THE ORLICZ-PETTIS THEOREM FAILS
FOR LUMER'S HARDY SPACES \((LH)^p(B)\)

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Abstract. In this paper we prove that if \(n > 1\) and \(0 < p < 1\) then the
Lumer's Hardy space \((LH)^p(B_n)\) of the unit ball \(B_n\) in \(C^n\) does not have the
Orlicz-Pettis property.

1. Introduction

If \(B = B_n\) is the unit ball in \(C^n\), \(n \geq 1\), and \(0 < p < \infty\), Lumer’s Hardy
space \((LH)^p(B_n)\) is defined to consist of all holomorphic functions \(f\) on \(B\)
such that \(|f|^p \leq u\) for some pluriharmonic function \(u\) on \(B_n\). \((LH)^p(B_n)\)
equipped with the quasi-norm

\[
\|f\|_p = \inf u(0)^{1/p},
\]

where the infimum is being taken over all pluriharmonic majorants \(u\) of \(|f|^p\), is
a complete locally bounded space (a Banach space if \(p \geq 1\)), whose topological
dual separates the points (see [6]). When \(n = 1\), pluriharmonic is the same as
harmonic, so \((LH)^p(B_1)\) coincides with the classical Hardy space \(H^p = H^p(U)\)
of the unit disc in \(C\).

W. Rudin [8] showed that, from the standpoint of functional analysis,
Lumer’s Hardy spaces have some pathological properties. For example, if \(p \geq 1\) and
\(n > 1\) then \((LH)^p(B_n)\) contains a subspace isomorphic to the space \(l^\infty\)
of all bounded complex sequences. In particular, \((LH)^p(B_n)\) is not separable
and \((LH)^2(B_n)\) is not a Hilbert space.

In the present paper we note that if \(0 < p < 1\), \((LH)^p(B_n)\) is still nonsepara-
able and it has another unexpected pathology; it does not have the Orlicz-Pettis
property.

2. The Orlicz-Pettis Property

We recall that a topological vector space \(X = (X, \tau)\) whose topological dual
\(X'\) separates the points is said to have the Orlicz-Pettis Property (OPP), if each
weakly subseries convergent series in \( X \) (i.e. a series \( \sum x_n \) in \( X \) such that weak-
\lim_{n \to \infty} \sum_{j=1}^{n} x_{k_j} \) exists for each increasing sequence \( \{k_j\} \) of positive integers) is \( \tau \)-convergent (i.e. the Orlicz–Pettis theorem holds in \( X \)). It is well known
that all locally convex spaces and all separable \( F \)-spaces (complete metrizable
t.v.s.) with separating duals have the OPP (see [3, 4]). For a long time it was left
open if the Orlicz-Pettis theorem can be extended to the class of nonseparable
\( F \)-spaces with separating duals. The first counterexample was given by J. H.
Shapiro [10], who proved that the harmonic \( h^p \) spaces, \( 0 < p < 1 \), do not have
the OPP. Recently, the author showed [6] that the Orlicz–Pettis theorem fails
for weak \( l^p \) sequence spaces \( l(p, \infty) \) if \( 0 < p \leq 1 \). In the present paper we
find the third largest class of spaces without the Orlicz–Pettis property.

In the sequel we will use the following equivalent version of the OPP; an
\( F \)-space \( (X, \tau) \) with separating dual has the OPP if and only if each series
in \( X \) which is subseries convergent in the Mackey topology \( \mu(X) \) of \( X \) is
\( \tau \)-convergent (see [7]).

Let us recall that the Mackey topology \( \mu(X) \) of an \( F \)-space \( (X, \tau) \) (i.e. the
strongest locally convex topology on \( X \) which produces the same topological
dual as \( X \)) coincides with the strongest locally convex topology on \( X \) which
is weaker than \( \tau \). Thus, if \( \mathcal{B} \) is a base of neighborhoods of zero for \( \tau \) then
the family \( \{\text{conv } U : U \in \mathcal{B}\} \) is a base of neighborhoods of zero for \( \mu(X) \) (see
[10]).

