TOPOLOGICAL RATIONALIZATION
OF A CLASS OF MEROMORPHIC FUNCTIONS

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Abstract. We prove that a class of germs at $0 \in \mathbb{C}^2$ of meromorphic functions can be transformed, via homeomorphisms, into rational functions.

0. Introduction

We prove in this note that if $M(x, y)$ is the germ at $0 \in \mathbb{C}^2$ of a pure meromorphic function whose blow-up $\tilde{M} = M \circ E$, where $E$ is the blow-up of $\mathbb{C}^2$ at 0, has no essential singularities on $E^{-1}(0)$, then $M$ is topologically conjugate to a rational function $R(x, y)$. That is, there is a homeomorphism $\Phi$ of a neighborhood of $0 \in \mathbb{C}^2$ such that $M \circ \Phi = R$.

D. Cerveau and J. F. Mattei have studied [1] finite determination for multiform functions. In particular, they prove rationalization for germs of meromorphic functions with critical set reduced to $\{0\}$. They also show that germs at $0 \in \mathbb{C}^2$ of holomorphic functions are conjugate to polynomials.

We begin this note by recalling briefly the blow-up of $\mathbb{C}^2$ at 0, and then we define the class of meromorphic functions that we will be dealing with. Next, we prove an intermediate result: a particular conjugation of germs of analytic varieties. Finally, we state and prove the main result.

1. Definitions

In this section, we recall very briefly the blow-up of $\mathbb{C}^2$ at 0 (see [4]) and define a class of meromorphic functions, called here class A. Some examples will follow.

The blow-up of $\mathbb{C}^2$ at 0 is the subset of $\mathbb{C}^2 \times \mathbb{CP}^1$, where $\mathbb{CP}^1$ is the complex projective space of dimension 1, given as

$$\mathbb{C}^2_0 = \{(p, [p]) \in (\mathbb{C}^2 - \{0\}) \times \mathbb{CP}^1\} \cup \{0\} \times \mathbb{CP}^1,$$

where $[\cdot] : \mathbb{C}^2 - \{0\} \rightarrow \mathbb{CP}^1$ is the quotient map. It is shown, in [4] for example, that $\mathbb{C}^2_0$ is a two-dimensional complex manifold. Moreover, $\mathbb{C}^2_0$ is covered by

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two charts \((x, t, (s, y))\) \((t, s)\) are the canonical coordinates of \(\mathbb{CP}^1\) related by
\[
s = \frac{1}{t} \quad \text{and} \quad y = xt.
\]

The first projection \(E: \mathbb{C}^2_0 \hookrightarrow \mathbb{C}^2\), called the blow-up map, is given in the charts \((x, t), (s, y)\) by
\[
E(x, t) = (x, xt) \quad \text{and} \quad E(s, y) = (sy, y).
\]

\(E\) is a biholomorphism from \(\mathbb{C}^2_{0} - E^{-1}(0)\) onto \(\mathbb{C}^2 - \{0\}\). The exceptional divisor \(E^{-1}(0)\) is identified with \(\mathbb{CP}^1\).

A meromorphic function \(M\) on a two-dimensional complex manifold \(\mathcal{M}\) is said to have an essential singularity at \(p_0 \in \mathcal{M}\) if, near \(p_0\), the function \(M\) is expressed as a quotient \(\alpha/\beta\), with \(\alpha, \beta\) germs at \(p_0\) of coprime holomorphic functions and \(\alpha(p_0) = \beta(p_0) = 0\). Equivalently, \(M\) has an essential singularity at \(p_0\) if, in a neighborhood of \(p_0\), all the level sets of \(M\) pass through \(p_0\). In this case, as in [1], we say that \(M\) is a pure meromorphic function at \(p_0\).

