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A PROPERTY OF POLYNOMIAL CURVES OVER A FIELD OF POSITIVE CHARACTERISTIC

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ABSTRACT. Let $\mathbf{k}$ be an algebraically closed field of characteristic $p > 0$. We show that if $F \in \mathbf{k}[X, Y]$ is a rational curve with one place at infinity and with nonprincipal bidegree, such that $\theta(F) \in \mathbf{k}[X^p, Y]$ for some automorphism θ of $\mathbf{k}[X, Y]$, then θ can be chosen to be either linear or of "de Jonquière" type. We also give consequences of that fact for the problem of classifying the embeddings of the line in the plane.

Let $\mathbf{k}$ be an algebraically closed field of characteristic $p > 0$, X, Y, T indeterminates over $\mathbf{k}$, $R = \mathbf{k}[X, Y]$, $R_0 = \mathbf{k}[X^p, Y]$ and $\mathbf{k}^* = \mathbf{k} \setminus \{0\}$. Let $\operatorname{Aut}_{\mathbf{k}} R$ be the group of $\mathbf{k}$-algebra automorphisms of R and say that F, $G \in R$ are *equivalent* $(F \sim G)$ if $G = \lambda\theta(F)$ for some $\theta \in \operatorname{Aut}_{\mathbf{k}} R$ and $\lambda \in \mathbf{k}^*$.

A *polynomial curve* is an irreducible $F \in R$ such that $F(x, y) = 0$ for some x, $y \in \mathbf{k}[T]$ with $\mathbf{k}(x, y) = \mathbf{k}(T)$. Such a pair (x, y) is then called a *parametrization* of F. The key result of this paper is

Theorem. *Suppose F is a polynomial curve with $0 < \deg_Y F < \deg_X F$ and $\deg_Y F \nmid \deg_X F$. If F is equivalent to an element of R_0 then either $F(Y, X) \in R_0$ or $F(X, Y + f(X)) \in R_0$, for some $f(X) \in \mathbf{k}[X]$ with $\deg f(X) \deg_Y F < \deg_X F + \deg_Y F$.*

It is interesting to note that this theorem does *not* generalize to plane curves with one place at infinity—see (1.8). Also, the theorem has interesting consequences for the problem of classifying the embeddings of the line in the affine plane, as will be discussed in the second section.

1. POLYNOMIAL CURVES

1.1. We begin by recalling some notations and definitions. We also state some basic facts concerning plane curves with one place at infinity; these facts can be either found in or deduced from [1].

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1. Given $x \in \mathbf{k}[T]$, the formal derivative dx/dT is denoted $\dot{x}$.
2. Let $F \in R$. The formal derivatives $\partial F/\partial X$ and $\partial F/\partial Y$ are denoted F_X and F_Y respectively. If $F = \sum a_{ij}X^iY^j$ $(a_{ij} \in \mathbf{k})$ then $F^{(p)} = \sum a_{ij}^p X^iY^j$. If $F \neq 0$, let $\deg_X F = \max\{i | a_{ij} \neq 0 \text{ for some } j\}$, $\deg_Y F = \max\{j | a_{ij} \neq 0 \text{ for some } i\}$; also define $\deg_X 0 = \deg_Y 0 = 0$ and $\deg 0 = 0$. The *bidegree* of F is $\operatorname{bideg} F = (\deg_X F, \deg_Y F)$.
3. To compare bidegrees, we partially order $\mathbf{Z}^2$ (where $\mathbf{Z}$ is the ring of integers) by declaring $(a, b) < (c, d)$ if either $\min(a, b) < \min(c, d)$, or $\min(a, b) = \min(c, d)$ and $\max(a, b) < \max(c, d)$.
4. $(a, b) \in \mathbf{Z}^2$ is said to be *nonprincipal* if either $ab = 0$, or $a \nmid b$ and $b \nmid a$. $F \in R$ *has nonprincipal bidegree* if $\operatorname{bideg} F$ is nonprincipal.
5. An integral domain which strictly contains $\mathbf{k}$ is said to have *one place at infinity* if it is contained in all but one of the valuation rings (containing $\mathbf{k}$) of its field of fractions. $F \in R$ is said to have *one place at infinity* if it is irreducible and R/FR has one place at infinity.
6. Suppose $F \in R$ has one place at infinity, $\deg_X F = n > 0$ and $\deg_Y F = m > 0$.
   - If $m|n$, say $md = n$, then $\operatorname{bideg} F(X, Y + \lambda X^d) < \operatorname{bideg} F$ for a unique $\lambda \in \mathbf{k}^*$.
   - If $n|m$, say $nd = m$, then $\operatorname{bideg} F(X + \lambda Y^d, Y) < \operatorname{bideg} F$ for a unique $\lambda \in \mathbf{k}^*$.