### 3. Notations

Throughout the paper we use standard notations as in [9]. \( C \) will denote
the complex field, \( Z_+ \) and \( R_+ \) the sets of all nonnegative integers and reals,
respectively. \( C^n, R^n, Z^n_+ \) will be the Cartesian products of \( n \) copies of \( C, R_+, \) and \( Z_+ \), respectively. We will treat \( R^n_+ \) as a subset in \( C^n \).

For \( z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in C^n, \alpha = (\alpha_1, \ldots, \alpha_n) \in Z^n_+ \)
and \( \lambda \in C \) we denote
\[
\langle z, w \rangle = \sum_{j=1}^{n} z_j w_j \quad \text{the standard inner product of } z \text{ and } w,
\]
\[
|z| = |z, z|^{1/2} \quad \text{the norm of } z \text{ in } C^n,
\]
\[
\alpha! = \alpha_1! \cdots \alpha_n!,
\]
\[
|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad zw = (z_1 w_1, \ldots, z_n w_n),
\]
\[
z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad \lambda z = (\lambda z_1, \ldots, \lambda z_n).
\]

\( B = B^n = \{z \in C^n : |z| < 1\} \) will denote the unit ball, \( S = \partial B \) the unit
sphere in \( C^n \), and \( U \) the unit disc in \( C \). For any region \( \Omega \) in \( C^n \), \( H(\Omega) \) is
the space of all holomorphic functions on \( \Omega \) while \( H^\infty(\Omega) \) is the subspace of
\( H(\Omega) \) consisting of all bounded functions, which is equipped with the sup-norm
\( \| \cdot \|_\infty \).
Finally, if \( f \) is any function on \( B^n \) and \( z \in \mathbb{S} \), the slice function \((f)_{z}\) on \( U \) is defined by \((f)_{z}(\lambda) = f(\lambda z)\).

4. Lemma. For \( \zeta \in \mathbb{S} \), \( 0 < p < 1 \) and \( k = 1, 2 \ldots \) define the functions \( v_{k, \zeta} \in H^{\infty}(B) \) by \( v_{k, \zeta}(z) = \langle z, \zeta \rangle^k \), \( z \in B \). Then
   \begin{enumerate}
   \item \( \|v_{k, \zeta}\|_p \geq 1 \) for each \( \zeta \) and \( k \),
   \item \( \mu((LH)^p(B)) = \lim_{k \to \infty} v_{k, \zeta} = 0 \) for every \( \zeta \in \mathbb{S} \).
   \end{enumerate}

**Proof.** (a) Fix \( k \in \mathbb{N} \) and \( \zeta \in \mathbb{S} \). Let \( u \) be any pluriharmonic majorant of \( |v_{k, \zeta}|^p \). The slice function \((u)_{z}\) is harmonic on \( U \), so
\[
(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (u)_{z}(\text{re}^{i\theta}) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} |v_{k, \zeta}(\text{re}^{i\theta} \zeta)|^p d\theta.
\]
Consequently, \( u(0) \geq 1 \) for every pluriharmonic majorant of \( |v_{k, \zeta}|^p \).

(b) For \( \zeta \in \mathbb{S} \) and \( f \in H^p(U) \) define the holomorphic function \( J_{\zeta}f \) on \( B \) by \((J_{\zeta}f)(z) = f((z, \zeta)), z \in B \). Fix \( \zeta \in U \) and an arbitrary \( f \in H^p(U) \). Let \( h = \text{Re} g \) be the smallest harmonic majorant of \( |f|^p \), where \( g \in H(U) \). Obviously, \( u = \text{Re} J_{\zeta}g \) is a pluriharmonic majorant of \( |J_{\zeta}f|^p \) and \( u(0) = h(0) = \|f\|_p \). Therefore, \( \|J_{\zeta}f\|_p \leq \|f\|_p \) for every \( f \in H^p(U) \), so \( J_{\zeta} \) is a continuous linear operator from \( H^p(U) \) into \( (LH)^p(B) \). Consequently, \( J_{\zeta} \) is continuous if we equip each \( H^p \) and \( (LH)^p \) with its own Mackey topology. However, \( v_{k, \zeta} = J_{\zeta}(\lambda^k) \), where \( \lambda^k \) is the standard monomial of degree \( k \) on \( U \), so for the proof of the lemma it is enough to show that \( \mu(H^p) - \lim_{k \to \infty} \lambda^k = 0 \).

