ON THETA PAIRS FOR A MAXIMAL SUBGROUP

N. P. Mukherjee and Prabir Bhattacharya

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Abstract. For a maximal subgroup $M$ of a finite group $G$, a $\Theta$-pair is any pair of subgroups $(C, D)$ of $G$ such that (i) $D \triangleleft G$, $D \subseteq C$, (ii) $(M, C) = G$, $(M, D) = M$ and (iii) $C/D$ has no proper normal subgroup of $G/D$. A natural partial ordering is defined on the family of $\Theta$-pairs. We obtain several results on the maximal $\Theta$-pairs which imply $G$ to be solvable, supersolvable, and nilpotent.

1. Introduction

There has been some interest in the past in investigating how some conditions imposed on a maximal subgroup of a finite group influence the structure of the group. Our objective is to associate a certain family of pairs of subgroups with any maximal subgroup and study how some conditions on the maximal elements (with respect to a natural partial ordering) of such a family imply that the group is solvable, supersolvable, or nilpotent. The family of subgroups we introduce is motivated by the interesting concept of the Index Complex defined in Deskins [4–5].

Definition. Given a maximal subgroup $M$ of a group $G$, let

$$S_M = \{(A, B) : A \leq G, B \triangleleft G, B \subseteq A, (M, A) = G, (M, B) = M\}.$$ 

Also, let

$$\Theta(M) = \{(C, D) \in S_M : C/D \text{ contains properly no normal subgroup of } G/D\}.$$ 

We call any pair $(C, D)$ in $\Theta(M)$ a $\Theta$-pair.

A partial order $\leq$ may be defined on $\Theta(M)$ as follows:

$$(C, D) \leq (C', D') \text{ if } C \leq C';$$

no condition is placed on the second component of the pairs. (One notices that using the definition of $\Theta(M)$, it follows that $D \subseteq D'$. Also, $C = C' \Rightarrow D = D'$). It is easy to verify that

$$(C, D) \leq (C', D') \text{ and } (C', D') \leq (C, D) \Leftrightarrow C = C', D = D'.$$

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Obviously, $\Theta(M)$ will contain maximal elements with respect to this ordering. We shall call a maximal element a maximal $\Theta$-pair. As a straightforward example, take $G = \text{Sym}(4)$. Then if $M \in \text{Syl}_2(G)$, $\text{Core}_G M$ is the Klein 4-group and it is easy to see that $(\text{Sym}(3), (e)) \in \Theta(M)$ and $(\text{Sym}(3), (e))$ is a maximal $\Theta$-pair. For any group $G$ and a maximal subgroup $M$ of $G$, if $(C, D) \in \Theta(M)$, then clearly $D \subseteq \text{Core}_G(M)$; but $D$ need not be equal to $\text{Core}_G M$ as the above example shows.

In §3 we shall obtain conditions on maximal pairs in $\Theta(M)$ which imply $G$ to be solvable, or supersolvable. In §4 we shall obtain conditions on maximal pairs in $\Theta(M)$ which imply $G$ to be nilpotent, or $p$-nilpotent.

All groups considered are finite. We use standard notation as in Huppert [7]. By a slight abuse of language, by a simple group we shall always mean a simple, non-Abelian group. Also, for convenience we denote $M \triangleleft G$ to indicate that $M$ is a maximal subgroup of $G$. If $M \triangleleft G$ and $[G : M]$ is composite, then $M$ is called a $c$-maximal subgroup of $G$.

2. Preliminaries

If $M \triangleleft G$ and $(M, A) = G$, then any normal subgroup $B$ of $G$ contained in $A$ with $(M, B) = M$ will produce a pair $(A, B) \in S_M$. In particular, $(A, (e))$ where $e$ denotes the identity of $G$, is also a pair in $S_M$. If $(A, B)$ is any pair in $S_M$, then clearly $B \subseteq \text{Core}_G M$.

If $M \triangleleft G$, then $\Theta(M)$ is nonempty. For, let $C \leq G$ such that $(M, C) = G$ and denote

$$D = \prod \{ Y : (C, Y) \in S_M \}.$$

Then, if $C/D$ has no proper normal subgroup of $G/D$, one has that $(C, D) \in \Theta(M)$. Otherwise, suppose that $L/D$ is a minimal normal subgroup of $G/D$ contained in $C/D$. Then it is easy to see that $(L, D) \in \Theta(M)$.

