HYPERCOMPLETIONS OF RIESZ SPACES

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Abstract. It is shown that each Riesz space with separating order continuous dual can be embedded in a unique “e-hypercompletion,” where e is a fixed weak unit of the extended order continuous dual.

1. Introduction

In [8] I investigated—in terms of an internal characterization—those Riesz spaces which can be represented as bands of spaces of measures; I call these spaces hypercomplete. The present paper is devoted to the question of how far Archimedean Riesz spaces can be embedded in a “hypercompletion.” It follows already from [7, Theorem A] that each Riesz space L with separating order continuous dual \( L^\sim \) (i.e. \( (L^\sim)^0 = \{0\} \)) embeds order densely in some hypercomplete Riesz space \( M \), but uniqueness is not yet guaranteed. The main result given here is that each Riesz space \( L \) with \( (L^\sim)^0 = \{0\} \) can be embedded order densely in a unique “e-hypercompletion” where e is some weak unit of the extended order continuous dual \( \Gamma(L) \) of \( L \). But we will see in an example that, in general, dependence on the weak unit e cannot be avoided.

2. Results

\( L \) always denotes an Archimedean Riesz space.

For convenience, the definitions of extended order continuous dual and hypercompleteness follow.

Let \( I_\Phi \) be the set of all order dense ideals of \( L \). Then \( \Gamma(L) := \bigcup I_\Phi \) is the extended order continuous dual of \( L \) (see [10]), where \( \xi \in I_n^\sim \) and \( \eta \in J_n^\sim \) are identified if they coincide on \( I \cap J \). With algebraic and order structure defined by using representatives, \( \Gamma(L) \) is a universally complete Riesz space [10, 1.5], and for each \( \xi \in \Gamma(L) \) there exists a greatest \( L[\xi]\in I_\Phi \) such that \( L[\xi]\sim^\sim \) [10, 1.3].

If e is a weak unit of \( \Gamma(L) \) and \( R \subset L^\sim \) is a set of components of e such that for each upward-directed family \( (x_i) \) from \( L^+ \) for which \( \sup g(x_i) < \infty \)
for every \( g \in R \) there exists \( \sup x_i \) in \( L \), then call \((e, R)\) an \( hc\)-pair of \( L \). Call \( L \) hypercomplete if there exists an \( hc\)-pair of \( L \).

In the sequel let \( e \) be a weak unit of \( \Gamma(L) \). I denote by \( R_e \) the set of all components of \( e \) in \( L \).

Before reading the appropriate definition, note that it would not be senseful to call a Riesz space \( M \) \( e\)-hypercompletion of \( L \) if \( L \) is order dense in \( M \) and \( e \) is the first component of an \( hc\)-pair of \( M \), since in this case \( M \) would not be unique: If \( M(\mathcal{R}) \) denotes the set of all real-valued \( \sigma\)-additive measures on some \( \delta\)-ring \( \mathcal{R} \) of subsets of a set \( X \), and \( M_b(\mathcal{R}) \) denotes the bounded measures on \( \mathcal{R} \), then for \( L = M_b(\mathcal{R}) \) and \( e = 1_X \), both \( M_1 = M_b(\mathcal{R}) \) and \( M_2 = M(\mathcal{R}) \) would satisfy this bad definition. One has to impose an extra condition for this reason.

If \( M \) is an Archimedean Riesz space and if \( \phi: L \to M \) is an injective Riesz homomorphism such that \( \phi L \) is order dense in \( M \), I denote by \( \psi_{M,\phi} \) the canonical Riesz isomorphism

\[
\Gamma(M) \to \Gamma(L), \xi \mapsto \xi \circ \phi|_{\psi^{-1}(M[\xi])}
\]

([10, 2.6]).

**Definition 1.** A hypercomplete Riesz space \( M \) is called \( e\)-hypercompletion of \( L \) if there exists an injective Riesz homomorphism \( \phi: L \to M \) such that

1. \( \phi L \) is an order dense Riesz subspace of \( M \);
2. we have \( \psi^{-1}_{M,\phi}(g) \in M_n \) for all \( g \in R_e \), and \( (\psi^{-1}_{M,\phi}(e), \psi^{-1}_{M,\phi}(R_e)) \) is an \( hc\)-pair of \( M \).

