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THE RELATIVE FORM OF GERSTEN'S CONJECTURE FOR POWER SERIES OVER A COMPLETE DISCRETE VALUATION RING

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ABSTRACT. A relative form of Gersten's Conjecture is established for a ring of formal power series over a complete discrete valuation ring. The main corollaries are that the absolute version of Gersten's Conjecture is valid for such a ring if it is valid for arbitrary discrete valuation rings, and, consequently, that the conjecture is true for such a ring if we use K-theory with finite coefficients of order prime to the characteristic of the residue field.

In [GL], Gillet and Levine established the relative version of Gersten's Conjecture for a regular local ring essentially of finite type and smooth over a discrete valuation ring (DVR). The main consequence is that the absolute version of Gersten's Conjecture is valid for such a ring if it is true for DVRs. These results were a relative analog of Quillen's theorem on the absolute version of the Conjecture for regular local rings essentially of finite type and smooth over a field (from which Quillen then deduced the Conjecture for all regular local rings essentially of finite type over a field [Q, Theorem 7.5.11]).

Quillen also proved the absolute version of the Conjecture for a ring of formal power series over a field [Q, Theorem 5.13] (in other words, by the Cohen Structure Theorem, the complete equicharacteristic case). In this note we show that the analogs of Gillet and Levine's results are valid for a ring of formal power series over a complete DVR. (By the Cohen Structure Theorem, in the unequal characteristic situation this includes the case of a complete unramified regular local ring.) The proof bears the same relation to Quillen's as Gillet and Levine's proof bears to the proof of Quillen's geometric result.

We will adopt the following notation: R will denote a complete DVR, with maximal ideal generated by π, and residue field k. We will put $A = R[[X_1, \dots, X_n]]$.

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Theorem. *Let $t \in A$, $t \neq 0$, be such that A/tA is flat over R (i.e.,* $\operatorname{Spec}(A/tA) \to \operatorname{Spec} A$ *is a "principal effective relative divisor"). Then the exact functors*

$$M^i(A/tA) \to M^i(A)$$
$$MF^i(A/tA) \to MF^i(A)$$

on the categories of finitely generated A-modules, and finitely generated A-modules flat over R, respectively, supported in codimension i, both induce the zero map on K-theory for all $i \geq 0$.

Proof. A standard result shows that A/tA is flat over R if and only if it is torsionfree. Then it is easy to see (using, for instance, the fact that A is a UFD) that the conditions on t are equivalent to the condition $\pi \nmid t$.

Thus $\bar{t}$, the image of t in $k[[X_1, \dots, X_n]]$, is nonzero, so by [B, Lemma 3 of Chapter 7, §3.7], there exist natural numbers $u(i)(1 \leq i \leq n-1)$ such that the k-automorphism of $k[[X_1, \dots, X_n]]$ defined by the substitutions

$$X_i \mapsto X_i + X_n^{u(i)}(1 \leq i \leq n-1), \; X_n \mapsto X_n$$

carries $\bar{t}$ to a power series $\bar{g}(X_1, \dots, X_n)$ such that $\bar{g}(0, 0, \dots, 0, X_n) \neq 0$. Clearly this automorphism lifts to an R-automorphism of $A = R[[X_1, \dots, X_n]]$ carrying t to an element $g \in A$ such that $g \notin (\pi, X_1, \dots, X_{n-1})$, and we may work with g instead of t.

For convenience, put $B = R[[X_1, \dots, X_{n-1}]]$, so that $A = B[[X_n]]$. Then, in the terminology of [B], the reduced series of g is nonzero, so by the Preparation Theorem [B, Proposition 6 of Chapter 7, §3.8], there is a polynomial $h \in B[X_n]$ and a unit $u \in B[[X_n]]$ such that $g = uh$. Furthermore, $B[X_n]/(h) \cong B[[X_n]]/(h) = A/gA$ by [B, Proposition 5 of Chapter 7, §3.9], so A/gA is finite over B. Then, as in Quillen's original argument, the kernel of $A \otimes_B (A/gA) \to A/gA$ is principal, generated by a nonzero divisor, from which the theorem follows as in the proof of [Q, Theorem 7.5.11]. □

This theorem is the analog of the main theorem of Gillet and Levine's paper, and we have the following analogs of their corollaries, with the same proofs (the only essential point being that if a finitely generated torsion A-module is flat over R, or if it has codimension 2 or higher, then it is annihilated by some $t \in A$ with $\pi \nmid t$).

Corollary 1. (a) *The map $K_q(MF^{i+1}(A)) \to K_q(MF^i(A))$ is zero for all $q \geq 0$ and all $i \geq 0$.* (b) *The map $K_q(M^{i+1}(A)) \to K_q(M^i(A))$ is zero for all $q \geq 0$ and all $i \geq 1$.* □

Corollary 2. *$K_0(M^i(A))$ is generated by the classes $[A/(f_1, \dots, f_i)]$, where $(f_1, \dots, f_i)$ is a regular sequence.* □

Finally, we have the following result on the absolute case of Gersten's Conjecture.

Corollary 3. *If Gersten's Conjecture is valid for the DVR* $A_{(\pi)}$, *then it is valid for* $A = R[[X_1, \dots, X_n]]$. □

Remark. This result (and its proof) is exactly analogous to [GL, Corollary 6] since $A_{(\pi)}$ is the local ring at the generic point of the closed fiber of $\operatorname{Spec} A \to \operatorname{Spec} R$.

Since Gillet has proved Gersten's Conjecture for DVR's for K-theory with finite coefficients of order prime to the characteristic of k [G], and since the proof of the Theorem and the proofs of Gillet and Levine's corollaries still go through for K-theory with coefficients, we have:

Corollary 4. *Gersten's Conjecture is valid for* $A = R[[X_1, \dots, X_n]]$, *if we use* K*-theory with finite coefficients of order prime to the characteristic of* k. □

One may also formulate results on the K-cohomology of $\operatorname{Spec} A$, analogous to Corollaries 7, 8, and 9 of [GL]. This is left to the interested reader.

References

[B] N. Bourbaki, *Commutative algebra*, Addison-Wesley, Reading, MA, 1972.

[G] H. Gillet, *Gersten's Conjecture for the K-theory with torsion coefficients of a discrete valuation ring*, J. Algebra **103** (1986), 377–380.

[GL] H. Gillet and M. Levine, *The relative form of Gersten's Conjecture over a discrete valuation ring: the smooth case*, J. Pure and Appl. Alg. **46** (1987), 59–71.

[Q] D. Quillen, *Higher algebraic K-theory*, I, Lecture Notes in Math., vol. 341, Springer, Berlin and New York, 1972.

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