PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 109, Number 3, July 1990

ON THETA PAIRS FOR A MAXIMAL SUBGROUP

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(Communicated by Warren J. Wong)

ABSTRACT. For a maximal subgroup M of a finite group G, a Θ-*pair* is any pair of subgroups (C, D) of G such that (i) $D \triangleleft G$, $D \subset C$, (ii) $\langle M, C\rangle = G$, $\langle M, D\rangle = M$ and (iii) C/D has no proper normal subgroup of G/D. A natural partial ordering is defined on the family of Θ-*pairs*. We obtain several results on the maximal Θ-*pairs* which imply G to be solvable, supersolvable, and nilpotent.

1. INTRODUCTION

There has been some interest in the past in investigating how some conditions imposed on a maximal subgroup of a finite group influence the structure of the group. Our objective is to associate a certain family of pairs of subgroups with any maximal subgroup and study how some conditions on the maximal elements (with respect to a natural partial ordering) of such a family imply that the group is solvable, supersolvable, or nilpotent. The family of subgroups we introduce is motivated by the interesting concept of the *Index Complex* defined in Deskins [4–5].

Definition. Given a maximal subgroup M of a group G, let

$$S_M = \{(A, B): A \leq G, B \triangleleft G, B \subset A, \langle M, A\rangle = G, \ \langle M, B\rangle = M\}.$$

Also, let

$$\Theta(M) = \{(C, D) \in S_M : C/D \text{ contains properly no normal subgroup of } G/D\}.$$

We call any pair (C, D) in $\Theta(M)$ a Θ-pair.

A partial order $\leq$ may be defined on $\Theta(M)$ as follows:

$$(C, D) \leq (C', D') \text{ if } C \leq C';$$

no condition is placed on the second component of the pairs. (One notices that using the definition of $\Theta(M)$, it follows that $D \subset D'$. Also, $C = C' \Rightarrow D = D'$). It is easy to verify that

$$(C, D) \leq (C', D') \text{ and } (C', D') \leq (C, D) \Leftrightarrow C = C', \ D = D'.$$

Received by the editors June 2, 1989 and, in revised form, September 27, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20D10, 20D25; Secondary 20D99.
Key words and phrases. Solvable, supersolvable, nilpotent.

Obviously, $\Theta(M)$ will contain maximal elements with respect to this ordering. We shall call a maximal element a *maximal* Θ*-pair*. As a straightforward example, take $G = \mathrm{Sym}(4)$. Then if $M \in \mathrm{Syl}_2(G)$, $\mathrm{Core}_G M$ is the *Klein* 4*-group* and it is easy to see that $(\mathrm{Sym}(3), \langle e\rangle) \in \Theta(M)$ and $(\mathrm{Sym}(3), \langle e\rangle)$ is a maximal Θ-pair. For any group G and a maximal subgroup M of G, if $(C, D) \in \Theta(M)$, then clearly $D \subset \mathrm{Core}_G(M)$; but D need not be equal to $\mathrm{Core}_G M$ as the above example shows.

In §3 we shall obtain conditions on maximal pairs in $\Theta(M)$ which imply G to be solvable, or supersolvable. In §4 we shall obtain conditions on maximal pairs in $\Theta(M)$ which imply G to be nilpotent, or p-nilpotent.

All groups considered are finite. We use standard notation as in Huppert [7]. By a slight abuse of language, by a *simple* group we shall always mean a simple, non-Abelian group. Also, for convenience we denote $M \lessdot G$ to indicate that M is a maximal subgroup of G. If $M \lessdot G$ and $[G : M]$ is composite, then M is called a *c-maximal* subgroup of G.

2. Preliminaries

If $M \lessdot G$ and $\langle M, A\rangle = G$, then any normal subgroup B of G contained in A with $\langle M, B\rangle = M$ will produce a pair $(A, B) \in S_M$. In particular, $(A, \langle e\rangle)$ where e denotes the identity of G, is also a pair in S_M. If (A, B) is any pair in S_M, then clearly $B \subset \mathrm{Core}_G M$.