Roughly, \( M \) is an \( e\)-hypercompletion of \( L \) iff \( L \) is order densely embedded in \( M \), \( R_e \subset M_n \), and \((e, R_e)\) is an \( hc\)-pair of \( M \).

It follows from condition 1 and [8, 2.3] that only spaces with \( (L_n)^0 = \{0\} \) can possess an \( e\)-hypercompletion. For this reason, we assume \( L_n \) to be separating from now on.

If \( Y \) is a locally compact Stonian space, let \( \mathcal{M}(Y) \) denote the set of normal Radon measures on \( Y \).

**Proposition 2.** If \( M_1, M_2 \) are \( e\)-hypercompletions of \( L \), then \( M_1 \) and \( M_2 \) are canonically Riesz isomorphic, i.e. if for \( k \in \{1, 2\} \), \( \phi_k: L \to M_k \) satisfies the definition above, then there exists a Riesz isomorphism \( \psi: M_1 \to M_2 \) such that \( \psi \circ \phi_1 = \phi_2 \).

**Proof.** Set \( \psi_k := \psi_{M_k,\phi_k} \). By the Ogasawara-Maeda representation theorem there exist a compact hyperstonian space \( Z \) and a Riesz isomorphism \( u: \Gamma(L) \to C_\infty(Z) \) with \( u(e) = 1_Z \). Then \( u \circ \psi_k: \Gamma(M_k) \to C_\infty(Z) \) is a Riesz isomorphism with \( u \circ \psi_k(\psi_k^{-1}(e)) = 1_Z \). Set

\[
Y_k := \bigcup_{g \in R_e} u \circ \psi_k(\psi_k^{-1}(g))
\]

\( \hat{u}_k: \Gamma(M_k) \to C_\infty(Y_k), \xi \mapsto (u \circ \psi_k)(\xi)|_{Y_k} \).
By [7, Theorem A], there exists a Riesz isomorphism $v_k: M_k \rightarrow \mathcal{M}(Y_k)$ such that $(Y_k, \hat{u}_k, v_k)$ is a $\psi^{-1}_k(e)$-mr of $M_k$ to which $\psi^{-1}_k(R_e)$ is associated; observe that $v_k$ is onto by the hypercompleteness of $M_k$. But $Y_1 = Y_2$, and hence $v := v_2 \circ v_1$ is a Riesz isomorphism.

Fix $x \in L$. Then for all $\xi \in \Gamma(L)$ for which $\xi(x)$ is defined:

\[
\int u\xi|_{Y_1} d\psi_1(\phi_1 x) = \int \hat{u}_1(\psi^{-1}_1 \xi) d\psi_1(\phi_1 x) = (\psi^{-1}_1 \xi)(\phi_1 x) = \xi(x) = \cdots = \int u\xi|_{Y_2} d\psi_2(\phi_2 x).
\]

Hence $v_1(\phi_1 x) = v_2(\phi_2 x)$ and thus $v \circ \phi_1 = \phi_2$. \qed

Let $G_e$ denote the ideal of $L_n$ generated by $R_e$.

**Theorem 3.** $(G_e)_n$ is the e-hypercompletion of $L$.

**Proof.** By [7, Theorem A] there exists an e-mr $(Y, \hat{u}, v)$ of $L$ to which $R_e$ is associated. Then $(1_Y, \{\hat{u}g|g \in R_e\})$ is an he-pair of $\mathcal{M}(Y)$, which implies that $\mathcal{M}(Y)$ is an e-hypercompletion of $L$.

Since obviously $G_e = C_e(Y)$ (the set of continuous functions on $Y$ with compact support), it follows from the Riesz representation theorem that $(G_e)_n = \mathcal{M}(Y)$.

The uniqueness was proved in Proposition 2. \qed

Of course, $L$ is its own e-hypercompletion if $(e, R_e)$ is an hc-pair of $L$.