It is well known that the Mackey topology of \( H^p \) coincides with the topology defined on \( H^p \) by the norm
\[
\|f\| = \int_U |f(\lambda)|(1 - |\lambda|)^{(1/p) - 2} A(\lambda),
\]
where \( A \) is the area measure on \( C \) (see [2]). Now, the \( \mu(H^p) \)-convergence of \( \lambda^k \) to zero immediately follows from the dominated convergence theorem.

5. Theorem. If \( n > 1 \) and \( 0 < p < 1 \), then the Lumer’s Hardy space \((LH)^p(B^n)\) does not have the Orlicz-Pettis property.

**Proof.** We will construct a sequence \((f_j)\) in \( H^\infty(B^n) \) such that the series \( \sum f_j \) is \( \mu((LH)^p) \)-subseries convergent but is not convergent in \((LH)^p \). This series will be almost the same as that used by W. Rudin [7] to build a copy of \( l^\infty \) in \((LH)^p \), \( p \geq 1 \).

Let \( \{U_j\} \) be a sequence of pairwise disjoint balls in \( \mathbb{R}^n_+ \) with the centers \( \zeta_j = (\zeta_{j1}, \ldots, \zeta_{jn}), |\zeta_j| = 1, j = 1, 2, \ldots, \) such that
\begin{enumerate}
   \item \( \zeta_{ji} \leq (2n)^{-1}, i = 2, \ldots, n, \)
   \item \( r_i \geq 2^{-1} \) for each \( r = (r_1, \ldots, r_n) \in U_j, \)
\end{enumerate}
and let
\[ \varepsilon_j = \sup \left\{ \sum_{i=1}^{n} r_i \zeta_{ji} \colon r \in \mathbb{R}^n_+ \setminus U_j, |r| \leq 1 \right\}, \]
\[ j = 1, 2, \ldots. \] Obviously, \( 0 < \varepsilon_j < 1, \ j = 1, 2, \ldots, \) so we can find a strictly increasing sequence \( \{n_j\} \) of positive integers such that
\begin{enumerate}
  \item[(c)] \( n_j - j > 0, \)
  \item[(d)] \( (2n)^j 2^{-n_j} \leq 2^{-j}, \)
  \item[(e)] \( n_j^{j-1} \varepsilon_j^{n_j-j} \leq 2^{-j}, \)
\end{enumerate}
\( j = 1, 2, \ldots \) ((e) is possible because \( \lim_{x \to \infty} x^a b^x = 0 \) for any \( a > 0 \) and \( 0 < b < 1 \)).

Let us define \( f_j(z) = v_{n_j, \zeta_j}(z) = (z, \zeta_j)^{n_j}, \ z \in \mathbb{B}_n. \) By Lemma (a) the series \( \sum f_j \) is not convergent in \((LH)^p(\mathbb{B}_n)\). We will show that this series is \( \mu((LH)^p(\mathbb{B}_n))\)-convergent. For \( k = 1, 2, \ldots \) and \( j = k, k+1, \ldots \) we define the sets \( A_{k,j} = \{ \alpha \in \mathbb{Z}^n_+ : \alpha_1 \geq k, |\alpha| = n_j \}, \ B_{k,j} = \{ \alpha \in \mathbb{Z}^n_+ : |\alpha| = n_j \} \setminus A_{k,j}, \) and the functions
\[
\begin{align*}
g_{k,j}(z) &= \zeta_{k,j}^{n_j} \left( \sum_{\alpha \in A_{k,j}} \frac{n_j!}{\alpha!} (z_1 \zeta_{k,j1})^{\alpha_1-k} (z_2 \zeta_{k,j2})^{\alpha_2} \cdots (z_n \zeta_{k,jn})^{\alpha_n}, \\
h_{k,j}(z) &= \sum_{\alpha \in B_{k,j}} \frac{n_j!}{\alpha!} (z \zeta_j)^{\alpha} \quad (z \in \mathbb{B}_n).\end{align*}
\]

