LOWER BOUNDS FOR THE EXTRINSIC TOTAL CURVATURES
OF A SPACE-LIKE CODIMENSION 2 SURFACE
IN MINKOWSKI SPACE

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Abstract. There are three invariant curvature functions defined on any smooth
space-like 2-surfaces in four-dimensional Minkowski space. (If the surface
lies in a Euclidean hyperplane then the functions agree with \( H^2, K^2, \) and
\( (H^2 - K^2)^2 \). For each of these functions we show that there exists a space-like
immersion of any oriented compact (or noncompact complete) surface with
associated total curvature arbitrarily small.

A two dimensional oriented surface \( M \) in four dimensional Minkowski space
which inherits a positive definite metric is called space-like. Such a surface
carries three functions \( H, K, \) and \( U \) that are determined by the first and
second fundamental forms. (If the surface lies in a Euclidean hyperplane then
these functions reduce to \( H^2, K^2, \) and \( (H^2 - K^2)^2 \) respectively.) It is natural to
ask for the infimum of the respective total curvatures \( \int_M |H| dA, \int_M |K|^{1/2} dA \)
and \( \int_M U^{1/2} dA \) over the set of embedded compact space-like surfaces. In this
paper we show that there exists an embedded smooth compact orientable surface
\( M_g \) of genus \( g \geq 0 \) with \( \int_{M_g} |H| dA \) arbitrarily small (Theorem 5). In the
process we relate \( H \) to the split mean curvatures in the manner of reference [2].
We then show that there exist compact (and complete noncompact) embedded
orientable surfaces of arbitrary genus with either \( K \) or \( U \) identically zero,
(Theorems 6 and 7). The author would like to thank P. Ehrlich, R. Howard, T.
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Throughout this paper \( \mathbb{M}^4 \) will denote Minkowski space, the real four di-

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a smooth immersion of a smooth compact oriented 2-manifold \( k: M \to M^4 \), such that the induced metric \( k^* \langle , \rangle \) is positive definite. It follows that the normal bundle \( M^\perp \to M \) is a trivial rank two bundle with fibres carrying a type \((1,1)\) metric. Now choose an orthogonal splitting of Minkowski space \( M^4 = E^3 + E^- \) with projections \( t: E^3 + E^- \to E^- \) and \( \pi_E: E^3 + E^- \to E^3 \) and let \( U_i^j, \ i = F, P \) denote the two unique sections of \( M^\perp \) which satisfy the following:

\[
\begin{align*}
(a) & \quad \langle U_i^j, U_i^j \rangle = 0, \quad i = F, P, \\
(b) & \quad t_s(U_i^j(x)) \text{ is of unit length for all } x \in M, \quad i = F, P, \\
(c) & \quad \langle U_i^j, \ FUT \rangle < 0, \\
(d) & \quad dV(-, -) = dA(-, -),
\end{align*}
\]
(Here \( dA \) is the induced area form on \( M \) and equivalence is in the sense of orientations. The subscript \( s \) refers to the choice of splitting.)

We also have the splitting dependent function \( \frac{1}{\sqrt{2}} \langle U_s^F, U_s^P \rangle: M \to R \), which we denote by \( \text{TILT}_s \), and which satisfies:

\[
\begin{align*}
(a) & \quad 0 < \text{TILT}_s \leq 1 \\
(b) & \quad \text{TILT}_s \text{ is constant iff } \text{TILT}_s = 1 \text{ and } M \text{ lies in a fiber of } t: E^3 + E^- \to E^-.
\end{align*}
\]

See Lemma 2 in [1].

Now for any choice of splitting \( M^4 = E^3 + E^- \) we get two split second fundamental forms \( \Pi_i^j: TM \times TM \to R, \ i = F, P \) defined by \( \Pi_i^j(X, Y) = \langle \nabla_X U_i^j, Y \rangle \) where \( X, Y \) are vector fields on \( M \) and \( \nabla \) is the ambient connection. We define the split curvatures by \( K_i^I = \frac{\text{DET} \Pi_i^I}{\text{DET} I} \), \( i = F, P \) and split mean curvatures by \( H_s^I = \frac{1}{2} \text{TR} \Pi_s^I \), \( i = F, P \) where \( I \) is the first fundamental form on \( M \).