As we have seen in the proof of the preceding theorem, the e-hypercompletion of $L$ can always be realized as the space $\mathcal{M}(Y)$, for some locally compact hyperstonian space $Y$. Another beautiful description of it in terms of extended $L^1(\mu)$-spaces—where $\mu$ is a normal measure on a compact Stonian space—was obtained by Y. A. Abramovich [15].

Copying the proof of Theorem 3, we get

**Corollary 4.** If $R \subset R_e$ with $e = \sup_{g \in R} g$, then $(G_R)_n$ is hypercomplete, where $G_R$ stands for the ideal of $L_n$ generated by $R$. \qed

But $(G_R)_n = (G_e)_n$ need not always hold: If $L = \mathcal{M}_b(\mathcal{R})$, $e = \int_A$, $R = \{\int_A|A \in \mathcal{R}\}$, then $(G_e)_n = \mathcal{M}_b(\mathcal{R})$ while $(G_R)_n = \mathcal{M}(\mathcal{R})$.

The following example shows that, in general, to different $e$'s there belong different e-hypercompletions.

**Example.** Let $L := \ell^\infty$. Then $L_n = \ell^1$ and $\Gamma(L) = \mathbb{R}^N$. For each weak unit $e$ of $\mathbb{R}^N$, set

\[
\mathcal{A}(e) := \left\{ A \subset \mathbb{N} \mid \sum_{n \in A} e(n) < \infty \right\},
\]

\[
M_e := \left\{ h \in \mathbb{R}^N \mid \sum_{n \in A} |h(n)|e(n) < \infty \text{ for all } A \in \mathcal{A}(e) \right\}.
\]
Then $M_e$ is the $e$-hypercompletion of $L$, where the embedding map $\phi$ is just the identity map on $L$.

It follows that if $v: M_{e_1} \rightarrow M_{e_2}$ is a Riesz isomorphism preserving the embedding of $L$, we must have $v|_L = id$, and hence, by the order continuity of $v$, $v = id$, which implies $M_{e_1} = M_{e_2}$. But if, for example, $e_1, e_2 \in L_n^\sim$, then the condition $M_{e_1} = M_{e_2}$ is easily seen to be equivalent to the existence of positive reals $C, D$ with $Ce_1 \leq e_2 \leq De_1$, which allows us to construct many nonisomorphic $e$-hypercompletions of $L$.  

Let us briefly consider the situation where $L_n^\sim$ possesses weak units.

If $e \in L_n^\sim$, then $e$ is the greatest element of $R_e$, and since $R_e \subseteq (\langle G_e \rangle_n)^\sim$ by Theorem 3, we have

**Proposition 5.** If $e \in L_n^\sim$ then $(e, \{e\})$ is an hc-pair of $(G_e)_n^\sim$.  

Let $M$ be a Dedekind complete Riesz space, and denote, for $x \in M$, by $S(x)$ the ideal of $M$ generated by $x$. Then, if $f, g$ are weak units of $M$, the Ogasawara-Maeda theorem implies the existence of a Riesz isomorphism $\phi: S(f) \rightarrow S(g)$ with $\phi f = g$.

An immediate consequence of this remark is that for weak units $e_1, e_2$ of $L_n^\sim$, the corresponding $e$-hypercompletions are Riesz isomorphic (but of course not canonically, i.e. not preserving the embedding of $L$):

**Proposition 6.** If $e_1, e_2$ are weak units of $L_n^\sim$, then $G_{e_1}$ and $G_{e_2}$ are Riesz isomorphic, and hence the same holds for $(G_{e_1})_n^\sim$ and $(G_{e_2})_n^\sim$.  

We can even say that an $e$-hypercompletion belonging to $e \in L_n^\sim$ is in some sense minimal. Consider another weak unit $f$ of $\Gamma(L)$. Then $e_1 := \inf\{e, f\}$ is a weak unit of $L_n^\sim$, and since $S(f)$ is isomorphic to $G_{e_1}$ (the remark above applied to $\Gamma(L)$), $G_f$ embeds order densely in $G_{e_1}$, which implies that $(G_{e_1})_n^\sim$ embeds in $(G_f)_n^\sim$, and hence $(G_e)_n^\sim$ embeds in $(G_f)_n^\sim$, too.