   In particular, F is equivalent to some G with nonprincipal bidegree.
7. We define an equivalence relation $\simeq$ on R by declaring $F \simeq G$ if there are λ, a, $c \in \mathbf{k}^*$ and b, $d \in \mathbf{k}$ such that $G = \lambda F(aX + b, cY + d)$. Note that if $F \simeq G$ then $F_X = 0 \Leftrightarrow G_X = 0$ and $F_Y = 0 \Leftrightarrow G_Y = 0$.
8. Suppose $F \in R$ (resp. $G \in R$) has one place at infinity, has nonprincipal bidegree and satisfies $\deg_Y F < \deg_X F$ (resp. $\deg_Y G < \deg_X G$). If $F \sim G$ then $\operatorname{bideg} F = \operatorname{bideg} G$ and $G \simeq F(X, Y + f(X))$ for some $f(X) \in \mathbf{k}[X]$ with $\deg f(X) \deg_Y F < \deg_X F$.
9. For $F \in R$, the following are equivalent:
   - F is a polynomial curve,
   - F has one place at infinity and has genus zero (which means that the field of fractions of R/FR is isomorphic to $\mathbf{k}(T)$).
10. If (x, y) is a parametrization of a polynomial curve F then $\deg_X F = \deg y$ and $\deg_Y F = \deg x$. Moreover, $F \in R_0 \Leftrightarrow y \in \mathbf{k}[T^p]$ and $F \in \mathbf{k}[X, Y^p] \Leftrightarrow x \in \mathbf{k}[T^p]$.

Definition 1.2. We define an injective ring homomorphism $\varphi\colon R \to R$ by $\varphi(F) = F^{(p)}(X^p, Y)$. Note that $\varphi(R) = R_0$.

1.3. Let $F \in R$ be such that $F_Y \neq 0$. Observe that

- F is irreducible if and only if $\varphi(F)$ is.
- If F is irreducible and x, y are the images of X, Y in R/FR, then $\mathbf{k}(x, y^p) = \mathbf{k}(x, y)$. Moreover, $\varphi(F)(x, y^p) = 0$, which allows us to identify $R/\varphi(F)R$ with the subring $\mathbf{k}[x, y^p]$ of $\mathbf{k}[x, y] = R/FR$.

- F has one place at infinity if and only if $\varphi(F)$ has one place at infinity.
- F is a polynomial curve if and only if $\varphi(F)$ is a polynomial curve.

Definitions 1.4.

1. An element θ of $\operatorname{Aut}_{\mathbf{k}} R$ is called a *φ-automorphism* of R if it satisfies the following equivalent conditions:
   - There exists $\theta' \in \operatorname{Aut}_{\mathbf{k}} R$ such that $\theta\varphi = \varphi\theta'$,
   - $\theta(R_0) = R_0$,
   - $\theta(Y) \in R_0$.

   The set of all φ-automorphisms of R is a subgroup of $\operatorname{Aut}_{\mathbf{k}} R$ denoted $\operatorname{Aut}_\varphi R$.
2. $F, G \in R$ are *φ-equivalent* ($F \approx G$) if $G = \lambda\theta(F)$ for some $\theta \in \operatorname{Aut}_\varphi R$ and $\lambda \in \mathbf{k}^*$. Note that if $F \approx G$ then $F \in R_0 \Leftrightarrow G \in R_0$.
3. Let $F \in R$ have one place at infinity. Say that F is *φ-stopped* if the following hold:
   - $\deg_X F \nmid \deg_Y F$,
   - $p \deg_Y F \nmid \deg_X F$,
   - $\deg_Y F | \deg_X F$.

   Note that if F is φ-stopped then it does not have nonprincipal bidegree.

