HALL SUBGROUP NORMALIZERS
AND CHARACTER CORRESPONDENCES IN \( M \)-GROUPS

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Abstract. Let \( H \) be a Hall subgroup of an \( M \)-group \( G \), and let \( N \) be its normalizer in \( G \). It is shown that the group \( N/H' \) is an \( M \)-group. This involves the construction of an explicit bijection between the sets of irreducible characters of \( G \) and of \( N \) with degrees coprime to the order of \( H \).

1. Introduction

Suppose \( G \) is a finite \( M \)-group: a group in which every irreducible complex character \( \chi \in \text{Irr}(G) \) can be written in the form \( \chi = \lambda^G \) for some linear character \( \lambda \) of some subgroup of \( G \). While it is far from true that every subgroup of \( G \) must be an \( M \)-group, there is some evidence that certain types of subgroups might necessarily be \( M \)-groups. It has been conjectured, for instance, that Hall subgroups of \( M \)-groups inherit the property. Perhaps it is even true that if \( H \) is a Hall subgroup of \( G \), then \( N = N_G(H) \) is automatically an \( M \)-group. (This would imply that \( H \) is an \( M \)-group since normal Hall subgroups of \( M \)-groups are known to be \( M \)-groups.)

In this paper, we obtain a much more modest result. If \( G, H, \) and \( N \) are as above, and \( H \) happens to be Abelian, then \( N \) is an \( M \)-group. In fact, we get the following.

**Theorem A.**  \( H \subseteq G \), where \( G \) is an \( M \)-group and \( H \) is a Hall subgroup. Then \( N/H' \) is an \( M \)-group, where \( N = N_G(H) \).

In above situation, it is easy to see that the irreducible characters of \( N/H' \) correspond precisely to those irreducible characters of \( N \) with degrees coprime to \( |H| \). Theorem A essentially says, therefore, that all of these characters are monomial.

In fact, we do not quite need \( G \) to be an \( M \)-group. For any set \( \pi \) of primes and any group \( G \), let us write \( \text{Irr}^\pi(G) \) to denote the subset of \( \text{Irr}(G) \) consisting of characters having \( \pi' \)-degree.
Theorem B. Let $G$ be solvable, let $\pi$ be a set of primes, and write $N = N_G(H)$, where $H$ is a Hall $\pi$-subgroup of $G$. If every character in $\text{Irr}^\pi(G)$ is monomial, then every character in $\text{Irr}^\pi(N)$ is also monomial.

Since $M$-groups are solvable, Theorem A is an immediate consequence of Theorem B.

By a result of T. R. Wolf [5], if $G$, $\pi$, and $N$ are as in Theorem B, then (without any monomiality assumption) we have $|\text{Irr}^\pi(G)| = |\text{Irr}^\pi(N)|$. (See also [2, Theorem 2.9].) Wolf's result is crucial in our proof of Theorem B and in return, our argument strengthens Wolf's theorem by constructing a canonical bijection $\text{Irr}^\pi(G) \rightarrow \text{Irr}^\pi(N)$ in the case that $\text{Irr}^\pi(G)$ consists entirely of monomial characters.

Theorem C. Let $(G)$, $\pi$, and $N$ be as in Theorem B and suppose every $\chi \in \text{Irr}^\pi(G)$ is monomial. Then there exists a bijection $\beta: \text{Irr}^\pi(G) \rightarrow \text{Irr}^\pi(N)$ such that if $\chi \in \text{Irr}^\pi(G)$, then $\beta(\chi) = (\lambda_{N \cap K})^N$ for every choice of a pair $(K, \lambda)$ with linear $\lambda \in \text{Irr}(K)$, such that $\lambda^G = \chi$ and $H \subseteq K \subseteq G$. Furthermore, for every $\chi \in \text{Irr}^\pi(G)$, there is such a pair $(K, \lambda)$.

Of course, from the statement of Theorem C, it is immediate that the bijection $\beta$ is unique. It follows, for instance, that if $\sigma$ is any automorphism of the complex numbers, then $\beta(\chi^\sigma) = \beta(\chi)^\sigma$, and thus the fields $\mathbb{Q}(\chi)$ and $\mathbb{Q}(\beta(\chi))$ are the same.

In the general case of Wolf's theorem, where no monomiality assumption is made, the equality of the numbers $|\text{Irr}^\pi(G)|$ and $|\text{Irr}^\pi(N)|$ is established without constructing a specific bijection. Indeed, in general, no well-behaved bijection exists. If $G = \text{GL}(2, 3)$ and $\pi = \{3\}$, for instance, then all members of $\text{Irr}^\pi(N)$ are rational valued, and yet $\text{Irr}^\pi(G)$ contains irrational characters. (It should be noted, however, that under suitable oddness conditions, but without assuming monomiality, one can get a canonical bijection. The case of this where $|\pi| = 1$ appears as [1, Theorem 10.9].)