If $M \lessdot G$, then $\Theta(M)$ is nonempty. For, let $C \leq G$ such that $\langle M, C\rangle = G$ and denote

$$D = \prod\{Y : (C, Y) \in S_M\}.$$

Then, if C/D has no proper normal subgroup of G/D, one has that $(C, D) \in \Theta(M)$. Otherwise, suppose that L/D is a minimal normal subgroup of G/D contained in C/D. Then it is easy to see that $(L, D) \in \Theta(M)$.

The following result will be used frequently in induction arguments.

Lemma 2.1. *If (C, D) is a maximal Θ-pair in $\Theta(M)$ and $N \triangleleft G$, $N \subset D$, then $(C/N, D/N)$ is a maximal Θ-pair in $\Theta(M/N)$. Conversely, if $(C/N, D/N)$ is a maximal Θ-pair in $\Theta(M/N)$, then (C, D) is a maximal Θ-pair in $\Theta(M)$.*

Proof. $\Rightarrow$: Since $\langle M, C\rangle = G$, $\langle M, D\rangle = M$, and C/D has no proper normal subgroup of G/D, it follows that $(C/N, D/N) \in \Theta(M/N)$. If $(C/N, D/N)$ is not a maximal Θ-pair in $\Theta(M/N)$, then suppose that $(C/N, D/N) \leq (X/N, Y/N)$, $C/N \subset X/N$. This implies that $C \subset X$. Now, one sees that $(X, Y) \in \Theta(M)$. Further, $(C, D) \leq (X, Y)$, since $C \subset X$. This violates the maximality of (C, D) in $\Theta(M)$.

$\Leftarrow$: It is easy to see that $(C, D) \in \Theta(M)$. If (C, D) is not a maximal Θ-pair, then suppose $(C, D) \leq (C_1, D_1)$ and $C \subset C_1$. This implies that $(C/N, D/N) \leq (C_1, D_1/N)$, violating the maximality of the pair $(C/N, D/N)$. We have used here the fact that $N \subset D_1$. To see this, first we observe that $\langle C_1, M\rangle = G$, $\langle M, D_1\rangle = M$, and $N \subset C \subset C_1$. If $N \not\subset D_1$, then ND_1/D_1

is a normal subgroup of G/D_1, and this, since $(C_1, D_1) \in \Theta(M)$, means $ND_1 = C_1$. However, in that case $G = \langle M, C_1\rangle = \langle M, ND_1\rangle = M$ since $\langle M, D_1\rangle = M$ and $N \subset D$, a contradiction.

3. Solvability conditions

Theorem 3.1. *A group G is solvable $\Leftrightarrow$ for each $M \lessdot G$, every maximal Θ-pair (C, D) in $\Theta(M)$ is such that C/D is solvable.*

Proof. $\Leftarrow$: Obviously, the hypothesis cannot hold if G is simple. Let N be a minimal normal subgroup of G. We use induction on the order of G. If $M/N \lessdot G/N$ and $(C/N, D/N)$ is a maximal Θ-pair in $\Theta(M/N)$, then by Lemma 2.1, (C, D) is a maximal Θ-pair in $\Theta(M)$. It follows by induction that G/N is solvable, and without loss in generality, N may be assumed to be the unique minimal normal subgroup of G. If $N \subseteq \Phi(G)$, the Frattini subgroup of G, then G is solvable and therefore assume that $G = M_1N$ for some $M_1 \lessdot G$. If $(N, \langle 1\rangle)$ is a pair in $\Theta(M)$ and if it is not a maximal Θ-pair, then since M_1 is core-free, $N, \langle 1\rangle) \leq (R, \langle 1\rangle)$ for some pair $(R, \langle 1\rangle)$ in $\Theta(M)$. But then $R/\langle 1\rangle$ has no proper normal subgroup of $G/\langle 1\rangle$, which is not possible since $N \subset R$. Thus $(N, \langle 1\rangle)$ is a maximal Θ-pair in $\Theta(M_1)$, and so N is solvable, implying that G is solvable.