For any normal vector \( u_x \in M_x \), we have an endomorphism of \( T_x M \) defined by \( v_x \to (\text{tangential projection of } \nabla_v u \text{ at } x) \). There are unique global sections \( R^F, R^P \) of \( M \) that satisfy \( \langle R^F, R^P \rangle = \sqrt{2} \) and \( a, c, d \) of (E1) above. For such a pair we have the \textit{quadric mean curvature} \( H = \frac{1}{4} \text{TR}(\nabla R^F) \cdot \text{TR}(\nabla R^P) \) and the \textit{quartic curvature} \( K = \text{DET}(\nabla R^F) \cdot \text{DET}(\nabla R^P) \). Observe that if we fix a splitting \( M^4 = E^3 + E^- \), then

\[
\begin{align*}
H & = \frac{H_s^F H_s^P}{\text{TILT}_s^2}, \quad \text{and} \\
K & = \frac{K_s^F K_s^P}{(\text{TILT}_s)^2}
\end{align*}
\]

From these splitting dependent descriptions of \( H \) and \( K \) it is apparent that

\[
U = \frac{(K_s^F - (H_s^F)^2)(K_s^P - (H_s^P)^2)}{(\text{TILT}_s)^2} \geq 0
\]
is also an invariant, which we will call the umbilic curvature. All three functions \( H, K, U: M \to \mathbb{R} \) are congruence invariants for \((M, k)\).

**Proposition 1.** Given \((M, k)\), then for all splittings \( M^4 = E^3 + E^- \),

\[
\int_M |H| \, dA \leq \left[ \sup_{\text{TILT}_s} \frac{1}{2} \right] \int_M (H_s^F)^2 + (H_s^P)^2 \, dA.
\]

Further if equality holds for some splitting then \((M, k)\) is an immersion into a space-like hyperplane \( E^3 \subset M^4 \). (The supremum is taken over all splittings \( M^4 = E^3 + E^- \)).

**Proof.** Choose a splitting. Then (E3) implies that

\[
2|H| \leq \frac{1}{\text{TILT}_s} \left[ (H_s^F)^2 + (H_s^P)^2 \right],
\]

from which follows the inequality

\[
2 \int_M |H| \, dA \leq \int_M \frac{(H_s^F)^2 + (H_s^P)^2}{\text{TILT}_s} \, dA \leq \left[ \sup_{\text{TILT}_s} \frac{1}{\text{TILT}_s} \right] \int_M (H_s^F)^2 + (H_s^P)^2 \, dA.
\]

Since equality implies \((\text{TILT}_s)\) is constant (E2b) implies our claim.

**Theorem 2.** Given \((M, k)\) then for all splittings \( M^4 = E^3 + E^- \),

\[
8\pi \leq \left[ \sup_{\text{TILT}_s} \frac{1}{\text{TILT}_s} \right] \int_M (H_s^F)^2 + (H_s^P)^2 \, dA.
\]

Further if equality holds for some splitting then \( M = S^2 \) and \( k \) embeds \( S^2 \) as a round sphere in a space-like hyperplane \( E^3 \subset M^4 \).

**Proof.** Recall that for any splitting we have:

\[
K_s^f = (H_s^f)^2 \quad i = F, P
\]

with equality at \( x \in M \) if and only if \( x \) is an umbilic point for \( \Pi_s^i, \ i = F, P \).

Now with (E7) of [1] we have that

\[
8\pi \leq \int_{\{K_s^f > 0\}} K_s^F \, dA + \int_{\{K_s^p > 0\}} K_s^P \, dA
\]

\[
\leq \int_{\{K_s^f > 0\}} (H_s^F)^2 \, dA + \int_{\{K_s^p > 0\}} (H_s^P)^2 \, dA
\]

\[
\leq \int_M (H_s^F)^2 + (H_s^P)^2 \, dA
\]

\[
\leq \left[ \sup_{\text{TILT}_s} \frac{1}{\text{TILT}_s} \right] \int_M (H_s^F)^2 + (H_s^P)^2 \, dA
\]

Finally if we have equality, then \( \text{TILT}_s = 1 \), hence \((M, k)\) is an immersion into an \( E^3 \) factor of the splitting with \( E^3 \)-mean curvature given by \( H_s^F = H_s^P \) and Gaussian curvature given by \( K_s^F = K_s^P \). Thus \((M, k)\) is a totally umbilic surface in a \( E^3 \subset M^4 \).
Next we examine how the split mean curvatures behave individually.