**Definition 1.1.** A germ at \(0 \in \mathbb{C}^2\) of a pure meromorphic function \(M(x, y)\) is said to be in the class \(A\) if the map \(\widetilde{M} = M \circ E: \mathbb{C}^2_0 \hookrightarrow \mathbb{C}\), defined in a neighborhood of \(E^{-1}(0) = \mathbb{CP}^1 \subset \mathbb{C}^2_0\), has no essential singularities.

**Remark 1.1.** It is easy to see, as a consequence of the desingularization of foliations defined by level sets of meromorphic functions (see [3] for example), that if \(M\) is the germ at \(0 \in \mathbb{C}^2\) of a pure meromorphic function, then there is a proper holomorphic map \(\Pi: \mathcal{M} \hookrightarrow \mathbb{C}^2\), obtained by composition of blow-ups, such that the lift \(M \circ \Pi\) of \(M\) to \(\mathcal{M}\) has no essential singularities. Hence, the class \(A\) is the set of all meromorphic functions for which \(\mathcal{M} = \mathbb{C}^2\).

**Example 1.** The function
\[
M(x, y) = \frac{x^3 + xy^2 + \alpha(x, y)}{y^3 + \beta(x, y)},
\]
where \(\alpha, \beta\) are holomorphic functions of order \(\geq 4\), is in the class \(A\). The blow-up of \(M\) is given in the charts \((x, t), (s, y)\) respectively by
\[
\widetilde{M}(x, t) = \frac{1 + t^2 + \left(\alpha(x, tx)/x^3\right)}{y^3 + \left(\beta(x, tx)/x^3\right)}, \quad \widetilde{M}(s, y) = \frac{s^3 + s + \left(\alpha(sy, y)/y^3\right)}{1 + \left(\beta(sy, y)/y^3\right)}.
\]
So \(\widetilde{M}\) has no essential singularities on \(\mathbb{CP}^1 = \{x = 0\} \cup \{y = 0\}\).

**Example 2.** The meromorphic function \(M(x, y) = (y^2 + x^3)/xy\) is not in the class \(A\). Its blow-up \(\widetilde{M}(x, t) = (t^2 + x)/t\) has an essential singularity at \((x = 0, t = 0)\). A blow-up of \(\mathbb{C}^2_0\) at \((x = 0, t = 0)\) will produce a meromorphic function with no essential singularities.
Remark 1.2. The functions of the form $x^r/y^s$, where $r$, $s$ are positive integers and $r > s$, are not in the class $A$. However, we characterized in [3] all meromorphic functions that are conjugate to the rational functions of the form $R(x^r/y^s)$, where $R$ is a rational function on $\mathbb{C}$.

The following proposition characterizes the class $A$.

**Proposition 1.1.** Let $M(x, y) = \alpha(x, y)/\beta(x, y)$ be the germ at $0 \in \mathbb{C}^2$ of a pure meromorphic function and let

$$\alpha(x, y) = \sum_{i \geq n} P_i(x, y)$$

and

$$\beta(x, y) = \sum_{j \geq m} Q_j(x, y)$$

be the Taylor expansions of $\alpha$ and $\beta$ ($P_i$ and $Q_j$ are homogeneous polynomials with respective degrees $i$ and $j$). Then $M$ is in the class $A$ if and only if $n = m$ and $P_n$, $Q_n$ are coprime.

**Proof.** We first show that if $n = m$ and $P_n$, $Q_n$ are coprime, then $M$ is in the class $A$. The blow-up $\widetilde{M} = M \circ E$ of $M$ is given in the charts $(x, t)$, $(s, y)$ by

$$\widetilde{M}(x, t) = \frac{P_n(1, t) + xP_{n+1}(1, t) + \ldots}{Q_n(1, t) + xQ_{n+1}(1, t) + \ldots},$$

$$\widetilde{M}(s, y) = \frac{P_n(s, 1) + yP_{n+1}(s, 1) + \ldots}{Q_n(s, 1) + yQ_{n+1}(s, 1) + \ldots}.$$

Since $P_n(1, t)$ and $Q_n(1, t)$ (resp. $P_n(s, 1)$ and $Q_n(s, 1)$) have no common root, then $\widetilde{M}$ has no essential singularities on $E^{-1}(0)$.