The following result will be used frequently in induction arguments.

**Lemma 2.1.** If $(C, D)$ is a maximal $\Theta$-pair in $\Theta(M)$ and $N \triangleleft G$, $N \subseteq D$, then $(C/N, D/N)$ is a maximal $\Theta$-pair in $\Theta(M/N)$. Conversely, if $(C/N, D/N)$ is a maximal $\Theta$-pair in $\Theta(M/N)$, then $(C, D)$ is a maximal $\Theta$-pair in $\Theta(M)$.

**Proof.** $\Rightarrow$: Since $(M, C) = G$, $(M, D) = M$, and $C/D$ has no proper normal subgroup of $G/D$, it follows that $(C/N, D/N) \in \Theta(M/N)$. If $(C/N, D/N)$ is not a maximal $\Theta$-pair in $\Theta(M/N)$, then suppose that $(C/N, D/N) \leq (X/N, Y/N)$, $C/N \subseteq X/N$. This implies that $C \subseteq X$. Now, one sees that $(X, Y) \in \Theta(M)$. Further, $(C, D) \leq (X, Y)$, since $C \subseteq X$. This violates the maximality of $(C, D)$ in $\Theta(M)$.

$\Leftarrow$: It is easy to see that $(C, D) \in \Theta(M)$. If $(C, D)$ is not a maximal $\Theta$-pair, then suppose $(C, D) \leq (C_1, D_1)$ and $C \subseteq C_1$. This implies that $(C/N, D/N) \leq (C_1/N, D_1/N)$, violating the maximality of the pair $(C/N, D/N)$. We have used here the fact that $N \subseteq D_1$. To see this, first we observe that $(C_1, M) = G$, $(M, D_1) = M$, and $N \subseteq C \subseteq C_1$. If $N \not\subseteq D_1$, then $ND_1/D_1=
is a normal subgroup of $G/D_1$, and this, since $(C_1, D_1) \in \Theta(M)$, means $ND_1 = C_1$. However, in that case $G = \langle M, C_1 \rangle = \langle M, ND_1 \rangle = M$ since $\langle M, D_1 \rangle = M$ and $N \subseteq D$, a contradiction.

### 3. Solvability conditions

**Theorem 3.1.** A group $G$ is solvable $\iff$ for each $M \leq G$, every maximal $\Theta$-pair $(C, D)$ in $\Theta(M)$ is such that $C/D$ is solvable.

**Proof.** $\Leftarrow$: Obviously, the hypothesis cannot hold if $G$ is simple. Let $N$ be a minimal normal subgroup of $G$. We use induction on the order of $G$. If $M/N < G/N$ and $(C/N, D/N)$ is a maximal $\Theta$-pair in $\Theta(M/N)$, then by Lemma 2.1, $(C, D)$ is a maximal $\Theta$-pair in $\Theta(M)$. It follows by induction that $G/N$ is solvable, and without loss in generality, $N$ may be assumed to be the unique minimal normal subgroup of $G$. If $N \subseteq \Phi(G)$, the Frattini subgroup of $G$, then $G$ is solvable and therefore assume that $G = M_1N$ for some $M_1 \leq G$. If $(N, \langle 1 \rangle)$ is a pair in $\Theta(M)$ and if it is not a maximal $\Theta$-pair, then since $M_1$ is core-free, $N, \langle 1 \rangle \leq (R, \langle 1 \rangle)$ for some pair $(R, \langle 1 \rangle)$ in $\Theta(M)$. But then $R/\langle 1 \rangle$ has no proper normal subgroup of $G/\langle 1 \rangle$, which is not possible since $N \subseteq R$. Thus $(N, \langle 1 \rangle)$ is a maximal $\Theta$-pair in $\Theta(M_1)$, and so $N$ is solvable, implying that $G$ is solvable.

The converse holds trivially.

We remark that it can be shown that Theorem 3.1 remains valid if the statement "$C/D$ is solvable" is replaced by "$C/D$ is solvable whenever $C/D < G/D$". The proof of the following result is analogous to the proof of Theorem 3.1 and is omitted.

**Theorem 3.2.** (i) A group $G$ is solvable $\iff$ for each maximal pair $(C, D)$ in $\Theta(M)$, $M \leq G$, one has that $C_{G/D}(C/D) \neq \langle 1 \rangle$ whenever $C/D \leq G/D$.