The converse holds trivially.

We remark that it can be shown that Theorem 3.1 remains valid if the statement "C/D is solvable" is replaced by "C/D is solvable whenever $C/D \triangleleft G/D$". The proof of the following result is analogous to the proof of Theorem 3.1 and is omitted.

Theorem 3.2. (i) *A group G is solvable $\Leftrightarrow$ for each maximal pair (C, D) in $\Theta(M)$, $M \lessdot G$, one has that $C_{G/D}(C/D) \neq \langle \bar{1}\rangle$ whenever $C/D \trianglelefteq G/D$.*

(ii) *A group G is solvable if for every $M \lessdot G$, each maximal pair (C, D) is such that $L(G/D) \neq \langle \bar{1}\rangle$.*

(*Here, for any group X, $L(X)$ denotes the intersection of all c-maximal subgroups of X; if there is no subgroup then set $L(X) = X$* ([2–3])).

(iii) *A group G is solvable $\Leftrightarrow$ for any two distinct maximal subgroups X and Y of G whenever $\Theta(X)$ and $\Theta(Y)$ have a common maximal pair (C, D) it follows that $[G : X] = [G : Y]$ if $C/D \triangleleft G/D$.*

We now give another characterization of solvable groups in terms of Θ-pairs.

Theorem 3.3. *A group G is solvable $\Leftrightarrow$ for each c-maximal subgroup M of G, there exists a maximal pair (C, D) in $\Theta(M)$ such that C/D is Abelian.*

Proof. $\Leftarrow$: Obviously, G cannot be simple. Let N be a minimal normal subgroup of G. We use induction on the order of G. If R/N is a c-maximal subgroup of G/N, then R is c-maximal in G; so $\Theta(R)$ contains a maximal pair (C, D) such that C/D is Abelian. If $N \subseteq D$ then $(C/N, D/N)$ is a maximal pair in $\Theta(R/N)$ and $(C/N)/(D/N)$ is Abelian. If $N \not\subset D$, then

if $ND \subset C$, C/D will contain a proper normal subgroup of G/D; and if $C = ND$, then

$$\langle R, C\rangle = G = \langle R, ND\rangle = R,$$

a contradiction. We may therefore assume that $N \not\subset C$. Considering the pair (CN, DN), one has that CN/DN is Abelian. Let K be the largest proper normal subgroup of G contained in CN such that $K \subseteq R$. If CN/K does not contain any proper normal subgroup of G/K, then (CN, K) is an element of $\Theta(R)$ and $(C, D) \leq (CN, K)$ implies that $C = CN$, a contradiction. If on the other hand CN/K contains a proper normal subgroup of G/K, then suppose that H/K is a minimal normal subgroup of G/K. Now, $H \subset CN$, $H \triangleleft G$, and $G = \langle R, H\rangle$. Therefore (H, K) is a pair in $\Theta(R)$, and it is easy to see that H/K is Abelian. If (H, K) is a maximal pair, then $(H/N, K/N)$ is a maximal pair in $\Theta(R/N)$ and $(H/K)/(K/N)$ is Abelian. If on the other hand (H, K) is not a maximal pair, then let $(H, K) \leq (H_1, K_1)$, where (H_1, K_1) is a maximal pair and consequently $H \leq H_1$. One sees that K_1 is the largest proper normal subgroup of G in H_1 that is contained in R; also H is not contained in K_1. If $HK_1 \neq H_1$, then HK_1/H_1 is a proper normal subgroup in H_1/K_1, a contradiction. Hence $HK_1 = H_1$. It follows that $K \subset K_1$ and HK_1/K_1 is Abelian. Thus $(H_1/N, K_1/N)$ is a maximal pair in $\Theta(R/N)$ such that $(H_1/N)/(K_1/N)$ is Abelian. By induction, G/N is solvable, and without loss in generality one may assume that N is the unique minimal normal subgroup of G. If $N \subseteq L(G)$ (where $L(G)$ is defined in the statement of Theorem 3.2), then G is solvable since $L(G)$ is supersolvable (a published proof appears in [3, Theorem 3]). Thus one may assume that $G = MN$, where M is c-maximal and core-free in G. By hypothesis, there exists a maximal pair $(X, \langle 1\rangle)$ in $\Theta(M)$ such that $X/\langle 1\rangle$ is Abelian. If $X = N$ then G is solvable. Let $\overline{X} \supset X$ and X be a maximal subgroup of $\overline{X}$. If $\overline{X}$ does not contain any proper normal subgroup of G, then $(X, \langle 1\rangle) \leq (\overline{X}, \langle 1\rangle)$ and $X = \overline{X}$, a contradiction. Thus $\overline{X}$ contains proper normal subgroup of G, and consequently $N \subset \overline{X}$. From a result of Huppert [6, Satz 2], it follows directly that if any group Z has a maximal subgroup which is Abelian, then Z is solvable. Since $\overline{X}$ has a maximal subgroup which is Abelian, it follows that $\overline{X}$ is solvable. So N is solvable and consequently G is solvable.