Conversely, let us show that if $M$ is in the class $A$, then $n = m$ and $P_n$, $Q_n$ are coprime. By contradiction, first if $n = m + k > m$, then

$$\widetilde{M}(x, t) = \frac{x^k(P_{m+k}(1, t) + xP_{m+k+1}(1, t) + \ldots)}{Q_m(1, t) + xQ_{m+1}(1, t) + \ldots},$$

$$\widetilde{M}(s, y) = \frac{y^k(P_{m+k}(s, 1) + yP_{m+k+1}(s, 1) + \ldots)}{Q_m(s, 1) + yQ_{m+1}(s, 1) + \ldots}.$$

So, if $t_0$ (resp. $s_0$) is a root of $Q_m(1, t)$ (resp. $Q_m(s, 1)$), then the point $(x = 0, t = t_0)$ (resp. $(s = s_0, y = 0)$) would be an essential singularity of $\widetilde{M}$. The second eventuality, $n = m$ and $P_n$, $Q_n$ are not coprime, also cannot occur. Indeed, $P_n(1, t)$ and $Q_n(1, t)$ would have a common root $t_0$, and then

$$\widetilde{M}(x, t) = \frac{P_n(1, t) + xP_{n+1}(1, t) + \ldots}{Q_n(1, t) + xQ_{n+1}(1, t) + \ldots} = \frac{(t-t_0)P_n^*(1, t) + xP_{n+1}(1, t) + \ldots}{(t-t_0)Q_n^*(1, t) + xQ_{n+1}(1, t) + \ldots}$$

would have an essential singularity at $(x = 0, t = t_0)$. 

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2. AN INTERMEDIATE RESULT

We prove in this section the existence of a particular homeomorphism between two germs of analytic varieties that coincide to high order at $0 \in \mathbb{C}^2$. We will denote by $C_c$ the complex line in $\mathbb{C}^2$ given by $y = cx$, where $c$ is a complex number.

**Proposition 2.1.** Let $f(x, y)$ be the germ at $0 \in \mathbb{C}^2$ of a holomorphic function and let $f_k(x, y)$ be its $k$th jet. Define

$$V(f) = \{(x, y); f(x, y) = 0\} \quad \text{and} \quad V(f_k) = \{(x, y); f_k(x, y) = 0\}.$$ 

Then, for $k$ large enough, there is a germ of a homeomorphism $\Phi$ at $0 \in \mathbb{C}^2$ such that

$$\Phi(V(f)) = V(f_k) \quad \text{and} \quad \Phi(C_c) = C_c$$

for every $c \in \mathbb{C}$.

The proof of this proposition is based upon some lemmas.

**Lemma 2.1.** Consider a family $D(p_i, r_i), i = 1, \ldots, N$, of $N$ disjoint discs in the plane $\mathbb{R}^2$, with respective centers $p_i$ and radii $r_i$. For every $i$, let $q_i$ be a point in $D(p_i, r_i)$. Then there is a homeomorphism $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that:

$$\phi(p) = p \quad \text{if} \quad p \notin \bigcup_{i=1}^{N} D(p_i, r_i)$$

$$\phi(p_i) = q_i \quad \text{for} \quad i = 1, \ldots, N.$$ 

We leave the elementary proof of this result to the reader.

**Lemma 2.2.** Let $f(x, y)$ be the germ at $0 \in \mathbb{C}^2$ of an irreducible holomorphic function of order $n$. Then, we can assume, after a linear change of coordinates, that

$$f(x, y) = y^n + P_{n+1}(x, y) + \ldots + P_{n+k}(x, y) + \ldots ,$$

where the $P_{n+j}$'s are homogeneous polynomials of degree $n + j$. Furthermore, if $n > 1$, then there is a $k$ such that $P_{n+k}(x, 0)$ is not identically zero.