(ii) A group $G$ is solvable if for every $M \leq G$, each maximal pair $(C, D)$ is such that $L(G/D) \neq \langle 1 \rangle$.

(Here, for any group $X$, $L(X)$ denotes the intersection of all c-maximal subgroups of $X$; if there is no subgroup then set $L(X) = X$ ([2-3])).

(iii) A group $G$ is solvable $\iff$ for any two distinct maximal subgroups $X$ and $Y$ of $G$ whenever $\Theta(X)$ and $\Theta(Y)$ have a common maximal pair $(C, D)$ it follows that $[G : X] = [G : Y]$ if $C/D \leq G/D$.

We now give another characterization of solvable groups in terms of $\Theta$-pairs.

**Theorem 3.3.** A group $G$ is solvable $\iff$ for each c-maximal subgroup $M$ of $G$, there exists a maximal pair $(C, D)$ in $\Theta(M)$ such that $C/D$ is Abelian.

**Proof.** $\Leftarrow$: Obviously, $G$ cannot be simple. Let $N$ be a minimal normal subgroup of $G$. We use induction on the order of $G$. If $R/N$ is a c-maximal subgroup of $G/N$, then $R$ is c-maximal in $G$; so $\Theta(R)$ contains a maximal pair $(C, D)$ such that $C/D$ is Abelian. If $N \subseteq D$ then $(C/N, D/N)$ is a maximal pair in $\Theta(R/N)$ and $(C/N)/(D/N)$ is Abelian. If $N \nsubseteq D$, then...
if \( ND \subset C \), \( C/D \) will contain a proper normal subgroup of \( G/D \); and if \( C = ND \), then

\[
\langle R, C \rangle = G = \langle R, ND \rangle = R,
\]
a contradiction. We may therefore assume that \( N \not\subset C \). Considering the pair \( (CN, DN) \), one has that \( CN/DN \) is Abelian. Let \( K \) be the largest proper normal subgroup of \( G \) contained in \( CN \) such that \( K \leq R \). If \( CN/K \) does not contain any proper normal subgroup of \( G/K \), then \( (CN, K) \) is an element of \( \Theta(R) \) and \( (C, D) \leq (CN, K) \) implies that \( C = CN \), a contradiction. If on the other hand \( CN/K \) contains a proper normal subgroup of \( G/K \), then suppose that \( H/K \) is a minimal normal subgroup of \( G/K \). Now, \( H \subset CN \), \( H \triangleleft G \), and \( G = \langle R, H \rangle \). Therefore \( (H, K) \) is a pair in \( \Theta(R) \), and it is easy to see that \( H/K \) is Abelian. If \( (H, K) \) is a maximal pair, then \( (H/N, K/N) \) is a maximal pair in \( \Theta(R/N) \) and \( (H/K)/(K/N) \) is Abelian. If on the other hand \( (H, K) \) is not a maximal pair, then let \( (H_1, K_1) \), where \( (H_1, K_1) \) is a maximal pair, and consequently \( H \leq H_1 \). One sees that \( K_1 \) is the largest proper normal subgroup of \( G \) in \( H \), that is contained in \( R \); also \( H \) is not contained in \( K_1 \). If \( HK_1 \not= H_1 \), then \( HK_1/H_1 \) is a proper normal subgroup in \( H_1/K_1 \), a contradiction. Hence \( HK_1 = H_1 \). It follows that \( K \subset K_1 \) and \( HK_1/K_1 \) is Abelian. Thus \( (H_1/N, K_1/N) \) is a maximal pair in \( \Theta(R/N) \) such that \( (H_1/N)/(K_1/N) \) is Abelian. By induction, \( G/N \) is solvable, and without loss in generality one may assume that \( N \) is the unique minimal normal subgroup of \( G \). If \( N \subset L(G) \) (where \( L(G) \) is defined in the statement of Theorem 3.2), then \( G \) is solvable since \( L(G) \) is supersolvable (a published proof appears in [3, Theorem 3]). Thus one may assume that \( G = MN \), where \( M \) is a-maximal and core-free in \( G \). By hypothesis, there exists a maximal pair \( (X, \langle 1 \rangle) \) in \( \Theta(M) \) such that \( X/\langle 1 \rangle \) is Abelian. If \( X = N \) then \( G \) is solvable. Let \( \overline{X} \supset X \) and \( X \) be a maximal subgroup of \( \overline{X} \). If \( \overline{X} \) does not contain any proper normal subgroup of \( G \), then \( (X, \langle 1 \rangle) \leq (\overline{X}, \langle 1 \rangle) \) and \( X = \overline{X} \), a contradiction. Thus \( \overline{X} \) contains proper normal subgroup of \( G \), and consequently \( N \subset \overline{X} \). From a result of Huppert [6, Satz 2], it follows directly that if any group \( Z \) has a maximal subgroup which is Abelian, then \( Z \) is solvable. Since \( \overline{X} \) has a maximal subgroup which is Abelian, it follows that \( \overline{X} \) is solvable. So \( N \) is solvable and consequently \( G \) is solvable.