Lemma 1.5. *If $F \in R$ has one place at infinity then $F \approx G$ for some G which either is φ-stopped or has nonprincipal bidegree.*

Proof. Suppose F is not φ-stopped and does not have nonprincipal bidegree. Write $\operatorname{bideg} F = (n, m)$; then $mn \neq 0$ and either $n|m$ or $pm|n$. In the first case, let $d = m/n$ and $H = F(X+\lambda Y^d, Y)$; in the second case, let $d = n/(pm)$ and $H = F(X, Y + \lambda X^{pd})$. Since $F \approx H$ and since, by (1.1.6), in each case $\lambda \in \mathbf{k}^*$ may be chosen so that $\operatorname{bideg} H < \operatorname{bideg} F$, we are done. □

Lemma 1.6. *Let $F_1, F_2 \in R$ and suppose that $F_2 \simeq F_1(X, Y + f_1(X))$ for some $f_1(X) \in \mathbf{k}[X]$. Then $F_1 \simeq F_2(X, Y + f_2(X))$ for some $f_2(X) \in \mathbf{k}[X]$ with $\deg f_2(X) = \deg f_1(X)$.*

Proof. Write $\lambda F_2(aX + b, cY + d) = F_1(X, Y + f_1(X))$, where λ, a, $c \in \mathbf{k}^*$ and b, $d \in \mathbf{k}$. Substituting $Y - f_1(X)$ for Y in that equation, we get $F_1 = \lambda F_2(aX + b, cY - cf_1(X) + d)$. Substituting $a^{-1}X - a^{-1}b$ for X and $c^{-1}Y - c^{-1}d$ for Y yields

$$F_1\left(a^{-1}X - a^{-1}b, c^{-1}Y - c^{-1}d\right) = \lambda F_2\left(X, Y - cf_1\left(a^{-1}X - a^{-1}b\right)\right),$$

hence $F_1 \simeq F_2(X, Y + f_2(X))$ with $f_2(X) = -cf_1(a^{-1}X - a^{-1}b)$. □

Proof of the main theorem. We may assume that $F \notin \mathbf{k}[X, Y^p]$, i.e., $F_Y \neq 0$. By applying (1.5) to an element of R_0 equivalent to F, we see that F is equivalent to some $G \in R_0$ such that G has nonprincipal bidegree or is φ-stopped.

If G has nonprincipal bidegree then, by (1.1.8), there exists $f(X) \in \mathbf{k}[X]$ with $\deg f(X) \deg_Y F < \deg_X F$ such that if we set $H = F(X, Y + f(X))$ then $H \simeq G$ or $H \simeq G(Y, X)$. Since $F_Y \neq 0$, we have $H_Y \neq 0$, thus $H \not\simeq G(Y, X)$. Therefore $H \simeq G \in R_0$, whence $H \in R_0$ and we are done.

If G is φ-stopped, write $\operatorname{bideg} G = (n, m)$. Then it follows from (1.1.6) that there exists $g(X) \in \mathbf{k}[X]$ of degree $d = n/m$ such that $H = G(X, Y + g(X))$ satisfies $\deg_Y H \nmid \deg_X H$. Note that $d \geq 2$ and $d \not\equiv 0 \pmod p$.

Claim. $0 < \deg_Y H < \deg_X H$. Clearly, $\deg_Y H = \deg_Y G > 0$. For the other inequality, consider a parametrization (x, y) of G; then $\deg x = m$, $\dot{y} = 0$ and $(x, y - g(x))$ is a parametrization of H. Now

$$\deg_X H = \deg(y - g(x)) > \deg \frac{d}{dT}(y - g(x)) = \deg\left(g'(x)\,\dot{x}\right) \geq \deg g'(x)$$

since $\dot{x} \neq 0$. Since $\deg g'(x) = (d-1)m = (d-1)\deg_Y H$ we have in fact

$$\deg_X H > (d-1)\deg_Y H,$$

from which the claim follows. In particular, H has nonprincipal bidegree. By (1.1.8), it follows that $\operatorname{bideg} F = \operatorname{bideg} H$ and $F \simeq H(X, Y + h(X))$, for some $h(X) \in \mathbf{k}[X]$ such that

$$\deg h(X) < \frac{\deg_X H}{\deg_Y H} = \frac{\deg_X H}{\deg_Y G} < \frac{\deg_X G}{\deg_Y G} = d.$$

So $\deg(g(X) + h(X)) = d$ and $F \simeq G(X, Y + g(X) + h(X))$. By the first part of (1.6), there exists $f(X) \in \mathbf{k}[X]$ of degree d such that $F(X, Y + f(X)) \simeq G \in R_0$, hence $F(X, Y + f(X)) \in R_0$. Since $\deg_X H > (d-1)\deg_Y H$ and $\operatorname{bideg} F = \operatorname{bideg} H$ we see that $\deg f(X)$ satisfies the desired condition, and this completes the proof. □

Note that the following fact was established in the above proof.