**Proof.** The Taylor expansion of $f$ is $f(x, y) = \sum_{j \geq n} P_j(x, y)$, where the $P_j$'s are homogeneous polynomials of degree $j$. Recall from [4] that the irreducibility of $f$ implies that the tangent cone of the variety $\{(x, y); f(x, y) = 0\}$ is reduced to a point. That is, the set of points on $\mathbb{CP}^1$ on which $P_n(1, t)$ or $P_n(s, 1) = 0$ is reduced to one point. We can assume, after a linear change of coordinates if necessary, that $s = 0$ is not a root of $P_n(s, 1)$. So, the irreducibility of $f$ implies that $P_n(1, t) = (t - t_0)^i$ for some $t_0 \in \mathbb{C}$ and $i \leq n$. By a linear change of coordinates, we can assume $t_0 = 0$. Now, if $n > i$, then $P_n(s, 1) = s^{n-i}$ would have a root at $s = 0$, and this is impossible. So $i = n$ and $P_n(x, y) = y^n$. Finally, if $n > 1$ and $P_{n+j}(x, 0) = 0$ for every...
$j \geq 1$, then $y$ would be a factor of $f$ and $f$ would be reducible. The lemma is proved. \hfill \Box

**Lemma 2.3.** Let $f(x, y) = y^n + P_{n+1}(x, y) + \ldots + P_{n+k}(x, y) + \ldots$, be an irreducible holomorphic function and $k$ be the first integer such that $P_{n+k}(x, 0) \neq 0$. Then there is a positive constant $K$ such that, for every $c \in \mathbb{C}$, $0 < |c| < K$, the variety $V(f) = \{f = 0\}$ intersects the line $C_c$ at $0$ and $k$ other points $p^1(c), \ldots, p^k(c)$ which vary continuously with $c$ and $p^1(0) = \ldots = p^k(0) = 0$.

*Proof.* Let $c$ be fixed in $\mathbb{C}$. Then

$$V(f) \cap C_c = \{(x, cx) \in \mathbb{C}^2 ; f(x, cx) = 0\},$$

i.e., $V(f) \cap C_c$ is given by the equation

$$x^n(c^n + P_{n+1}(1, c)x + \ldots + P_{n+k}(1, c)x^k + \ldots) = 0.$$

Since $P_{n+j}(1, 0) = 0$ if $j < k$ and $P_{n+k}(1, 0) \neq 0$, then similar arguments as that used in the proof of Theorem 4.13 of Chapter 2 in [2] and the irreducibility of $f$ show that the equation

$$c^n + P_{n+1}(1, c)x + \ldots + P_{n+k}(1, c)x^k + \ldots = 0$$

has $k$ distinct and continuous roots $p^1(c), \ldots, p^k(c)$, provided that $c \neq 0$ is sufficiently small and $p^1(0) = \ldots = p^k(0) = 0$. \hfill \Box

*Proof of the proposition.* We start by proving the proposition in the case where the tangent cone of $V(f)$ is reduced to one point. Let $K > 0$ be such that, for every $c \in \mathbb{C}$, $0 < |c| < K$, we have

$$V(f) \cap C_c = \{0, p^1(c), \ldots, p^k(c)\}$$

(see Lemma 2.3). Consider the homotopy

$$g_t = f + t(f_1 - f), \quad 0 \leq t \leq 1,$$

where $f_1$ is the $l$th jet of $f$. Since $f$ and $g_t$ coincide to order $l$ at $0$, then, for $l$ sufficiently large, we also have

$$V(g_t) \cap C_c = \{0, p^1_t(c), \ldots, p^k_t(c)\},$$

and the functions $p^j_t(c)$'s are continuous in $c$ and $t$. So, for every $j$, we can find a continuous function $r_j(c)$, $r_j(c) > 0$ and $r_j(0) = 0$ such that