The key to the proof of Theorem C, and so to all of the results of this paper is the following.

Lemma D. Let $K \subseteq G$ and suppose $\lambda$ is a linear character of $K$ such that $\lambda^G$ is irreducible. If $S \subseteq K$ is pronormal (i.e. $S$ and $S^k$ are conjugate in $(S, S^k)$, for all $k \in K$), then $(\lambda_{K \cap N})^N$ is irreducible, where $N = N_G(S)$.

We will use Lemma D in the case where $G$ is solvable and $S$ is a Hall $\pi$-subgroup of $K$. Note that if $G$ is arbitrary and $S$ is a Sylow subgroup of $K$, then $S$ is pronormal in $K$ and the lemma would apply. In this form, with $S$ a Sylow $p$-subgroup, Lemma D was first proved by G. R. Robinson [4] using module-theoretic methods in characteristic $p$. (Actually, for Robinson's result, the condition that $\lambda$ is linear is unnecessarily strong. He needed only that $\lambda(1) \leq (p - 1)/2$. I am doubly indebted to Robinson: First, for challenging me to find a character theoretic proof of his result, and second, for suggesting, over...
a year later, that this result (which I had forgotten) might be useful for solving the problem about $M$-groups on which I was then working.

The $M$-group result, Theorem A, is a generalization of, and was motivated by, a theorem of T. Okuyama [3] which asserts that $N/H$ is an $M$-group when $H$ is a Sylow $p$-subgroup of an $M$-group $G$ and $N = N_G(H)$. Okuyama's proof depends on characteristic $p$ module theory, and so does not readily generalize to the case where $H$ is a Hall $\pi$-subgroup.

2. Proofs

We begin with a lemma which is an immediate consequence of Mackey's theorem. If $K \subseteq G$ and $\alpha$ is a character of $K$, then for $g \in G$, we write $\alpha^g$ to denote the character of $K^g$ defined by $\alpha^g(k^g) = \alpha(k)$, for $k \in K$.

2.1. Lemma. Let $K \subseteq G$ and suppose $\alpha \in \text{Irr}(K)$. Then
(a) $\alpha^g$ is irreducible iff for all $g \in G - K$, we have
$$[\alpha_{K \cap K^g}, \alpha_{K \cap K^g}] = 0.$$ 
(b) If $\beta \in \text{Irr}(L)$ with $L \subseteq G$ and $\alpha^g$ and $\beta^g$ are both irreducible, then $\alpha^g = \beta^g$ iff
$$[\beta_{K \cap L^g}, \alpha_{K \cap L^g}] \neq 0$$
for some $g \in G$.

Proof. We have
$$[\alpha^g, \alpha^g] = [(\alpha^g)_K, \alpha] = \sum_{t \in T} (\alpha^t_{K \cap K^t})_K = \sum_{t \in T} [\alpha^t_{K \cap K^t}, \alpha_{K \cap K^t}],$$
where $T$ is a set of representatives for the double cosets $HgH$ in $G$. Thus
$$[\alpha^g, \alpha^g] = 1 + \sum_{t \in T_0} [\alpha^t_{K \cap K^t}, \alpha_{K \cap K^t}],$$
where $T_0 = T - K$. Therefore, $\alpha^g$ is irreducible iff
$$[\alpha^t_{K \cap K^t}, \alpha_{K \cap K^t}] = 0$$
for all $t \in T_0$. Since we are free to choose double coset representatives as we will, any element $g \in G - K$ can appear in $T_0$, and part (a) follows.

Similarly, using Mackey's theorem, we have
$$[\beta^g, \alpha^g] = [(\beta^g)_K, \alpha] = \sum_{s \in S} [\beta^s_{K \cap L^s}, \alpha_{K \cap L^s}],$$
where $S$ is any set of representatives for the double cosets $LgK$. Thus $\beta^g = \alpha^g$ iff
$$[\beta^s_{K \cap L^s}, \alpha_{K \cap L^s}] \neq 0$$
for some $s \in S$. Since any element of $G$ can be in $S$, part (b) is complete. 

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2.2. **Lemma.** Let \( S \subseteq G \) be a pronormal subgroup and suppose that \( \lambda, \mu \) are linear characters of \( G \). Then \( \lambda = \mu \) iff \( \lambda_N = \mu_N \), where \( N = N_G(S) \).

**Proof.** Since "only if" is trivial, assume \( \lambda_N = \mu_N \) and let \( M = \ker(\lambda \mu) \). We must show that \( M = G \).