The converse holds trivially.

**Theorem 3.4.** If the index of each maximal pair in \( \Theta(M) \) is a prime, for every \( M < G \), then \( G \) is supersolvable. (The index of a pair \( (C, D) \) refers to \( [C : D] \)).

**Proof.** If \( G \) is simple, then the assertion follows trivially. Now, let \( N \) be a maximal normal subgroup of \( G \). We use induction on the order of \( G \). If \( M/N < G/N \) and \( (X/N, Y/N) \) is a maximal pair in \( \Theta(M/N) \), then by Lemma 2.1 \( (X, Y) \) is a maximal pair in \( \Theta(M) \) and the index of \( (X/N, Y/N) \), being equal to \( [X, Y] \), is a prime by using the hypothesis. By induction, \( G/N \) is supersolvable and \( N \) may be assumed to be the unique minimal normal
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If \( N \not\subset \Phi(G) \), then \( G = M_1N \) for some core-free maximal subgroup \( M_1 \) of \( G \). Then \( (N, \langle 1 \rangle) \) is a maximal pair in \( \Theta(M_1) \) and by hypothesis \( o(N) \) is a prime. Hence \( G \) is supersolvable.

**Corollary 3.5.** A group \( G \) is supersolvable \( \iff \) the index of each maximal pair \( (C, D) \) in \( \Theta(M) \) is a prime whenever \( C/D \triangleleft G/D \).

We omit the proof of the following result, which is analogous to Theorem 3.2(iii).

**Proposition 3.6.** A group \( G \) is supersolvable \( \iff \) for any two maximal subgroups \( X, Y \) of \( G \) whenever \( \Theta(X) \) and \( \Theta(Y) \) have a common maximal pair \( (C, D) \), one has that \( C/D \) is cyclic if \( C/D \triangleleft G/D \).

Let \( Q(G) \) denote the characteristic subgroup of \( G \) generated by all \( x \in G \) with the property that
\[
(x) \cdot (g) = (g) \cdot (x)
\]
for every \( g \in G \) (where as usual \( (g) \) denotes the subgroup generated by \( g \)). Let \( Q^*(G) \) denote the terminal member in the ascending series defined by:
\[
Q_0(G) = \langle 1 \rangle, \ Q_1(G) = Q(G) \quad \text{and for } i > 1, \quad Q_i(G/Q_{i-1}(G)) = Q_i(G)/Q_{i-1}(G).
\]
These subgroups were introduced in Mukherjee [8]; also see [10] for an exposition.

**Theorem 3.7.** A group \( G \) is supersolvable \( \iff \) for every maximal pair \( (C, D) \) in \( \Theta(M) \), one has that \( Q^*(G/D) \neq \langle 1 \rangle \), for every \( M < G \).

**Proof.** \( \Leftarrow \): It is obvious from the hypothesis that \( G \) cannot be simple. Let \( N \) be a minimal normal subgroup of \( G \). We use induction on \( o(G) \). As in the proof of Theorem 3.4, it follows that \( G/N \) is supersolvable. Without loss in generality \( N \) may now be assumed to be the unique minimal normal subgroup of \( G \), \( G = MN \), \( M < G \); and \( M \) is core-free. Then \( (N, \langle 1 \rangle) \) is a maximal pair in \( \Theta(M) \), and therefore \( Q^*(G) \neq \langle 1 \rangle \). By the minimality of \( N \), \( N \subset Q^*(G) \). Therefore \( N \) is a supersolvably embedded subgroup of \( G \) and \( o(N) \) is a prime ([10, Theorem 7.10, p. 32]). Hence \( G \) is supersolvable.

The converse holds trivially.