The converse holds trivially.

Theorem 3.4. *If the index of each maximal pair in $\Theta(M)$ is a prime, for every $M \lessdot G$, then G is supersolvable.* (The index of a pair (C, D) refers to $[C : D]$).

Proof. If G is simple, then the assertion follows trivially. Now, let N be a maximal normal subgroup of G. We use induction on the order of G. If $M/N \lessdot G/N$ and $(X/N, Y/N)$ is a maximal pair in $\Theta(M/N)$, then by Lemma 2.1 (X, Y) is a maximal pair in $\Theta(M)$ and the index of $(X/N, Y/N)$, being equal to $[X, Y]$, is a prime by using the hypothesis. By induction, G/N is supersolvable and N may be assumed to be the unique minimal normal

subgroup of G. If $N \not\subseteq \Phi(G)$, then $G = M_1 N$ for some core-free maximal subgroup M_1 of G. Then $(N, \langle 1 \rangle)$ is a maximal pair in $\Theta(M_1)$ and by hypothesis $o(N)$ is a prime. Hence G is supersolvable.

Corollary 3.5. *A group G is supersolvable $\Leftrightarrow$ the index of each maximal pair (C, D) in $\Theta(M)$ is a prime whenever $C/D \triangleleft G/D$.*

We omit the proof of the following result, which is analogous to Theorem 3.2(iii).

Proposition 3.6. *A group G is supersolvable $\Leftrightarrow$ for any two maximal subgroups X, Y of G whenever $\Theta(X)$ and $\Theta(Y)$ have a common maximal pair (C, D), one has that C/D is cyclic if $C/D \trianglelefteq G/D$.*

Let $Q(G)$ denote the characteristic subgroup of G generated by all $x \in G$ with the property that

$$\langle x \rangle \cdot \langle g \rangle = \langle g \rangle \cdot \langle x \rangle$$

for every $g \in G$ (where as usual $\langle g \rangle$ denotes the subgroup generated by g). Let $Q^*(G)$ denote the terminal member in the ascending series defined by: $Q_0(G) = \langle 1 \rangle$, $Q_1(G) = Q(G)$ and for $i > 1$,

$$Q_i(G/Q_{i-1}(G)) = Q_i(G)/Q_{i-1}(G).$$

These subgroups were introduced in Mukherjee [8]; also see [10] for an exposition.

Theorem 3.7. *A group G is supersolvable $\Leftrightarrow$ for every maximal pair (C, D) in $\Theta(M)$, one has that $Q^*(G/D) \neq \langle \bar{1} \rangle$, for every $M \lessdot G$.*

Proof. $\Leftarrow$: It is obvious from the hypothesis that G cannot be simple. Let N be a minimal normal subgroup of G. We use induction on $o(G)$. As in the proof of Theorem 3.4, it follows that G/N is supersolvable. Without loss in generality N may now be assumed to be the unique minimal normal subgroup of G, $G = MN$, $M \lessdot G$; and M is core-free. Then $(N, \langle 1 \rangle)$ is a maximal pair in $\Theta(M)$, and therefore $Q^*(G) \neq \langle 1 \rangle$. By the minimality of N, $N \subset Q^*(G)$. Therefore N is a supersolvably embedded subgroup of G and $o(N)$ is a prime ([10, Theorem 7.10, p. 32]). Hence G is supersolvable.