Lemma 1.7. *Suppose G is a polynomial curve, G is φ-stopped and $G \in R_0$. Then, for some $g(X) \in \mathbf{k}[X]$, the polynomial $H = G(X, Y + g(X))$ satisfies*

$$0 < \deg_Y H < \deg_X H \quad \text{and} \quad \deg_Y H \nmid \deg_X H.$$

Moreover, the degree d of any such $g(X)$ satisfies $d = \deg_X G / \deg_Y G$, $d \geq 2$ and $d \not\equiv 0 \pmod p$.

As mentioned in the introduction, the theorem does not generalize to all $F \in R$ with one place at infinity. This is shown in

Example 1.8. Let $\operatorname{char} \mathbf{k} = 3$; then $F = -X^5 - X^4 - X^3Y + X^2Y + X + Y^3 - Y^2$ has one place at infinity but is not a polynomial curve (one can see that the genus is 4). It satisfies $0 < \deg_Y F < \deg_X F$ and $\deg_Y F \nmid \deg_X F$. We have $F \sim F_1 \sim F_2 \sim F_3 \in R_0$, where

$$F_1 = F\left(X, Y + X^3 - X^2\right) = X^9 + X + Y^3 - Y^2,$$

$$F_2 = F_1\left(X + Y^2, Y\right) = X^9 + X + Y^{18} + Y^3,$$

$$F_3 = F_2(Y, X) = X^{18} + X^3 + Y^9 + Y.$$

On the other hand, $F(Y, X) \notin R_0$ and if, for some $f(X) \in \mathbf{k}[X]$, the polynomial $G = F(X, Y + f(X))$ is in R_0, then

$$0 = G_X = F_X(X, Y + f(X)) + F_Y(X, Y + f(X)) f'(X).$$

Looking at G_X as a polynomial in Y, we see that the coefficient of Y is $f'(X) - X$. Hence $f'(X) = X$ and the above equation becomes

$$0 = F_X(X, Y + f(X)) + F_Y(X, Y + f(X)) X = 1,$$

which is absurd. Hence the theorem does not hold for F.

One can prove that, for the F given in (1.8), $\operatorname{bideg} G > \operatorname{bideg} F$ for all G such that $\varphi(G) \sim F$. Hence the next result cannot be generalized either.

Corollary 1.9. *Let $F \in R$ be a polynomial curve not equivalent to Y. If there exists $G \in R$ such that $F \sim \varphi(G)$, then there exists $H \in R$ such that $F \sim \varphi(H)$ and* $\operatorname{bideg} H < \operatorname{bideg} F$.

Proof. We first consider the case where F has nonprincipal bidegree. Replacing, if necessary, F by $F(Y, X)$, we may assume that $0 < \deg_Y F < \deg_X F$ (we have $\deg_Y F > 0$ because $F \not\sim Y$). By the main theorem, either $F(Y, X) \in R_0$ or $F(X, Y + f(X)) \in R_0$ for some $f(X) \in \mathbf{k}[X]$ with $\deg f(X) \deg_Y F < \deg_X F + \deg_Y F$.

In the first case, write $F(Y, X) = \varphi(H)$; then clearly $\operatorname{bideg} H < \operatorname{bideg} F$.

In the second case, write $F(X, Y + f(X)) = \varphi(H)$. So $\deg_Y H = \deg_Y \varphi(H) = \deg_Y F$ and, by the condition on $\deg f(X)$,

$$\deg_X H = \frac{1}{p} \deg_X \varphi(H) < \deg_X F,$$

i.e., $\operatorname{bideg} H < \operatorname{bideg} F$.

