

NONOSCILLATORY SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH INTEGRABLE COEFFICIENTS

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ABSTRACT. The asymptotic behavior of nonoscillatory solutions of the equation $x'' + a(t)|x|^\gamma \operatorname{sgn} x = 0$, $\gamma > 0$, is discussed under the condition that $A(t) = \lim_{T \rightarrow \infty} \int_t^T a(s) ds$ exists and $A(t) \geq 0$ for all t . For the sublinear case of $0 < \gamma < 1$, the existence of at least one nonoscillatory solution is completely characterized.

1. INTRODUCTION

Consider the second-order differential equation

$$(1.1) \quad x'' + a(t)|x|^\gamma \operatorname{sgn} x = 0,$$

where a is continuous on $[t_0, \infty)$ and γ is a positive constant. Equation (1.1) is called the generalized Emden–Fowler equation, and is classified as superlinear or sublinear according to $\gamma > 1$ or $0 < \gamma < 1$. If $\gamma = 1$, then (1.1) reduces to a familiar linear equation.

In this paper we study the asymptotic properties of nonoscillatory solutions of equation (1.1) under the condition that

$$(1.2) \quad \lim_{t \rightarrow \infty} \int_{t_0}^t a(s) ds \quad \text{exists and is finite.}$$

If condition (1.2) is satisfied, then we can introduce the function A by

$$(1.3) \quad A(t) = \int_t^\infty a(s) ds, \quad t \geq t_0.$$

Our purpose here is to extend the well-known results for the case where $a(t)$ is nonnegative to the more general case where $A(t)$ is nonnegative. In the case where $A(t)$ is nonnegative, we can obtain a necessary and sufficient oscillation criterion for the sublinear equation.

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If $a(t) \geq 0$ for $t \geq t_0$, then it is easily seen that a nonoscillatory solution x of (1.1) satisfies exactly one of the following three asymptotic conditions:

$$(1.4) \quad x(t) = c + o(1) \quad \text{as } t \rightarrow \infty, \quad \text{where } c \neq 0;$$

$$(1.5) \quad x(t) = o(t) \quad \text{and} \quad \lim x(t) = \pm\infty \quad \text{as } t \rightarrow \infty;$$

$$(1.6) \quad x(t) = ct + o(t) \quad \text{as } t \rightarrow \infty, \quad \text{where } c \neq 0.$$

We can prove that this fact remains valid even for the case where $A(t) \geq 0$ for $t \geq t_0$:

Theorem 1.1. *Let $\gamma > 0$. Suppose that (1.2) holds and that $A(t) \geq 0$ for $t \geq t_0$. Then, for each nonoscillatory solution x of (1.1), exactly one of the three asymptotic conditions (1.4)–(1.6) is satisfied.*

For the superlinear case, the next theorem is already known.

Theorem 1.2. *Let $\gamma > 1$. Suppose that (1.2) holds and that $A(t) \geq 0$ for $t \geq t_0$. Then the following three statements are equivalent:*

- (i) *equation (1.1) has a nonoscillatory solution x satisfying (1.4);*
- (ii) *equation (1.1) has a nonoscillatory solution;*
- (iii) *the two integral conditions below are satisfied:*

$$(1.7) \quad \int_{t_0}^{\infty} A(t) dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} tA^2(t) dt < \infty.$$

In fact, the equivalence of (ii) and (iii) can be obtained as the contrapositive form of the result of Butler [1, Theorem 2.3]. Note that Butler's result is applicable to the more general case. The equivalence of (i) and (iii) has been proved by Naito [6, Theorem 2.4]. In light of the "duality" between superlinear and sublinear equations for the case of $a(t) \geq 0$ (see Coffman and Wong [3]) and the result of Naito [6, Theorem 3.5], it is natural to conjecture that the next theorem is true.

Theorem 1.3. *Let $0 < \gamma < 1$. Suppose that (1.2) holds and that $A(t) \geq 0$ for $t \geq t_0$. Then the following three statements are equivalent:*

- (i) *equation (1.1) has a nonoscillatory solution x satisfying (1.6);*
- (ii) *equation (1.1) has a nonoscillatory solution;*
- (iii) *the two integral conditions below are satisfied:*

$$(1.8) \quad \int_{t_0}^{\infty} t^{\gamma-1} A(t) dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} t^{2\gamma-1} A^2(t) dt < \infty.$$

In the succeeding section we show that Theorems 1.1 and 1.3 are indeed true. It is easily seen that the equivalence of (ii) and (iii) in Theorem 1.3 can be restated as follows:

Corollary 1.4. *Let $0 < \gamma < 1$. Suppose that (1.2) holds and that $A(t) \geq 0$ for $t \geq t_0$. Then all solutions of (1.1) are oscillatory if and only if*

$$\int_{t_0}^{\infty} \left(t^{\gamma-1} A(t) + \int_t^{\infty} [s^{\gamma-1} A(s)]^2 ds \right) dt = \infty.$$

The problem of oscillation of solutions to the Emden–Fowler equation (1.1) and the more general equation $x'' + a(t)f(x) = 0$ has attracted a great deal of attention, and numerous results have been obtained. For a general discussion on this problem, we refer to the survey article of Wong [10] (the nonlinear case) and the book of Swanson [7] (the linear case). The case where (1.2) holds has been studied by several authors including Butler [1, 2], Coles [4], Kwong and Wong [5], Naito [6], Willett [8], and Wong [9].