**Corollary 3.8.** A group \( G \) is supersolvable if for every maximal pair \( (C, D) \) in \( \Theta(M) \) for each \( c \)-maximal subgroup \( M \) of \( G \), \( Q^*(G) \neq \langle 1 \rangle \).

4. Nilpotency conditions

**Theorem 4.1.** A group \( G \) is nilpotent \( \iff \) for each maximal pair \( (C, D) \) in \( \Theta(M) \) for every \( M < G \), one has \( Z(G/D) \neq \langle 1 \rangle \).

**Proof.** \( \Leftarrow \): We use induction on \( o(G) \) to show that every maximal subgroup of \( G \) is normal. Clearly, \( G \) cannot be simple. Let \( M < G \) and \( N \) be a minimal normal subgroup of \( G \) contained in \( M \). If \( (C/N, D/N) \) is a maximal pair
in \( \Theta(M/N) \), then by Lemma 2.1, \((C, D)\) is a maximal pair in \( \Theta(M) \) and \( Z(G/D) \neq \langle 1 \rangle \). This implies that \( Z((G/N)/(D/N)) \neq \langle 1 \rangle \). By induction, \( M/N \triangleleft G/N \) and so \( M \triangleleft G \). If \( M \) is core-free, then a maximal pair in \( \Theta(M) \) is of the form \((X, \langle 1 \rangle)\) so that \( Z(G) \neq \langle 1 \rangle \). Thus \( G = MZ(G) \) and therefore \( M \triangleleft G \). The result now follows.

The converse holds trivially.

**Corollary 4.2.** A group \( G \) is nilpotent \( \iff \Theta(M) \), for every \( M \triangleleft G \), contains a maximal pair \((C, D)\) such that \( G/D \) is nilpotent.

**Corollary 4.3.** Let \( M \triangleleft G \) and \( \Theta(M) \) contain a maximal pair such that \( G/D \) is nilpotent. Then \( M \triangleleft G \).

**Theorem 4.4.** A solvable group \( G \), whose order is divisible by at least two primes, is nilpotent \( \iff \) the index of each maximal pair in \( \Theta(M) \), for every \( M \triangleleft G \), is the same.

**Proof.** \( \Leftarrow \): We use induction on the order of \( G \). Let \( N \) be a minimal normal subgroup of \( G \); \( N \) is an elementary Abelian \( p \)-group for some prime \( p \). By induction it follows that \( G/N \) is nilpotent, and without loss in generality it may be assumed that \( N \) is the unique minimal normal subgroup of \( G \). If \( N \not\subseteq \Phi(G) \), then for some \( M \triangleleft G \), \( G = MN \) and \( M \) is core-free. Now, \((N, \langle 1 \rangle)\) is a maximal element in \( \Theta(M) \), and so, by hypothesis, the index of any maximal pair in \( \Theta(M) \) is a power of \( p \). If \( q \) is another divisor of the order of \( G \), then all the elements of order \( q \) in \( G \) cannot lie in \( M \), as they will generate a characteristic subgroup of \( G \). Let \( y \in G \setminus M \) such that \( o(y) = q \). Then \( \langle M, y \rangle = G \). If \( \langle y \rangle \triangleleft G \), then \( G = M \langle y \rangle \) and

\[
[G : M] = o(y) = q = o(N) = p^m
\]

for some \( m \geq 1 \), a contradiction to the fact that \( q \neq p \). Thus we may assume that \( \langle y \rangle \) is not normal in \( G \). So \((\langle y \rangle, \langle 1 \rangle) \in \Theta(M) \). Either \((\langle y \rangle, \langle 1 \rangle)\) is a maximal pair in \( \Theta(M) \), or if not, then \((\langle y \rangle, \langle 1 \rangle) \leq (X, \langle 1 \rangle)\), where \((X, \langle 1 \rangle)\) is a maximal pair in \( \Theta(M) \). But the index of \((X, \langle 1 \rangle)\) is divisible by \( q \), a contradiction. Thus \( N \subseteq \Phi(G) \), and therefore \( G \) is nilpotent.

The converse holds trivially.

We remark that Theorem 4.4 does not remain valid if in its statement one omits the condition that \( G \) is solvable. For example, if \( G \) is simple, then \((G, \langle 1 \rangle)\) is the unique minimal pair in \( \Theta(M) \) for every \( M \triangleleft G \), but \( G \) is not nilpotent unless \( o(G) \) is a prime.