Now \( N \subseteq M < G \) and the "Frattini argument" proves the result. ■

Our next result contains Lemma D of the Introduction.

2.3. **Lemma.** Let \( K \subseteq G \) and suppose \( \lambda^G = \chi \in \text{Irr}(G) \), where \( \lambda \) is a linear character of \( K \). Let \( S \) be a pronormal subgroup of \( K \) and write \( N = N_G(S) \). Then

(a) \( (\lambda_{K \cap N})^N \) is irreducible.

Suppose all that \( \mu^G = \psi \in \text{Irr}(G) \) where \( \mu \) is a linear character of some subgroup \( L \) which contains \( S \) pronormally.

(b) If \( (\lambda_{K \cap N})^N = (\mu_{L \cap N})^N \), then \( \chi = \psi \).

**Proof.** Let \( U = K \cap N \). By 2.1 (a), it will follow that \( (\lambda_U)^N \) is irreducible if we can show that

\[
\left[ \lambda_{U \cap U^n}, \lambda_{U \cap U^n} \right] = 0
\]

for every element \( n \in N - U \). In other words, we need

\[
\lambda_{U \cap U^n} \neq \lambda_{U \cap U^n}.
\]

Now

\[
U \cap U^n = N \cap (K \cap K^n) = N_{K \cap K^n}(S)
\]

and so by 2.2, it suffices to show that

\[
\lambda_{K \cap K^n} \neq \lambda_{K \cap K^n}.
\]

This follows, however, by 2.1 (a), since \( \lambda^G \) is irreducible and \( n \notin K \) (because \( n \notin U \)). This proves (a).

In this situation of (b), write \( V = L \cap N \). Since \( (\lambda_U)^N = (\mu_V)^N \), Lemma 2.1 (b) gives

\[
\left[ \mu_{U \cap V^n}, \lambda_{U \cap V^n} \right] \neq 0
\]

for some \( n \in N \), and so

\[
\mu_{U \cap V^n} = \lambda_{U \cap V^n}.
\]

Since

\[
U \cap V^n = N \cap (K \cap L^n) = N_{K \cap L^n}(S),
\]

we deduce from 2.2 that

\[
\mu_{K \cap L^n} = \lambda_{K \cap L^n}
\]

and thus \( \psi = \chi \) by 2.1 (b). ■

**Proof of Theorem C.** Let \( \chi \in \text{Irr}^*(G) \). Then \( \chi \) is monomial and we can write \( \chi = \lambda^G \) for some linear character of a subgroup \( K \subseteq G \). Also, \(|G:K| = \chi(1)|\) is a \( \pi' \)-number and so, since \( G \) is solvable, \( K \) must contain some \( G \)-conjugate
of the Hall \( \pi \)-subgroup \( H \). Replacing \( K \) by a conjugate, we may assume that \( K \supseteq H \).

Since the Hall subgroup \( H \) is pronormal in \( K \), Lemma 2.3 applies and we conclude that \( \phi = (\lambda_{K \cap N})^N \) is irreducible. Since \( \phi(1) = |N : K \cap N| \) which divides \( |N : H| \), we have \( \phi \in \text{Irr}^\pi(N) \). We shall say in this situation, that \( \phi \) arises from \( \chi \), but we do not yet claim that \( \phi \) is the only element of \( \text{Irr}^\pi(N) \) which arises from \( \chi \). By 2.3 (b), however, we know that \( \phi \) cannot arise from any element \( \psi \in \text{Irr}^\pi(G) \) with \( \psi \neq \chi \).

If \( \mathcal{A} \subseteq \text{Irr}^\pi(N) \) is the set of characters which arise from some \( \chi \in \text{Irr}^\pi(G) \), then we have a surjective map \( \alpha: \mathcal{A} \rightarrow \text{Irr}^\pi(G) \), defined by \( \phi \mapsto \chi \) if \( \phi \) arises from \( \chi \). Thus \( |\mathcal{A}| \geq |\text{Irr}^\pi(G)| \).

Since \( |\text{Irr}^\pi(G)| = |\text{Irr}^\pi(N)| \), by the result of Wolf referred to in the introduction, it follows that \( \mathcal{A} = \text{Irr}^\pi(N) \) and that \( \alpha \) is injective. The bijection \( \beta \) which we seek is just the inverse of \( \alpha \).

Theorem B is now immediate and to prove Theorem A, it suffices to observe that if \( N = N_G(H) \), where \( H \) is a Hall \( \pi \)-subgroup of \( G \), and \( \phi \in \text{Irr}(N) \), then \( \phi \) has \( \pi' \)-degree iff \( \phi_H \) has linear constituents. This occurs iff the derived subgroup \( H' \subseteq \ker \phi \).

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**References**


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