We will show that \( g_{k,j}, h_{k,j} \in H^\infty(\mathbb{B}_n) \) and
\[
(*) \quad |g_{k,j}(z)| \leq \begin{cases} 2 & \text{if } z \in V_j, \\ 2^{-j} & \text{if } z \in \mathbb{B}_n \setminus V_j, \end{cases}
\]
\[ \|h_{k,j}\|_\infty \leq 2^{-j}, \]
for all \( k, j, \) where \( V_j = \mathcal{F}^{-1}(U_j) \cap \mathbb{B}_n \) and \( \mathcal{F} \) is the function acting from \( \mathbb{C}^n \) into \( \mathbb{R}^n_+ \) defined by \( \mathcal{F}(z_1, \ldots, z_n) = (|z_1|, \ldots, |z_n|). \) Indeed, for \( z \in V_j \) we have
\[
|g_{k,j}(z)| \leq |z_1|^{-k} \sum_{|\alpha| = n_j} \frac{n_j!}{\alpha!} |z_1 \zeta_{k,j1}|^{\alpha_1} \cdots |z_n \zeta_{k,jn}|^{\alpha_n}
\leq |z_1|^{-j} \left( \sum_{i=1}^{n} |z_i| \zeta_{ji} \right)^{n_j} \leq |z_1|^{-j} \leq 2 \quad (\text{by (b)})
\]
and
\[
|h_{k,j}(z)| \leq \sum_{\alpha \in B_{k,j}} \frac{n_j!}{\alpha!} |\zeta_j^1|^{\alpha_1} \cdots |\zeta_j^n|^{\alpha_n}
\]
\[
\leq \sum_{\alpha \in B_{k,j}} \frac{n_j!}{\alpha!} (2n)^{k-n_j} \quad \text{(by (a))}
\]
\[
\leq (2n)^{j-n_j} \sum_{|\alpha| = n_j} \frac{n_j!}{\alpha!}
\]
\[
\leq (2n)^{j-n_j} n_j \leq 2^{-j} \quad \text{(by (d)).}
\]

Suppose now that \( z \in B_n \setminus V_j \). Then,
\[
|g_{k,j}(z)| \leq n_j \sum_{\alpha \in A_{k,j}} \frac{(n_j-k)!}{(\alpha-k)! \alpha_1! \cdots \alpha_n!} |z_1 \zeta_j^1|^{\alpha_1-k} |z_2 \zeta_j^2|^{\alpha_2} \cdots |z_n \zeta_j^n|^{\alpha_n}
\]
\[
\leq n_j \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| = n_j-k} \frac{(n_j-k)!}{\alpha!} |z_1 \zeta_j^1|^{\beta_1} \cdots |z_n \zeta_j^n|^{\beta_n}
\]
\[
\leq n_j \left( \sum_{i=1}^n |z_i| |\zeta_j^i| \right)^{n_j} \leq n_j \varepsilon_j^{n_j} \leq 2^{-j} \quad \text{(by (e))}
\]

and
\[
|h_{k,j}(z)| \leq \sum_{|\alpha| = n_j} \frac{n_j!}{\alpha!} |z_1 \zeta_j^1|^{\alpha_1} \cdots |z_n \zeta_j^n|^{\alpha_n}
\]
\[
\leq \left( \sum_{i=1}^n |z_i| |\zeta_j^i| \right)^{n_j} \leq \varepsilon_j^{n_j} \leq 2^{-j} \quad \text{(by (e)).}
\]

Now, define
\[
g_k(z) = \sum_{j=k}^{\infty} g_{k,j}(z) \quad \text{and} \quad h_k(z) = \sum_{j=k}^{\infty} h_{k,j}(z)
\]
for \( z \in B_n \). Obviously, \( \|h_k\|_\infty \leq 2^{1-k} \) (see (*)), so \( \lim_{k \to \infty} \|h_k\|_p = 0 \).