$$\{p^j_t(c), 0 \leq t \leq 1\} \subset D\left(p^j_t(c), r_j(c)\right)$$

and

$$D\left(p^j_t(c), r_j(c)\right) \cap D\left(p^i_t(c), r_i(c)\right) = \emptyset \quad \text{if } j \neq i.$$
Next, for \( c \) fixed, \( |c| < K \), consider, as in Lemma 2.1, a homeomorphism \( \phi_c \) of the complex line \( \mathbb{C}_c \) such that

\[
\phi_c(p) = p \quad \text{if } p \in \mathbb{C}_c - \bigcup_{i=1}^{k} D\left(p^i(c), r_i(c)\right),
\]
\[
\phi_c\left(p^i(c)\right) = p^i_1(c) \quad \text{for } i = 1, \ldots, k.
\]

Note that if \( U \) is a small neighborhood of \( 0 \in \mathbb{C}^2 \) and \( c \) is such that \( K/2 < |c| < K \), then \( \phi_c(p) = p \) if \( p \in U \cap \mathbb{C}_c \). So, we can define a homeomorphism \( \Phi: (\mathbb{C}^2, 0) \leftrightarrow (\mathbb{C}^2, 0) \) by

\[
\Phi(p) = p \quad \text{if } p \notin \{(x, y) : y = cx \text{ and } |c| < K\}
\]
\[
\Phi(p) = \phi_c(p) \quad \text{if } p \in \mathbb{C}_c \text{ and } |c| < K.
\]

In this case \( \Phi \) is the desired homeomorphism of the proposition.

In the general case, the tangent cone of \( V(f) \) is not necessarily reduced to a point; we decompose

\[
V(f) = V^1 \cup \ldots \cup V^N,
\]

where each \( V^j \) has a tangent cone reduced to a point \( a_j \). Similarly, we decompose

\[
V(f_k) = V^1_k \cup \ldots \cup V^N_k,
\]

where the tangent cone of \( V^j_k \) is the point \( a_j \). By reducing the neighborhood \( U \) of \( 0 \in \mathbb{C}^2 \), we can assume that, for every \( j \), there is a \( K_j \) such that \( V^j \cap \mathbb{C}_c = \{0\} \) if \( |c - a_j| \geq K_j \) and the discs \( D(a_j, K_j) \) are mutually disjoint. Now we can define a homeomorphism \( \Phi \) as follows:

\[
\Phi(p) = p \quad \text{if } p \notin \bigcup_{j=1}^{N} \{(x, y) : y = cx \text{ and } |c - a_j| \geq K_j\}
\]
\[
\Phi(p) = \Phi_j(p) \quad \text{if } p \in \{(x, y) : y = cx \text{ and } |c - a_j| < K_j\},
\]

where \( \Phi_j \) is a homeomorphism, as in the previous case, corresponding to \( V^j \). The proposition is therefore proved.

3. Statement and proof of the main result

**Theorem.** Let \( M(x, y) \) be the germ at \( 0 \in \mathbb{C}^2 \) of a meromorphic function in the class \( A \). Then \( M \) is topologically conjugate to a rational function.

The proof of this result needs two lemmas. First, we adopt the following notation: if \( V(f) \) is the germ of the variety \( \{f = 0\} \), we will denote by \( \tilde{V}(f) \) the closure in \( \mathbb{C}^2_0 \) of \( E^{-1}(V(f) - \{0\}) \). Also, we will say that two varieties \( V(f) \) and \( V(g) \) coincide to order \( k \) if their defining functions \( f \) and \( g \) coincide to order \( k \) at \( 0 \).
Lemma 3.1. Let $f$ and $g$ be germs of irreducible holomorphic functions of order $j$ at $0 \in C^2$. Suppose that $f$ and $g$ coincide to order $k$ ($k > j$) at $0$. Then $\widetilde{V}(f)$ and $\widetilde{V}(g)$ coincide to order $k-j$ on $E^{-1}(0)$.