2. PROOFS OF THEOREMS 1.1 AND 1.3

Proof of Theorem 1.1. Let x be a nonoscillatory solution of (1.1). There is no loss of generality in supposing that $t_0 > 0$ and $x(t) > 0$ for $t \geq t_0$. It is known (Kwong and Wong [5, Theorem 1]) that $x(t)$ satisfies the equality

$$x'(t) = A(t)x^\gamma(t) + \gamma x^\gamma(t) \int_t^\infty x^{-\gamma-1}(s)x'^2(s) ds$$

for $t \geq t_0$, where A is defined by (1.3). Therefore we have

$$(2.1) \quad x'(t) \geq A(t)x^\gamma(t), \quad t \geq t_0.$$

Further, by the nonnegativity of $A(t)$ we have $x'(t) \geq 0$ for $t \geq t_0$. An integration by parts of (1.1) gives

$$(2.2) \quad x'(\tau) - A(\tau)x^\gamma(\tau) + \gamma \int_t^\tau A(s)x^{\gamma-1}(s)x'(s) ds = x'(t) - A(t)x^\gamma(t)$$

for $\tau \geq t$ ($\geq t_0$). Let t be fixed. Since $A(s)x^{\gamma-1}(s)x'(s)$ is nonnegative, the integral term in (2.2) has a finite limit or diverges to ∞ as $\tau \rightarrow \infty$. If the latter case occurs, then $x'(\tau) - A(\tau)x^\gamma(\tau) \rightarrow -\infty$ as $\tau \rightarrow \infty$, which is a contradiction to (2.1). Thus the former case occurs:

$$(2.3) \quad \int_t^\infty A(s)x^{\gamma-1}(s)x'(s) ds < \infty.$$

This implies that the function K_1 can be defined by

$$(2.4) \quad K_1(t) = \int_t^\infty A(s)x^{\gamma-1}(s)x'(s) ds, \quad t \geq t_0,$$

and that $x'(\tau) - A(\tau)x^\gamma(\tau)$ converges to a finite limit as $\tau \rightarrow \infty$. Let α be the limit:

$$\alpha = \lim_{\tau \rightarrow \infty} [x'(\tau) - A(\tau)x^\gamma(\tau)].$$

Then equality (2.2) yields

$$(2.5) \quad x'(t) = \alpha + A(t)x^\gamma(t) + \gamma K_1(t)$$

for $t \geq t_0$. Observe by (2.1) that $\alpha \geq 0$. From (2.1) and (2.3) it follows that

$$(2.6) \quad \int_t^\infty A^2(s)x^{2\gamma-1}(s) ds < \infty.$$

Therefore the function K_2 can be defined by

$$(2.7) \quad K_2(t) = \int_t^\infty A^2(s)x^{2\gamma-1}(s) ds, \quad t \geq t_0.$$

Integrating (2.5) over $[t_0, t]$, we have

$$(2.8) \quad x(t) = x(t_0) + \alpha(t - t_0) + \int_{t_0}^t A(s)x^\gamma(s) ds + \gamma \int_{t_0}^t K_1(s) ds$$

for $t \geq t_0$. By Schwarz's inequality and the fact that $x'(t) \geq 0$ for $t \geq t_0$, the first integral term in (2.8) can be estimated as follows:

$$(2.9) \quad \begin{aligned} & \int_{t_0}^t A(s)x^\gamma(s) ds \\ & \leq \left(\int_{t_0}^t A^2(s)x^{2\gamma-1}(s) ds \right)^{1/2} \left(\int_{t_0}^t x(s) ds \right)^{1/2} \\ & \leq K_2^{1/2}(t_0)(t - t_0)^{1/2} x^{1/2}(t), \quad t \geq t_0. \end{aligned}$$

Thus we obtain

$$x(t) \leq x(t_0) + \alpha(t - t_0) + K_2^{1/2}(t_0)(t - t_0)^{1/2} x^{1/2}(t) + \gamma K_1(t_0)(t - t_0)$$

for $t \geq t_0$. The above inequality may be regarded as a quadratic inequality with respect to $x^{1/2}(t)$. Then we find that

$$x^{1/2}(t) \leq [K_2^{1/2}(t_0)(t - t_0)^{1/2} + D^{1/2}(t)]/2, \quad t \geq t_0,$$

where

$$D(t) = K_2(t_0)(t - t_0) + 4[x(t_0) + \alpha(t - t_0) + \gamma K_1(t_0)(t - t_0)], \quad t \geq t_0.$$