Consequently, \( (\mu((LH)_p) - \lim_{k \to \infty} h_k = 0 \). Moreover, since no two sets \( V_j \) intersect, \( |g_k(z)| \leq 2 + \sum_{j=1}^{\infty} 2^{-j} = 3 \) for each \( z \in B_n \).

Now, for each \( k = 1, 2, \ldots \) we define the multiplication operator \( M_k : (LH)_p \to (LH)_p \) by \( (M_k f)(z) = f(z)g_k(z), \ z \in B \). Obviously, \( \|M_k f\|_p \leq 3 \|f\|_p \) for each \( f \in (LH)_p \) and \( k \), so the family \( \mathcal{M} = \{M_k : k = 1, 2, \ldots \} \) is equicontinuous. This implies that \( \mathcal{M} \) is equicontinuous if we equip \( (LH)_p \) with its Mackey topology.

We are ready to prove that the series \( \sum f_j \) is \( \mu((LH)_p) \)-convergent. Since the sets \( V_j, j = 1, 2, \ldots \), are pairwise disjoint, \( |f_j(z)| \leq 2^{-j} \) for \( z \in B_n \setminus V_j \).
(see (e)), and \( \|f_j\|_\infty \leq 1 \), so the series \( \sum f_j \) is pointwise convergent and its sum \( f \) belongs to \( H^\infty(B_n) \subset (LH)^p(B_n) \). Moreover, we have

\[
f(z) - \sum_{j=1}^{k-1} f_j(z) = \sum_{j=k}^{\infty} \sum_{|\alpha|=n_j} \frac{n_j!}{\alpha!} (z_1^{\alpha_1} \cdots z_n^{\alpha_n})
\]

where \( e_1 \) is the first unit vector in \( \mathbb{C}^n \). Consequently,

\[
f - \sum_{j=1}^{k-1} f_j = M_k(v_k, e_1) + h_k \xrightarrow{k \to \infty} 0 \quad (\mu((LH)^p)),
\]

since the operators \( \{M_k\} \) are \( \mu((LH)^p) \)-equicontinuous and both sequences \( \{v_k, e_1\} \) and \( \{h_k\} \) tend to zero in the Mackey topology of \( (LH)^p \). The series \( \sum f_j \) is \( \mu((LH)^p) \)-convergent to \( f \).

Finally, let \( \sum f_{ij} \) be any subseries of \( \sum f_j \). The proof that this subseries is \( \mu((LH)^p) \)-convergent is quite the same as the above one for the entire series \( \sum f_j \). It is enough to replace \( j \) by \( i_j \) everywhere except the sums \( \sum_{j=1}^{k-1} \) and \( \sum_{j=k}^{\infty} \). The proof is finished.

6. Remark. Let \( \{f_j\} \) be as in the proof of the theorem. It is easily seen that for each sequence \( \gamma = \{\gamma_j\} \in l^\infty \) the series \( \sum \gamma_j f_j \) converges pointwise to a function \( f_\gamma \in H^\infty(B) \) such that \( \|f_\gamma\|_p \leq 2\|\gamma\|_\infty \) (see [8]). Therefore, the linear operator \( T: \gamma \to f_\gamma \) is continuous. Since \( Te_k = v_{n_k, \zeta_k} \), where \( e_k \) is the \( k \)th unit vector in \( l^\infty \), does not tend to zero in \( (LH)^p(B_n) \), so there is an infinite subset \( M \) of \( N \) such that \( T|l^\infty(M) \) is an isomorphism of \( l^\infty(M) \approx l^\infty \) into \( (LH)^p(B_n) \), where \( l^\infty(M) \) is the subspace of \( l^\infty \) consisting of all sequences with supports contained in \( M \) (see [11]).

The proof of this remark is due to Lech Drewnowski.

References


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