Proof. Without loss of generality, we can assume that the tangent cone of $V(f)$ or $V(g)$ is not the point $(s = 0, y = 0)$. So the varieties $\widetilde{V}(f)$ and $\widetilde{V}(g)$ admit, respectively, $\tilde{f}(x, t) = f(x, tx)/x^j$ and $\tilde{g}(x, t) = g(x, tx)/x^j$ as defining functions. Therefore, it follows from the fact that $f$ and $g$ coincide to order $k$ at $0$, that $\tilde{f}(x, t)$ and $\tilde{g}(x, t)$ coincide to order $k-j$ on $E^{-1}(0)$.

Lemma 3.2. Suppose that $A, B$ are germs of holomorphic functions that coincide to order $k_1$ and $V(f), V(g)$ are germs of analytic varieties that coincide to order $k_2$ at $0 \in C^2$. Then the images via the maps $F(x, y) = (x, A(x, y))$ and $G(x, y) = (x, B(x, y))$ of, respectively, $V(f)$ and $V(g)$ coincide to an order $\nu(k_1, k_2)$, where $\nu$ increases with $k_1, k_2$.

Proof. Let $p$ be the order of $f$ and $g$ at $0$. Then a parametrization of $V(f)$ (resp. $V(g)$) is (see [4])

$$
\begin{align*}
(x = f^p, y = a(t)) & \quad \text{(resp. } (x = f^p, y = b(t))) \end{align*}
$$

where $a, b$ are holomorphic functions of order $\geq p + 1$. Moreover, it follows from the fact that $f$ and $g$ coincide to order $k_2$ that $a$ and $b$ coincide to an order $\eta(k_2)$, where $\eta$ increases with $k_2$. The parametrization of $F(V(f))$ (resp. $G(V(g))$) is

$$
\begin{align*}
(x = f^p, y = A(t^p, a(t))) & \quad \text{(resp. } (x = f^p, y = B(t^p, b(t)))) \end{align*}
$$

The conclusion follows from the fact that $A, B$ coincide to order $k_1$ and $a, b$ coincide to order $\eta(k_2)$. □

Proof of the theorem. We write

$$
M(x, y) = \frac{a(x, y)}{\beta(x, y)} = \frac{\sum_{j \geq n} P_j(x, y)}{\sum_{j \geq n} Q_j(x, y)},
$$

where $P_j, Q_j$ are homogeneous polynomials of degree $j$ and $P_n, Q_n$ coprime. For $k > n$, define the rational function $R_k$ by

$$
R_k(x, y) = \frac{a_k(x, y)}{\beta_k(x, y)} = \frac{\sum_{j = n}^k P_j(x, y)}{\sum_{j = n}^k Q_j(x, y)}.
$$

Note that the blow-up $\widetilde{M}$ of $M$ coincides on $E^{-1}(0)$ with the blow-up $\widetilde{R}_k$ of $R_k$, it is

$$
\widetilde{M}(0, t) = \widetilde{R}_k(0, t) = \frac{P_n(1, t)}{Q_n(1, t)}.
$$
Consider the maps $\mathcal{P}: \mathbb{C}^2_0 \to \mathbb{C}^2_0$ and $\mathcal{P}_k: \mathbb{C}^2_0 \to \mathbb{C}^2_0$, defined in the charts $(x, t), (s, y)$ by

$$\mathcal{P}(x, t) = \left(x, \tilde{M}(x, t)\right) \quad \text{and} \quad \mathcal{P}_k(s, y) = \left(\frac{1}{\tilde{M}(sy, 1/s)}, sy\tilde{M}(sy, 1/s)\right),$$

$$\mathcal{P}_k(x, t) = \left(x, \tilde{R}_k(x, t)\right) \quad \text{and} \quad \mathcal{P}_k(s, y) = \left(\frac{1}{\tilde{R}_k(sy, 1/s)}, sy\tilde{R}_k(sy, 1/s)\right).$$