It is obvious that $D(t) = O(t)$ as $t \rightarrow \infty$, and consequently, there exists a positive constant M such that

$$(2.10) \quad x(t) \leq Mt \quad \text{for } t \geq t_0.$$

Let $T (\geq t_0)$ be an arbitrary number. It is clear that

$$(2.11) \quad 0 \leq \frac{1}{t} \int_{t_0}^t A(s)x^\gamma(s) ds = \frac{1}{t} \int_{t_0}^T A(s)x^\gamma(s) ds + \frac{1}{t} \int_T^t A(s)x^\gamma(s) ds$$

for $t \geq T$. Arguing as in (2.9), we have

$$\int_T^t A(s)x^\gamma(s) ds \leq K_2^{1/2}(T)(t - T)^{1/2} x^{1/2}(t), \quad t \geq T,$$

which, combined with (2.10), yields

$$(2.12) \quad \int_T^t A(s)x^\gamma(s) ds \leq M^{1/2} K_2^{1/2}(T) t^{1/2} (t - T)^{1/2}, \quad t \geq T.$$

Taking the upper limit as $t \rightarrow \infty$ in (2.11) and using (2.12), we see that

$$(2.13) \quad 0 \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t A(s)x^\gamma(s) ds \leq M^{1/2} K_2^{1/2}(T).$$

Since T is arbitrary and $K_2(T)$ tends to zero as $T \rightarrow \infty$, letting $T \rightarrow \infty$ in (2.13), we find that

$$(2.14) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t A(s)x^\gamma(s) ds = 0.$$

In view of (2.8), (2.14), and the fact that $K_1(t) \rightarrow 0$ as $t \rightarrow \infty$, we get

$$\lim_{t \rightarrow \infty} x(t)/t = \alpha.$$

Recall that $x(t)$ is nondecreasing for $t \geq t_0$. Then there are three possibilities: (i) $\alpha = 0$ and $x(t)$ is bounded above; (ii) $\alpha = 0$ and $x(t)$ is unbounded; (iii) $\alpha > 0$ (and hence $x(t)$ is unbounded). Case (i) implies (1.4) with $c = \lim_{t \rightarrow \infty} x(t) > 0$, while case (iii) implies (1.6) with $c = \alpha > 0$. It is also clear that case (ii) implies (1.5). The proof of Theorem 1.1 is complete.

Proof of Theorem 1.3. It is trivial that (i) implies (ii). The equivalence of (i) and (iii) has been proved in [6, Theorem 3.5]. Here we claim that (ii) implies (iii). Butler [2] has showed that if

$$\int^{\infty} t^{\gamma-1} A(t) dt = \infty,$$

then all solutions of (1.1) are oscillatory. This means that if (1.1) has a nonoscillatory solution, then the first integral condition in (1.8) is satisfied. Therefore it is enough to show that if (1.1) has a nonoscillatory solution, then the second integral condition in (1.8) is satisfied:

$$(2.15) \quad \int^{\infty} t^{2\gamma-1} A^2(t) dt < \infty.$$

Suppose that equation (1.1) has a solution $x(t)$ which is positive on $[t_0, \infty)$. The equalities and the inequalities in the proof of Theorem 1.1 remain valid. Consider the case where $2\gamma - 1 \leq 0$, then the desired condition (2.15) follows from (2.6) and (2.10). Next consider the case where $2\gamma - 1 > 0$. In (2.8), the first three terms of the right-hand side are nonnegative, and $K_1(t)$ is nonincreasing on $[t_0, \infty)$. Thus we have

$$x(t) \geq \gamma(t - t_0)K_1(t), \quad t \geq t_0.$$

Hence, in view of (2.1), (2.4), and (2.7), we get

$$x(t) \geq \gamma(t - t_0)K_2(t), \quad t \geq t_0,$$

and so, by the assumption $2\gamma - 1 > 0$,

$$(2.16) \quad A^2(t)x^{2\gamma-1}(t)K_2^{-2\gamma+1}(t) \geq \gamma^{2\gamma-1}(t - t_0)^{2\gamma-1}A^2(t), \quad t \geq t_0.$$

Since $K_2'(t) = -A^2(t)x^{2\gamma-1}(t)$ for $t \geq t_0$, an integration of (2.16) gives

$$-\frac{1}{2-2\gamma}K_2^{-2\gamma+2}(t) + \frac{1}{2-2\gamma}K_2^{-2\gamma+2}(t_0) \geq \gamma^{2\gamma-1} \int_{t_0}^t (s-t_0)^{2\gamma-1} A^2(s) ds$$

for $t \geq t_0$. Note that the left-hand side of the above inequality is bounded on $[t_0, \infty)$. Then it is easy to see that the desired condition (2.15) is also satisfied. The proof of Theorem 1.3 is complete.

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