3. AN APPLICATION

I now briefly sketch an application; a more detailed presentation will be given in a subsequent paper.

Several authors (cf. [1-5, 9, 11-14]) have studied so-called Hellinger integrals to get integral representations of the duals of spaces of finitely additive or $\sigma$-additive measures; here a real-valued function $\phi$, defined on some ring of sets, $\mathcal{R}$, is called Hellinger integrable with respect to a measure $\mu$ having domain $\mathcal{R}$ if $\lim \sum_{A \in \mathcal{A}} \phi(A) \mu(A)$ exists in $\mathbb{R}$, where $\mathcal{A}$ runs through the set of partitions of $\mathcal{R}$, directed under refinement. But the representation theorems these authors get are not very satisfying in some sense, because set theoretical assumptions have to be made (see e.g. [12]), and one must necessarily restrict to “small” spaces of measures as Keisler has shown [9, Theorem 4]. A method proposed here to overcome these difficulties is to work with generalized parti-
tions as described below; this method works not only in spaces of measures but in Riesz spaces $L$ with separating dual $L_\sim$

$\mathcal{R}$ denotes the set of all countable disjoint systems from $R_e$. Setting $g_{\mathcal{A}} := \sup_{g \in \mathcal{A}} g$ for $\mathcal{A} \in \mathcal{R}$, one can define an order relation on $\mathcal{R}$ by

$$\mathcal{A} \preceq \mathcal{B} \Leftrightarrow g_{\mathcal{A}} \leq g_{\mathcal{B}}, \text{ and for all } h \in \mathcal{B} \text{ either }$$

$$\inf \{ h, g_{\mathcal{A}} \} = 0 \text{ or there is } g \in \mathcal{A} \text{ with } h \leq g.$$

Let $\mathcal{F}$ be the section filter of $\mathcal{R}$ with respect to $\preceq$. For $x \in L$ and $\phi: R_e \to \mathbb{R}$ set $\phi_x(\mathcal{A}) := \sum_{g \in \mathcal{A}} \phi(g) g(x)$ if the series is summable, and $\phi_x(\mathcal{A}) := \infty$, otherwise ($\mathcal{A} \in \mathcal{F}$). One can call $\phi$ $x$-integrable if $\phi_x(\mathcal{F})$ converges in $\mathbb{R}$; in this case, set

$$\int \phi \, dx := \lim \phi_x(\mathcal{F}).$$

The following representation theorem may serve to justify the definition of generalized Hellinger integral just given (observe that this theorem applies in particular to spaces of measures since they are hypercomplete by [8, 2.5]):

**Theorem 7.** For each $\xi \in \Gamma(L)$ which is extendable to $M[e]$ (where $M$ denotes the $e$-hypercompletion of $L$), there exists a bounded map $\phi: R_e \to \mathbb{R}$ such that $\phi$ is $x$-integrable and $\xi(x) = \int \phi \, dx$ for all $x \in L[\xi]$.

**Proof** (idea of). Let $(Y, \mu, \nu)$ be the $e$-mr of $M$ to which $R_e \subset M_\sim$ is associated; then $\nu M = \mathcal{M}(Y)$, by [7, Theorem A] and the hypercompleteness of $M$.

For $g \in R_e$, $g \neq 0$, set $\phi(g) := \hat{\mu}\xi(y_g)$, where $y_g$ is a point of supp $\hat{\mu} g$ in which $|\hat{\mu}\xi|$ attains its minimum on supp $\hat{\mu} g$; for $g = 0$, set $\phi(g) := 0$.

Since $\hat{\mu}\xi$ is defined on $\mathcal{M}_b(Y) = \nu(M[e])$, $\hat{\mu}\xi$ is bounded by [6, 1.6.1b)], and hence $\phi$ is likewise. The rather technical verification that $\phi$ has the required properties is omitted here. □

Every $\xi \in \Gamma(L)$ can be represented in the same way, but $\phi$ will no longer be bounded in general. The advantage of the boundedness of $\phi$ is that in this case one obtains the same Hellinger integral when working only with finite disjoint systems from $R_e$ (at least for $x \in L[e]$).

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**References**


