near
the intersecting circle we have that $LL^F(FLT^F) \cap LL^P(FLT^P)$ is an immersed space-like cylinder $S^1 \times (-\delta, \delta)$. (It will be embedded if the $FLT^i \subset E^3$, $i = F, P$ are embedded.) It follows that this small cylinder is transverse to the fibers of both $\pi^i: LL'(FLT^i) \to FLT^i$. We may view the original $FLT^i, i = F, P$ as “zero sections” of $\pi^i$ and observe that the intersection cylinder is transverse to both of these zero sections in $LL'(FLT^i)$. Thus we may extend the $S^1 \times (-\delta, \delta)$ so that one boundary edge $S^1 \times \{-\delta\}$ extends so as to agree with the zero section of $\pi^F$ outside of a compact set and the other boundary edge $S^1 \times \{\delta\}$ extends so as to agree with the zero section of $\pi^P$ outside a compact set. We may construct this extension so that it is everywhere transverse to (at least) one of the fibrations $\pi^i$, $i = F, P$, thence this extended intersection $(FLT^F \# FLT^P, k)$ is a space-like surface in $M^4$.

In Figure 3 we illustrate our construction and include the image of the extended intersection under the projection $\pi_E: M^4 \to E^3$. Now (E10) of [2] tells us that any section of either $LL'(FLT^i), i = F, P$ must have $K$ identically zero. Using the construction on flat surfaces as represented by Figures 2 and 3 we can build spheres or tori and take connected sums so that the resulting surfaces have $K$ identically zero. We have our surfaces.

Theorem 7. Given $g \geq 0$ there exists an orientable embedded compact (or non-compact complete) space-like surface of genus $g$ with $U = 0$.

Proof. The proof parallels that of Theorem 6 with the modification that flat surfaces in $E^3$ are replaced by round spheres $S^2_i, i = F, P$ in $E^3$. The key point is again (E10) of [2], which in this context tells us that any section of either $LL'(S^2_i), i = F, P$ must have $U$ identically zero.

It should be apparent to the reader that these subumbilic immersions are highly nonrigid. So that $F(M, k) = \int_M U^{1/2} \, dA$ defines a geometric functional

![Figure 2](https://www.ams.org/journal-terms-of-use)
on $\operatorname{Immer}^\infty(M, \mathbb{M}^4)$ with highly degenerate second variation at the minima constructed above.

**BIBLIOGRAPHY**


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