If F does not have nonprincipal bidegree, there exists $F_0 \sim F$ such that F_0 has nonprincipal bidegree and $\operatorname{bideg} F_0 < \operatorname{bideg} F$. The result follows by applying the case proved above to F_0. □

Lemma 1.10. *If $\theta \in \operatorname{Aut}_k R$ satisfies $R_0 \cap \theta^{-1}(R_0) \not\subseteq R^p$ then $\theta \in \operatorname{Aut}_\varphi R$. In particular, for $F, G \in R_0 \setminus R^p$ we have $F \sim G \Rightarrow F \approx G$.*

Proof. Choose $F \in R_0 \cap \theta^{-1}(R_0)$ with $F \notin R^p$. Then $F_X = 0$, $F_Y \neq 0$, and $\theta(F) \in R_0$, so

$$0 = \tfrac{\partial}{\partial X}(\theta(F)) = \tfrac{\partial}{\partial X} F(\theta(X), \theta(Y)) = F_Y(\theta(X), \theta(Y)) \tfrac{\partial}{\partial X}(\theta(Y)),$$

which implies that $\frac{\partial}{\partial X}(\theta(Y)) = 0$, and hence that $\theta \in \operatorname{Aut}_\varphi R$. The last assertion is clear. □

Corollary 1.11. *Suppose $F, G \in R$ are such that $F_Y \neq 0$ and $G_Y \neq 0$. If $\varphi(F) \sim \varphi(G)$ then $F \sim G$.*

Proof. Observe that, for $F, G \in R$, $\varphi(F) \approx \varphi(G) \Rightarrow F \sim G$. Hence this follows immediately from (1.10). □

Remark. $F \sim G \not\Rightarrow \varphi(F) \sim \varphi(G)$.

2. Lines

The results of the first section have interesting consequences for the classification of lines problem. We now recall that problem (see [2, 3, 5, 6]).

A *line* is an $F \in R$ such that $R/FR \cong \mathbf{k}[T]$; hence, in particular, a line is a polynomial curve. Two lines F_1, F_2 are *equivalent* if $F_1 \sim F_2$ as defined in the introduction. It can be seen that the set of equivalence classes of lines has cardinality $|\mathbf{k}|$ (in contrast with the situation in characteristic zero, where all lines are equivalent—see [2]). A line F is a *coordinate line* if $F \sim Y$; otherwise it is said to be *wild*. Every example of wild line currently known, say F, can be obtained as $F = F_N$ for some sequence of lines $(F_0, \dots, F_N)$ such that $F_0 = Y$ and, for each i with $0 \le i < N$, either $F_{i+1} \sim F_i$ or $F_{i+1} = \varphi(F_i)$. (Note that, for $F \in R$, $\varphi(F)$ is a line $\Leftrightarrow F$ is a line and $F_Y \in \mathbf{k}^*$. Equivalently, if x, $y \in \mathbf{k}[T]$ then $\mathbf{k}[x, y^p] = \mathbf{k}[T] \Leftrightarrow \mathbf{k}[x, y] = \mathbf{k}[T]$ and $\dot{x} \in \mathbf{k}^*$. See Proposition 1 of [6] for this.)

Let us consider an oriented graph $\mathscr{L}$ whose vertices are the equivalence classes of lines and whose links are defined as follows:

- There is no link from a vertex to itself.
- If A and B are distinct vertices then there is at most one link going from A to B, and there is a link if and only if there exists $F \in A$ such that $\varphi(F) \in B$.

We write $A \to B$ to indicate that there is a link from A to B. If C_0 denotes the vertex containing the coordinate lines, then the fact stated above says that if A is any vertex of $\mathscr{L}$ currently known then there exists a path $C_0 = A_0 \to \cdots \to A_n = A$. It is conjectured that *all* vertices have this property. As far as we know, the strongest version of that conjecture is Conjecture 1 of [6], which can be stated as

Conjecture 1. If $F \in R$ is a wild line then there exists $\theta \in \operatorname{Aut}_{\mathbf{k}} R$ such that

1. $\theta(F) \in \mathbf{k}[X^p, Y]$, i.e., $\theta(F) = \varphi(G)$ for some line G;
2. $\operatorname{bideg} G < \operatorname{bideg} F$.

A seemingly weaker version is obtained by dropping the second requirement:

Conjecture 2. If $F \in R$ is a line then there exists $\theta \in \operatorname{Aut}_{\mathbf{k}} R$ such that $\theta(F) \in \mathbf{k}[X^p, Y]$.

However, (1.9) shows that the second conjecture is in fact equivalent to the first one.