**Corollary 4.5.** (i) A group \( G \) is nilpotent \( \iff \) the index of each maximal pair in \( \Theta(M) \) for every \( M \triangleleft G \) is the same and is equal to a prime.

(ii) A solvable group \( G \) is nilpotent \( \iff \Theta(M) \), for all \( M \triangleleft G \), contains exactly one maximal pair.

We omit the proof of the following result.
Proposition 4.6. A group $G$ is nilpotent $\iff$ for maximal pairs $(C, D)$ in $\Theta(M)$ and $(C_1, D_1)$ in $\Theta(M_1)$ with $C^G = C_1$, $M$ and $M_1$ are both normal in $G$.

For a group $G$ and a prime $p$, let $O^p(G)$ denote the subgroup generated by all the $p'$-elements of $G$.

Theorem 4.7. Let $G$ be a group in which $\Theta(M)$, for each $M \lhd G$, contains a maximal pair $(C, D)$ depending on $M$ such that

$$O^p(G/D) \subseteq C_{G/D}(C/D).$$

Then $G$ is $p$-nilpotent.

Proof. The assertion follows trivially if $G$ is simple. So, assume now that $G$ is not simple and let $N$ be a minimal normal subgroup of $G$. We use induction on $o(G)$. By inductive argument, it follows that $G/N$ is $p$-nilpotent; and one may also assume that $N$ is the unique minimal normal subgroup of $G$. If $N \subseteq \Phi(G)$, the assertion follows. So, let $G = MN$ for some core-free, maximal subgroup $M$ of $G$. By hypothesis, $\Theta(M)$ contains a maximal pair $(C, D)$ such that $C^G(G/D) \supseteq O^p(G/D)$. Then $D = \langle 1 \rangle$ and $N \subseteq C_G(O^p(G))$.

Let $R/N$ be a normal $p'$-complement in $G/N$. If $N$ is a $p'$-group, then $R$ is a normal $p'$-complement of $G$ and the assertion follows. Now, suppose that $p$ divides $o(N)$ Then, $N$ is an elementary Abelian $p$-group and is a normal Sylow $p$-subgroup of $R$. By the Schur-Zassenhaus theorem, $R = LQ$, where $LQ$ is a normal $p$-complement of $N$. It is now easy to see that $LQ$ is a normal $p$-complement in $G$. Therefore $G$ is $p$-nilpotent.

For a group $G$ and a prime $p$, let $Z_{p'}(G)$ denote the subgroup generated by all $x \in G$ which commute with every $p'$-element of $G$. We omit the proof of the following result whose proof is analogous to that of Theorem 4.7.

Proposition 4.8. A group $G$ is $p$-nilpotent if for every $M \lhd G$, $\Theta(M)$ contains a maximal pair $(C, D)$ such that $Z_{p'}(G/D) \neq \langle 1 \rangle$.

We now introduce two new characteristic subgroups of $G$ based on the concept of $\Theta$-pairs. Let

$$\mathcal{T} = \{M \lhd G : \exists \text{ a maximal pair } (C, D) \in \Theta(M), \ Z(G/D) = \langle 1 \rangle \}.$$ 

Define $A(G)$ to be the intersection of all the elements of $\mathcal{T}$. Again, define $\mathcal{T}$ exactly in the same way as $\mathcal{T}$ is defined but only replacing the condition that “$Z(G/D) = \langle 1 \rangle$” by the condition that “$Q^*(G) = \langle 1 \rangle$”. Define $B(G)$ to be the intersection of all the elements of $\mathcal{T}$. (If either of the families $\mathcal{T}$, $\mathcal{T}$ is empty, the corresponding subgroup is chosen to be $G$ itself).

The proof of the following result is left as an exercise.

Proposition 4.9. (i) $A(G)$ is a characteristic subgroup of $G$ which is nilpotent, and $A/\Phi(G) = Z_{\infty}(G/\Phi(G))$.

(ii) $B(G)$ is a characteristic subgroup of $G$, and $B(G)$ is supersolvable.
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SCHOOL OF COMPUTER AND SYSTEM SCIENCES, JAWAHARLAL NEHRU UNIVERSITY, NEW DELHI 110067, INDIA

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING, UNIVERSITY OF NEBRASKA–LINCOLN, LINCOLN, NEBRASKA 68588-0115