A NOTE ON WEINSTEIN’S CONJECTURE

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Abstract. We prove that the contact foliation of a compact contact manifold
(M, α) has at least one compact leaf in the following two cases: (i) α is a
K-contact form and M is simply connected, (ii) α is C²-close to a regular
contact form. This solves the Weinstein conjecture in those particular cases.

1. The CONJECTURE

Let S be a hypersurface in a symplectic manifold (M, Ω). There are a Riemannian metric g on M and an almost-complex structure J on M such that
Ω(U, V) = g(U, JV) and g(JU, JV) = g(U, V) for all vector fields U, V.
The characteristic distribution X_S of S is the 1-dimensional distribution on
S defined by X_S(x) = JN_x, where N_x is a unit outward normal vector to S
at x. An integral curve of the distribution X_S is called a characteristic of S.
In a famous paper [6], Rabinowitz proved that if S is a strongly star-shaped
hypersurface of R^{2n}, with its standard symplectic structure ω_0, then S has at
least one closed characteristic. In [8], Weinstein conjectured that if S is simply
connected and carries a contact form α such that i * Ω = dα (he calls such
submanifolds “hypersurfaces of contact type”), then S should have at least one
closed characteristic. Here i: S → M is the inclusion map.

The conjecture has been proved by Viterbo [7] in the particular case M = R^{2n}
with its standard symplectic form, but without the assumption that S is simply
connected. Viterbo’s trick is to change the problem into one of finding periodic
orbits of a Hamiltonian system and use the now-familiar variational method:
closed orbits correspond to critical points of an action-functional on a loop
space.

Let S be a hypersurface of contact type in a symplectic manifold (M, Ω). Then
i * Ω = dα, where i: S → M is the inclusion map. Let X_α be the Reeb
vector field of α: this is the unique vector field on S such that i(X_α)α = 1
and i(X_α)dα = 0; here i(•) is the interior product operation. It is clear that

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the characteristic field $X_S$ of $S$ spans the kernel of $i \ast \Omega = d\alpha$. Hence, there exists a nowhere-vanishing function $f$ on $S$ such that $X_S(x) = f(x) \cdot X_\alpha(x)$ for all $x \in S$. Therefore, characteristics of $S$ are just reparametrized flow lines of the Reeb field. If $(S, \alpha)$ is a contact manifold, the foliation $F$ of $S$ by integral curves of the Reeb field will be called the "contact foliation" of $(S, \alpha)$.

The Weinstein conjecture can be rephrased as follows: Let $(S, \alpha)$ be a compact, simply connected hypersurface of contact type of a symplectic manifold. Then, its contact foliation has a compact leaf.

We may forget about the embedding $S \hookrightarrow M$ and ask if the contact flow of any compact, simply connected contact manifold has a compact leaf? Posed this way, the problem becomes trivial for some types of contact structure: for instance, if $\alpha$ is a regular contact form.

Recall that a contact form $\alpha$ on $S$ is said to be a regular contact form if each point $x \in S$ has an open neighborhood $U$ such that the integral curves of its Reeb field $X_\alpha$ passing through $U$ pass through that neighborhood only once (see [1, p. 6]). One knows that if $\alpha$ is a regular contact form on a compact manifold $S$, then there exists a smooth nowhere-vanishing function $\lambda$ on $S$ such that $X' = \lambda X_\alpha$ generates a free action of the circle $S^1$ on $S$ (Boothby–Wang's theorem; see [1, p. 14]). Therefore, in that case all the leaves of the contact foliation are compact.

One would like to know what happens if one starts with a regular contact form $\alpha$ and adds to it a small perturbation. Since the set of contact forms is open in the set of all 1-forms, the resulting form $\alpha'$ is a contact form. However, $\alpha'$ is not necessarily a regular contact form. In the next section, we show that the contact foliation of $\alpha'$ has compact leaves. Finally, in §3, we show that the contact foliation $F_\alpha$ of a contact form $\alpha$ on a compact, simply connected manifold $M$ has a compact leaf provided that there exists a Riemannian metric on $M$ which leaves invariant the Reeb field of $\alpha$. Such contact forms are called $K$-contact forms [1].

2. A "DEFORMATION" RESULT

Let $\pi: M \to B$ be an oriented $S^1$-bundle over a compact, oriented even-dimensional manifold $B$, and let $F$ be an oriented 1-dimensional foliation on $M$ with a transverse symplectic form $\omega$, i.e., $d\omega = 0$ and $\text{Ker } \omega$ generates the tangent space to the leaves of $F$. That is, for all $x \in M$, $\text{Ker } \omega(x) = \{x \in T_x M \mid \omega(x)(X, \xi) = 0, \forall \xi \in T_x M\}$ is a 1-dimensional vector space isomorphic with the tangent space to the leaf of $F$ through $x$.