The converse holds trivially.

Corollary 3.8. *A group G is supersolvable if for every maximal pair (C, D) in $\Theta(M)$ for each c-maximal subgroup M of G, $Q^*(G) \neq \langle \bar{1} \rangle$.*

4. Nilpotency conditions

Theorem 4.1. *A group G is nilpotent $\Leftrightarrow$ for each maximal pair (C, D) in $\Theta(M)$ for every $M \lessdot G$, one has $Z(G/D) \neq \langle \bar{1} \rangle$.*

Proof. $\Leftarrow$: We use induction on $o(G)$ to show that every maximal subgroup of G is normal. Clearly, G cannot be simple. Let $M \lessdot G$ and N be a minimal normal subgroup of G contained in M. If $(C/N, D/N)$ is a maximal pair

in $\Theta(M/N)$, then by Lemma 2.1, (C, D) is a maximal pair in $\Theta(M)$ and $Z(G/D) \neq \langle \bar{1} \rangle$. This implies that $Z((G/N)/(D/N)) \neq \langle \bar{1} \rangle$. By induction, $M/N \triangleleft G/N$ and so $M \triangleleft G$. If M is core-free, then a maximal pair in $\Theta(M)$ is of the form $(X, \langle 1 \rangle)$ so that $Z(G) \neq \langle \bar{1} \rangle$. Thus $G = MZ(G)$ and therefore $M \triangleleft G$. The result now follows.

The converse holds trivially.

Corollary 4.2. *A group G is nilpotent $\Leftrightarrow$ $\Theta(M)$, for every $M \lessdot G$, contains a maximal pair (C, D) such that G/D is nilpotent.*

Corollary 4.3. *Let $M \lessdot G$ and $\Theta(M)$ contain a maximal pair such that G/D is nilpotent. Then $M \triangleleft G$.*

Theorem 4.4. *A solvable group G, whose order is divisible by at least two primes, is nilpotent $\Leftrightarrow$ the index of each maximal pair in $\Theta(M)$, for every $M \lessdot G$, is the same.*

Proof. $\Leftarrow$: We use induction on the order of G. Let N be a minimal normal subgroup of G; N is an elementary Abelian p-group for some prime p. By induction it follows that G/N is nilpotent, and without loss in generality it may be assumed that N is the unique minimal normal subgroup of G. If $N \nsubseteq \Phi(G)$, then for some $M \lessdot G$, $G = MN$ and M is core-free. Now, $(N, \langle 1 \rangle)$ is a maximal element in $\Theta(M)$, and so, by hypothesis, the *index* of any maximal pair in $\Theta(M)$ is a power of p. If q is another divisor of the order of G, then all the elements of order q in G cannot lie in M, as they will generate a characteristic subgroup of G. Let $y \in G \backslash M$ such that $o(y) = q$. Then $\langle M, y \rangle = G$. If $\langle y \rangle \triangleleft G$, then $G = M\langle y \rangle$ and

$$[G : M] = o(y) = q = o(N) = p^m$$

for some $m \geq 1$, a contradiction to the fact that $q \neq p$. Thus we may assume that $\langle y \rangle$ is not normal in G. So $(\langle y \rangle, \langle 1 \rangle) \in \Theta(M)$. Either $(\langle y \rangle, \langle 1 \rangle)$ is a maximal pair in $\Theta(M)$, or if not, then $(\langle y \rangle, \langle 1 \rangle) \leq (X, \langle 1 \rangle)$, where $(X, \langle 1 \rangle)$ is a maximal pair in $\Theta(M)$. But the index of $(X, \langle 1 \rangle)$ is divisible by q, a contradiction. Thus $N \subseteq \Phi(G)$, and therefore G is nilpotent.