**Theorem 3.**

(a) Given \((M, k)\) with \(M = S^2\), then for all splittings \(M^4 = E^3 + E^-\)

\[
4\pi \leq \int_{S^2} (H^i_j)^2 \, dA \quad i = F, P
\]

Further equality for \(i = F\) or \(P\) implies that \(k\) embeds \(S^2\) in a light cone \(LC \subset M^4\). Equality in both \(i = F\) and \(P\) implies that \((M, k)\) is congruent to a round \(S^2\) in a space-like hypersurface \(E^3 \subset M^4\).

(b) Given \(\varepsilon > 0\) and an integer \(g > 0\), then there exists an orientable compact space-like embedded surface of genus \(g\) and a splitting \(M^5 = E^3 + E^-\) such that for \(i = F\) or \(P\)

\[
0 \leq \int_{M^g} (H^i_j)^2 \, dA \leq \varepsilon.
\]

(Note that because of (E5) above this holds for only one of the split mean curvatures.)

**Proof.** (a) Referring to Theorem 1 of [1] we have that

\[
4\pi \leq \int_{\{K^i_j > 0\}} K^i_j \, dA , \quad i = F, P
\]

where the integration is over the subset of \(M\) where \(K^i_j > 0\). As a consequence we have that for \(i = F, P\),

\[
4\pi \leq \int_{\{K^i_j > 0\}} K^i_j \, dA
\]

\[
\leq \int_{\{K^i_j > 0\}} (H^i_j)^2 \, dA \leq \int_{S^2} (H^i_j)^2 \, dA.
\]

Now suppose there is equality for \(i = F\) or \(P\). From (E6) of [2] it follows that as conformal structures on \(M\), \(\Pi^j_i\) for \((M, k)\) agrees with \(\overline{\Pi}\) for the light-like hypersurface \(LL^i(M, k)\). Thus if we choose any (affine) space-like hyperplane \(E^3 \subset M^4\), at any immersed point in \(\{E^3 \cap LL^i(M, k)\}\) this intersection is locally an umbilic surface in \(E^3\), and hence is a piece of a round sphere or of a plane. Since we are assuming \(M\) is compact, we will have our result if we show that there exists an \(E^3 \subset M^4\) with intersection \(\{E^3 \cap LL^i(M, k)\}\) consisting of only immersive points (i.e., \(LL^i(M, k)\) is an icon). To prove this choose a hyperplane \(\widehat{E}^3 \subset M^4\) so that the intersection \(\{\widehat{E}^3 \cap LL^i(M, k)\}\) contains an immersive point. Recall that \(LL^i(M, k) \xrightarrow{\pi} M\) is fibered by null lines and each fiber intersects the chosen \(\widehat{E}^3\) in a unique point. (See [1] for details.) Since \(k : M \to M^4\) has image, which is a section of this fibering of \(LL^i(M, k)\),
we now have smooth maps

\[ S^2 = M \xrightarrow{k} \text{LL}^i(M, k) \subset M^4 \]

\[ \tilde{E}^3 \subset M^4 \]

where \( p \circ k(x) \) is intersection point of the \( \pi^i \)-fibered through \( k(x) \) and the chosen \( \tilde{E}^3 \). We know the immersive points of \( p \circ k \) are open in \( S^2 = M \), we need only show that they are closed. Let \( x_n \rightarrow x \) be a convergent sequence of points in \( S^2 \) with \( p \circ k \) immersive at \( x_n \). Now consider the \( E^3 \) parallel to the above chosen \( \tilde{E}^3 \) that contains \( k(x) \). We know that \( \{E^3 \cap \text{LL}^i(M, k)\} = V \) is a piece of a sphere or a plane. Hence \( (\pi^i)^{-1}(\pi^i(V)) \) is a subset of a light cone or a null 3-plane. But for any affine \( E^3 \subset M^4 \), the intersection of \( E^3 \) with a light cone or a null 3-plane consists entirely of immersive points or entirely of nonimmersive points. Thus the intersection of \( (\pi^i)^{-1}(\pi^i(V)) \subset \text{LL}^i(M, k) \) with the chosen \( \tilde{E}^3 \) contains \( p \circ k(x_n) \) (for \( n \) sufficiently large). These are immersive points for \( p \circ k \) and hence immersive points for the intersection with \( \tilde{E}^3 \). Since \( p \circ k(x) \) lies in this intersection it is also an immersive point. We conclude \( \text{LL}^i(M, k) \) is congruent to a light cone. For the last claim in Theorem 3 (a) we now know that both \( \text{LL}^i(M, k) \) are congruent to light cones. Hence \((M, k)\) is space-like, compact, and lies in their intersection. It must be a round \( S^2 \subset \tilde{E}^3 \subset M^4 \).