Under the hypothesis that (i) $\omega = \pi^* \omega_0 + d\omega_1$ for some closed 2-form $\omega_0$ on $B$ and (ii) $\text{Ker } \omega$ is $C^1$-close to the vertical, Ginzburg [2] has proved that the number of compact leaves of $F$ is at least equal to the number $k_\pi$ of critical points of smooth functions on $M$ such that their critical manifolds are smooth curves whose projections are homologous to zero in $B$. Clearly $k_\pi \geq 2$. 
Consider now a compact manifold $M$ equipped with a regular contact form $\alpha_0$, and consider a contact form $\alpha$ $C^2$-close to $\alpha_0$. By Boothby–Wang's theorem [1], $M$ is the total space of a principal $S^1$-bundle $\pi: M \to B$, where the action of $S^1$ on $M$ is generated by a multiple of the Reeb field $X_{\alpha_0}$ of $\alpha_0$. Hence the vertical direction is spanned by $X_{\alpha_0}$. The contact foliation $F$ of $(M, \alpha)$ admits $\omega = d\alpha$ as a transverse symplectic structure which satisfies hypothesis (i) in Ginzburg's theorem.

Since $\alpha$ and $\alpha_0$ are $C^2$-close, $(d\alpha)^n$ and $(d\alpha_0)^n$ are $C^1$-close. But the Reeb field $X_{\beta}$ of any contact form $\beta$ is uniquely determined by the equation $i(X_{\beta})(\beta \wedge (d\beta)^n) = (d\beta)^n$ when the dimension of the manifold is $2n + 1$. Therefore if $\alpha$ and $\alpha_0$ are $C^2$-close, the corresponding Reeb fields are $C^1$-close. Hence Ginzburg's theorem implies the following result, a form of which was already pointed out by Ginzburg in the case where $M$ is the unit cosphere bundle over a compact, oriented surface:

**Theorem 1.** Let $(M, \alpha)$ be a compact manifold where the contact form $\alpha$ is $C^2$-close to some regular contact form on $M$. Then the contact foliation $F_\alpha$ of $(M, \alpha)$ has at least two compact leaves.

### 3. $K$-CONTACT FOLIATIONS

A contact form $\alpha$ on a smooth manifold $S$ is called a $K$-contact form if there exists a Riemannian metric $g$ on $S$ which is invariant by the Reeb field $X_{\alpha}$ of $\alpha$, i.e., if $L_{X_{\alpha}}g = 0$, where $L$ is the Lie derivative. The corresponding contact foliation is called a $K$-contact foliation.

**Theorem 2.** Let $(S, \alpha)$ be a compact, simply connected manifold with a $K$-contact form $\alpha$. The $K$-contact foliation of $(S, \alpha)$ has at least one compact leaf.

**Proof.** Let $g$ be a Riemannian metric on $S$ such that $L_{X_{\alpha}}g = 0$, where $X_{\alpha}$ is the Reeb field of $\alpha$. Monna [5] has shown that the $K$-contact condition is equivalent to the existence of an invariant transverse metric for the contact foliation $F$, i.e., that $F$ is a Riemannian foliation with a bundle-like transverse metric in the sense of Rienhart. We refer to the excellent book of Molino [3]. For each vector field $V$ on $S$, let $\overline{V}$ be the normal field to $F$ such that $\overline{V}(x)$ is the projection on the subspace of $T_xS$ orthogonal to $X_{\alpha}(x)$. Monna [5] defines a transverse metric $\overline{g}$ by the equation

$$\overline{g}(\overline{U}, \overline{V}) = g(-U + \alpha(U) \cdot X_{\alpha}, -V + \alpha(V) \cdot X_{\alpha}).$$

An easy calculation shows that indeed $\overline{g}$ is a transverse metric invariant by $X_{\alpha}$. In fact, Monna proves that a contact foliation is transversally Riemannian with an invariant bundle-like metric if and only if the contact form is a $K$-contact form.

Since the kernel of $\omega = d\alpha$ is one-dimensional and spanned by $X_{\alpha}$, the 2-form $\omega = d\alpha$ is a transverse symplectic structure for the foliation $F$. We
are now in position to apply the geometric results of Molino on complete Riemannian foliations with transverse symplectic structures: according to Molino [4], if \((S, \alpha)\) is a compact, simply connected \(K\)-contact manifold and \(p\) is the dimension of the structural algebra of the complete Riemannian foliation \(F\) (the \(K\)-contact foliation), there exists a (momentum) map \(I: S \rightarrow \mathbb{R}^p\), constant on the leaves of \(F\), such that \(I(S) \subseteq \mathbb{R}^p\) is a closed convex polytope whose vertices correspond to compact leaves; in this case, these are closed curves. Since the polytope \(I(S)\) necessarily has one or more vertices, the contact foliation has necessarily at least one compact leaf. □

Remarks. There are examples of \(K\)-contact forms which are not regular (see for instance [1, pp. 90–91]). On the other hand, one knows that a regular contact form is a \(K\)-contact form.

Let us finally point out that the contact form \(\alpha\) on the 3-torus \(T^3\) induced by the contact form \(\tilde{\alpha} = \cos(2\pi x_3) \, dx_3 + \sin(2\pi x_3) \, dx_2\) on \(\mathbb{R}^3\) is neither regular nor \(K\)-contact. Indeed there are closed or nonclosed leaves, according to the rationality of \(\tan(2\pi x_3)\). Thus the contact foliation has compact and noncompact leaves; some of the compact leaves are not isolated. Monna pointed out to me that results of Molino prohibit this foliation from being a foliation with a bundle-like metric. Hence the contact form is not a \(K\)-contact form. Obviously it is not a regular contact form either, since the leaves are nonclosed.

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