We will now derive some properties of $\mathscr{L}$ from the results of the first section. First, observe that $\mathscr{L}$ is a subgraph of the oriented graph $\mathscr{P}$ whose vertices are the equivalence classes of polynomial curves, and whose links are defined exactly as in the case of $\mathscr{L}$. Further, define a partial order on the set of vertices of $\mathscr{P}$ by declaring $A < B$ if there exists $F_0 \in A$ such that $\operatorname{bideg} F_0 < \operatorname{bideg} F$ for all $F \in B$. Then it is clear that $C_0 < B$ whenever B is any vertex of $\mathscr{P}$ other than C_0. Also, note that C_0 is not the target of any link; indeed, if F

is a polynomial curve and $\varphi(F) \in C_0$ then $\varphi(F) \sim Y = \varphi(Y) \Rightarrow F \sim Y$ by (1.11). Some corollaries of the results of the first section are gathered in

Theorem 2.1.

1. *No vertex of* $\mathcal{P}$ *is the target of more than one link.*
2. *For distinct vertices* A *and* B *of* $\mathcal{P}$, *we have* $A < B$ *whenever* $A \to B$.
3. *No sequence* $(A_i)_{i=1}^{\infty}$ *of vertices of* $\mathcal{P}$ *satisfies* $A_i \leftarrow A_{i+1}$ *for all* i.
4. $\mathcal{P}$ *has no loops.*
5. *Each connected component of* $\mathcal{P}$ *contains exactly one vertex which is not the target of any link.*

Proof. (1) is just (1.11). For (2), consider a link $A \to B$. Choose $F \in B$ with nonprincipal bidegree; for some $G \in A$ we have $\varphi(G) \in B$, hence $F \sim \varphi(G)$. Since B is a target we have $B \neq C_0$, hence $F \not\sim Y$ and by (1.9) $F \sim \varphi(H)$ for some H such that $\operatorname{bideg} H < \operatorname{bideg} F$; by (1.11), $H \in A$. Clearly, $\operatorname{bideg} H < \operatorname{bideg} F'$ for all $F' \in B$ and (2) is proved.

To prove (3), observe that such a sequence $(A_i)_{i=1}^{\infty}$ would give rise, by (2), to an infinite sequence

$$(a_1, b_1) > (a_2, b_2) > \cdots$$

in $\mathbf{Z}^2$, with $(a_i, b_i) > (0, 0)$ for all i. Now, such a sequence does not exist.

If there is a loop in $\mathcal{P}$, then that loop either contains a vertex which is the target of two links, or gives rise to a sequence $(A_i)_{i=1}^{\infty}$ of vertices such that $A_i \leftarrow A_{i+1}$ for all i. So (4) follows from (1) and (3).

By (3), each connected component of $\mathcal{P}$ contains at least one vertex which is not the target of any link; uniqueness follows from (1), so (5) is proved. □

Clearly, this theorem is still true if we replace "$\mathcal{P}$" by "$\mathcal{L}$" everywhere. (R. Ganong pointed out that the third assertion of the theorem (for $\mathcal{L}$) can also be deduced from [4, Theorem 1.4], first assertion.) So, in particular, from the last assertion of the theorem we see that the conjectures are equivalent to connectedness of $\mathcal{L}$. We also point out that the main theorem gives us a lot of information about the conjectured automorphism.

More generally, one may dream of classifying the polynomial curves, i.e., *understanding which vertices of* $\mathcal{P}$ *are not targets of links.* For this, too, the main theorem gives interesting information. For instance, let m, n be relatively prime integers such that $1 < m < n$ and $mn \not\equiv 0 \pmod p$, and let $C_{m,n}$ be the vertex of $\mathcal{P}$ which contains $X^n + Y^m$. Then it follows from the main theorem that $C_{m,n}$ is not a target. Hence, by (2.1), $\mathcal{P}$ has infinitely many connected components.

To conclude, we note that the main theorem can be stated in terms of parametrizations as follows.

Corollary 2.2. *Let* F *be a polynomial curve with parametrization* (x, y) *such that* $0 < \deg x < \deg y$ *and* $\deg x \nmid \deg y$. *If there exists* $\theta \in \mathrm{Aut}_{\mathbf{k}} R$ *such that* $\theta(F) \in \mathbf{k}[X^p, Y]$ *then one of the following conditions holds*:

1. $x \in \mathbf{k}[T^p]$,
2. $y - f(x) \in \mathbf{k}[T^p]$ *for some* $f(X) \in \mathbf{k}[X]$ *with* $\deg f(X) \deg x < \deg x + \deg y$.

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