(b) The construction of \((M_n, k)\) is but a slight modification of the construction presented in Appendix 2 of [1]. Consider a fixed splitting \( M^4 = E^3 + E^- \) with projection \( t: E^3 + E^- \rightarrow E^- \) and the torus of revolution \((T^2, k)\) obtained by rotating an “embedded Figure 8” about a \( M^2 \) in \( M^4 \) so that a linear segment in the figure 8 generates a cylinder in \( E^3 \); and the \( S^2 \) valued Gauss maps are not surjective. For this torus and splitting \( \text{TR}_s = \frac{1}{2} \text{TR}(\nabla U^F_i, -) \)

\[ \frac{1}{\text{det} I} \]

and hence a torus with \( \text{TR}_s \) arbitrary small. Note that the key step of constructing \( b_{\theta_n} \) requires only that the Gauss map be nonsurjective. Thus to construct higher genus examples take two copies of \((T^2, k)\) and shrink one
copy so that the two copies do not intersect, and the cylinders in $E^3$ mentioned at the outset are concentric. We can connect these two cylinders in the $E^3$ with catanoidal necks in such a way that removing the $E^3$-minimal parts of the necks results in two punctured tori with $M^3$-Gauss maps non-surjective. If we denote this connected sum by $(M, k)$ then for any splitting $M^4 = E^3 + E^-$, both the $H^i, i = F, P$ will vanish on the $E^3$-minimal part of the necks. Thus we may repeat the above argument on $(M - \{H^i = 0\}, b_\theta \circ k)$ and we have surfaces which satisfy our claim for $i = F$. The same argument adapts to construct an example for $i = P$.

By a space-like graph in $M^4$ we mean a smooth space-like hypersurface which, relative to some (hence any) splitting of $M^4 = E^3 + E^-$ is globally the graph of a function $f: E^3 \to E^-$. Theorem 4. If $(M, k)$ is an embedding into a space-like graph, then for all splittings of $M^4 = E^3 + E^-$

$$4\pi \leq \int_M (H^i_s)^2 dA, \quad i = F, P.$$ Further equality for $i = F$ or $P$ implies that $M = S^2$ so that Theorem 3, Part (a) applies.

Proof. Recall from the proof of Theorem 9 in [1] that for $(M, k)$ a surface embedded in a graph we must have

$$4\pi \leq \int_{\{(K^i_s > 0)\}} K^i_s dA, \quad i = F, P.$$ (This is a restatement of the first line in the proof of Theorem 9.) Now for the case of equality we must have by Theorem 1 of [1] that $M = S^2$.

Theorem 5. Given $\varepsilon > 0$ and an integer $g \geq 0$, there exists an orientable compact embedded space-like surface of genus $g$ with,

$$0 < \int_{M^3} |H| dA < \varepsilon.$$ Proof. We first construct an example with $g = 0$. Take two round spheres $S^2$ and $S^2_2$ in $E^3 \subset M^4$ and smoothly connect them with a “Hopf neck” $HN \subset E^3$ of small total squared $E^3$-mean curvature to yield an embedded sphere, $S^2 \# S^2$. This can be done as in Figure 1 so that the transition region between round sphere and Hopf neck consist of a pair of planar parallel anuli in $E^3$. (For details concerning the Hopf neck see Lemma 6b in [3].) Let $P_1$ and $P_2$ be the parallel planes in which these annuli lie. Now extend the $S^2 \# S^2 \subset E^3$ to a Light-like hypersurface LL$(S^2 \# S^2)$ in $M^4$ so that the vertex of round spherical parts lie in the “future” of the $E^3$ fixed at the outset. Similarly extend the round spheres $S^2_1$ and $S^2_2$ to light cones $LC_1$ and $LC_2$ so that their vertices are in the future of the $E^3$. Next extend the 2-planes $P_1$ and $P_2$ to null 3-planes $N^3_1$.
and $N_2$ so that $N_2$ intersects the future of $LC_{\alpha}$ in a bounded set. Let $\overline{N}_2^F$, $\alpha = 1, 2$ denote the closure of the subsets of $N_2^F$ that lie in the future of $E^3$ and let $D_\alpha = \text{LL}(S^2 \# S^2) \cap \overline{N}_2^F$, $\alpha = 1, 2$. These $D_\alpha$ are embedded space-like 2-disks. Thus by construction $\{D_1 \cup H N \cup D_2\}$ is an embedded space-like two-sphere that intersects the fixed $E^3$ in the Hopf neck along with its pair of planar annuli. Now since each $D_\alpha$ lies in a null 3-plane $H$ vanishes on both $D_\alpha$. Thus the support of $H$ lies entirely in the Hopf neck where it agrees with the $E^3$-mean curvature squared. We have our sphere. To construct higher genus examples we need only connect the parallel annuli with additional Hopf necks with small total squared $E^3$-mean curvature and we are finished.