Note that the meromorphic map $\mathcal{P}$ (resp. $\mathcal{P}_k$) is holomorphic outside the variety $\{\tilde{M} = \infty\}$ (resp. $\{\tilde{R}_k = \infty\}$). Since the jacobian of $\mathcal{P}$ (resp. $\mathcal{P}_k$), in the chart $(x, t)$, vanishes only on the variety

$$X = \left\{(x, t); \frac{\partial \tilde{M}}{\partial t} = 0\right\} \quad \text{(resp. } X_k = \left\{(x, t); \frac{\partial \tilde{R}_k}{\partial t} = 0\right\}),$$

then $\mathcal{P}$ (resp. $\mathcal{P}_k$) is a covering outside $X$ (resp. $X_k$) whose number of sheets is the multiplicity of $\tilde{M}(0, t)$. Since

$$E(X) = \left\{(x, y); \beta \frac{\partial \alpha}{\partial y} - \alpha \frac{\partial \beta}{\partial y} = 0\right\}$$

and

$$E(X_k) = \left\{(x, y); \beta_k \frac{\partial \alpha_k}{\partial y} - \alpha_k \frac{\partial \beta_k}{\partial y} = 0\right\}$$

and the functions $\beta \frac{\partial \alpha}{\partial y} - \alpha \frac{\partial \beta}{\partial y}, \beta_k \frac{\partial \alpha_k}{\partial y} - \alpha_k \frac{\partial \beta_k}{\partial y}$ coincide to order $2k - 1$, then $X$ and $X_k$ coincide to an order $\eta(k)$ (see Lemma 3.1). Therefore the varieties $W = \mathcal{P}(X)$ and $W_k = \mathcal{P}_k(X_k)$ coincide to a certain order $\mu(k)$, and $\mu$ increases with $k$ (see Lemma 3.2).

Now, it follows from Proposition 2.1 that if $k$ is large enough, then there is a homeomorphism $\Psi: (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that

$$\tilde{\Psi}(W) = W_k \quad \text{and} \quad \tilde{\Psi}\left(\{(x, t); t = t_0\}\right) = \{(x, t); t = t_0\},$$

for every constant $t_0$. Next, since $\mathcal{P}$ (resp. $\mathcal{P}_k$) maps the analytic varieties $\tilde{M}^{-1}(\tilde{M}(x_0, t_0))$ (resp. $\tilde{R}_k^{-1}(\tilde{R}_k(x_0, t_0))$) into the complex line $t = t_0$, there is a homeomorphism $\Phi$ such that the diagram

$$\begin{array}{ccc}
\mathbb{C}^2_0, X & \xrightarrow{\Phi} & \mathbb{C}^2_0, X_k \\
\downarrow & & \downarrow \mathcal{P}_k \\
\mathbb{C}^2_0, W & \xrightarrow{\tilde{\Psi}} & \mathbb{C}^2_0, W_k
\end{array}$$

commutes, i.e., $\tilde{\Phi}$ conjugates the foliation defined by the level sets of $\tilde{M}$ to that defined by the level sets of $\tilde{R}_k$. Since $\tilde{M} = \tilde{R}_k$ on $E^{-1}(0)$ and $\tilde{\Psi}$ is the identity on $E^{-1}(0)$, then $\Phi$ can be chosen to be the identity on $E^{-1}(0)$. Finally, the homeomorphism $\Phi = E \circ \tilde{\Phi} \circ E^{-1}$, defined in a neighborhood of $0 \in \mathbb{C}^2$, conjugates $R_k$ with $M$, i.e., $R_k \circ \Phi = M$. The theorem is proved. \(\square\)
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