The converse holds trivially.

We remark that Theorem 4.4 does not remain valid if in its statement one omits the condition that G is solvable. For example, if G is simple, then $(G, \langle 1 \rangle)$ is the unique minimal pair in $\Theta(M)$ for every $M \lessdot G$, but G is not nilpotent unless $o(G)$ is a prime.

Corollary 4.5. (i) *A group G is nilpotent $\Leftrightarrow$ the index of each maximal pair in $\Theta(M)$ for every $M \lessdot G$ is the same and is equal to a prime.*

(ii) *A solvable group G is nilpotent $\Leftrightarrow$ $\Theta(M)$, for all $M \lessdot G$, contains exactly one maximal pair.*

We omit the proof of the following result.

Proposition 4.6. *A group G is nilpotent $\Leftrightarrow$ for maximal pairs (C, D) in $\Theta(M)$ and (C_1, D_1) in $\Theta(M_1)$ with $C^x = C_1$, M and M_1 are both normal in G.*

For a group G and a prime p, let $O^p(G)$ denote the subgroup generated by all the p'-elements of G.

Theorem 4.7. *Let G be a group in which $\Theta(M)$, for each $M \lessdot G$, contains a maximal pair (C, D) depending on M such that*

$$O^p(G/D) \subseteq C_{G/D}(C/D).$$

Then G is p-nilpotent.

Proof. The assertion follows trivially if G is simple. So, assume now that G is not simple and let N be a minimal normal subgroup of G. We use induction on $o(G)$. By inductive argument, it follows that G/N is p-nilpotent; and one may also assume that N is the unique minimal normal subgroup of G. If $N \subseteq \Phi(G)$, the assertion follows. So, let $G = MN$ for some core-free, maximal subgroup M of G. By hypothesis, $\Theta(M)$ contains a maximal pair (C, D) such that $C_G(G/D) \supseteq O^p(G/D)$. Then $D = \langle 1 \rangle$ and $N \subseteq C_G(O^p(G))$. Let R/N be a normal p-complement in G/N. If N is a p'-group, then R is a normal p-complement of G and the assertion follows. Now, suppose that p divides $o(N)$ Then, N is an elementary Abelian p-group and is a normal Sylow p-subgroup of R. By the Schur–Zassenhaus theorem, $R = L_0 N$, where L_0 is a p-complement of N. It is now easy to see that L_0 is a normal p-complement in G. Therefore G is p-nilpotent.

For a group G and a prime p, let $Z_{p'}(G)$ denote the subgroup generated by all $x \in G$ which commute with every p'-element of G. We omit the proof of the following result whose proof is analogous to that of Theorem 4.7.

Proposition 4.8. *A group G is p-nilpotent if for every $M \lessdot G$, $\Theta(M)$ contains a maximal pair (C, D) such that $Z_{p'}(G/D) \neq \langle \bar{1} \rangle$.*

We now introduce two new characteristic subgroups of G based on the concept of Θ-pairs. Let

$$\mathscr{F} = \{M \lessdot G \colon \exists \text{ a maximal pair } (C, D) \in \Theta(M),\ Z(G/D) = \langle \bar{1} \rangle\}.$$

Define $A(G)$ to be the intersection of all the elements of $\mathscr{F}$. Again, define $\mathscr{T}$ exactly in the same way as $\mathscr{F}$ is defined but only replacing the condition that "$Z(G/D) = \langle \bar{1} \rangle$" by the condition that "$Q^*(G) = \langle 1 \rangle$". Define $B(G)$ to be the intersection of all the elements of $\mathscr{T}$. (If either of the families $\mathscr{F}$, $\mathscr{T}$ is empty, the corresponding subgroup is chosen to be G itself).

The proof of the following result is left as an exercise.

Proposition 4.9. (i) *$A(G)$ is a characteristic subgroup of G which is nilpotent, and $A/\Phi(G) = Z_\infty(G/\Phi(G))$.*

(ii) *$B(G)$ is a characteristic subgroup of G, and $B(G)$ is supersolvable.*

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