We now construct space-like surfaces with $K$ and $U$ identically zero. We remark that the existence of $K$-flat surfaces is relevant to the conjecture discussed in Appendix 3 of [1] in that these surfaces indicate that there need not exist a point $x \in M$ where both split curvatures are positive $K^i(x) > 0$ $i = F, P$.

**Theorem 6.** Given $g \geq 0$, there exist orientable embedded compact (or noncompact complete) space-like surfaces of genus $g$ with $K = 0$.

**Proof.** Consider any two $E^3$-flat surfaces $\tilde{k}^i: \text{FLT}^i \to E^3 \subset M^4$, $i = F, P$ that intersect transversely on an immersed circle $\tilde{S}^1 \to \text{FLT}^F \cap \text{FLT}^P$. (The reason for indexing these surfaces by $F$ and $P$ will be apparent below.) Next choose $E^3$-normals for each surface $n^i: \text{FLT}^i \to S^2 \subset E^3$, $i = F, P$. As in the proof of Proposition 6 of [2] we use these $E^3$-normals to construct two light-like hypersurfaces $j^i: \text{LL}^i(\text{FLT}^i) \to M^4$, $i = F, P$. Now two null 3-planes in $M^4$ intersect in an $E^2 \subset M^4$ or coincide. Since $\text{FLT}^i$ intersect transversely in $E^3$ we have that $\text{LL}^F(\text{FLT}^F)$ intersects $\text{LL}^P(\text{FLT}^P)$ transversely in $M^4$. Thus near
the intersecting circle we have that $LL^F(FLT^F) \cap LL^P(FLT^P)$ is an immersed space-like cylinder $S^1 \times (-\delta, \delta)$. (It will be embedded if the $FLT^i \subset E^3$, $i = F, P$ are embedded.) It follows that this small cylinder is transverse to the fibers of both $\pi^i$: $LL'(FLT^i) \to FLT^i$. We may view the original $FLT^i$, $i = F, P$ as "zero sections" of $\pi^i$ and observe that the intersection cylinder is transverse to both of these zero sections in $LL'(FLT^i)$. Thus we may extend the $S^1 \times (-\delta, \delta)$ so that one boundary edge $S^1 \times \{-\delta\}$ extends so as to agree with the zero section of $\pi^F$ outside of a compact set and the other boundary edge $S^1 \times \{\delta\}$ extends so as to agree with the zero section of $\pi^P$ outside a compact set. We may construct this extension so that it is everywhere transverse to (at least) one of the fibrations $\pi^i$, $i = F, P$, thence this extended intersection $\langle FLT^F \# FLT^P, k \rangle$ is a space-like surface in $M^4$.

In Figure 3 we illustrate our construction and include the image of the extended intersection under the projection $\pi_E: M^4 \to E^3$. Now (E10) of [2] tells us that any section of either $LL'^i(FLT^i)$, $i = F, P$ must have $K$ identically zero. Using the construction on flat surfaces as represented by Figures 2 and 3 we can build spheres or tori and take connected sums so that the resulting surfaces have $K$ identically zero. We have our surfaces.

**Theorem 7.** Given $g \geq 0$ there exists an orientable embedded compact (or non-compact complete) space-like surface of genus $g$ with $U = 0$.

**Proof.** The proof parallels that of Theorem 6 with the modification that flat surfaces in $E^3$ are replaced by round spheres $S^2_i$, $i = F, P$ in $E^3$. The key point is again (E10) of [2], which in this context tells us that any section of either $LL'(S^2_i)$, $i = F, P$ must have $U$ identically zero.

It should be apparent to the reader that these subumbilic immersions are highly nonrigid. So that $F(M, k) = \int_M U^{1/2} dA$ defines a geometric functional.
on $\text{Immer}^{\infty}(M^1, M^4)$ with highly degenerate second variation at the minima constructed above.

**BIBLIOGRAPHY**


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