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THE ORLICZ-PETTIS THEOREM FAILS FOR LUMER'S HARDY SPACES $(LH)^p(B)$

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ABSTRACT. In this paper we prove that if $n > 1$ and $0 < p < 1$ then the Lumer's Hardy space $(LH)^p(\mathbf{B}_n)$ of the unit ball $\mathbf{B}_n$ in $\mathbf{C}^n$ does not have the Orlicz–Pettis property.

1. INTRODUCTION

If $\mathbf{B} = \mathbf{B}_n$ is the unit ball in $\mathbf{C}^n$, $n \geq 1$, and $0 < p < \infty$, *Lumer's Hardy space* $(LH)^p(\mathbf{B}_n)$ is defined to consist of all holomorphic functions f on $\mathbf{B}$ such that $|f|^p \leq u$ for some pluriharmonic function u on $\mathbf{B}_n$. $(LH)^p(\mathbf{B}_n)$ equipped with the quasi-norm

$$|||\, f \,|||_p = \inf u(0)^{1/p},$$

where the infimum is being taken over all pluriharmonic majorants u of $|f|^p$, is a complete locally bounded space (a Banach space if $p \geq 1$), whose topological dual separates the points (see [6]). When $n = 1$, pluriharmonic is the same as harmonic, so $(LH)^p(\mathbf{B}_1)$ coincides with the classical Hardy space $H^p = H^p(\mathbf{U})$ of the unit disc in $\mathbf{C}$.

W. Rudin [8] showed that, from the standpoint of functional analysis, Lumer's Hardy spaces have some pathological properties. For example, if $p \geq 1$ and $n > 1$ then $(LH)^p(\mathbf{B}_n)$ contains a subspace isomorphic to the space l^∞ of all bounded complex sequences. In particular, $(LH)^p(\mathbf{B}_n)$ is not separable and $(LH)^2(\mathbf{B}_n)$ is not a Hilbert space.

In the present paper we note that if $0 < p < 1$, $(LH)^p(\mathbf{B}_n)$ is still nonseparable and it has another unexpected pathology; it does not have the Orlicz–Pettis property.

2. THE ORLICZ–PETTIS PROPERTY

We recall that a topological vector space $X = (X, \tau)$ whose topological dual X' separates the points is said to have the *Orlicz-Pettis Property* (OPP), if each

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weakly subseries convergent series in X (i.e. a series $\sum x_n$ in X such that weak-$\lim_{n\to\infty}\sum_{j=1}^{n} x_{k_j}$ exists for each increasing sequence $\{k_j\}$ of positive integers) is τ-convergent (i.e. the Orlicz–Pettis theorem holds in X). It is well known that all locally convex spaces and all separable F-spaces (complete metrizable t.v.s.) with separating duals have the OPP (see [3, 4]). For a long time it was left open if the Orlicz-Pettis theorem can be extended to the class of nonseparable F-spaces with separating duals. The first counterexample was given by J. H. Shapiro [10], who proved that the harmonic h^p spaces, $0 < p < 1$, do not have the OPP. Recently, the author showed [6] that the Orlicz–Pettis theorem fails for weak l^p sequence spaces $l(p, \infty)$ if $0 < p \le 1$. In the present paper we find the third largest class of spaces without the Orlicz–Pettis property.

In the sequel we will use the following equivalent version of the OPP; an F-space (X, τ) with separating dual has the OPP if and only if each series in X which is subseries convergent in the Mackey topology $\mu(X)$ of X is τ-convergent (see [7]).

Let us recall that the Mackey topology $\mu(X)$ of an F-space (X, τ) (i.e. the strongest locally convex topology on X which produces the same topological dual as X) coincides with the strongest locally convex topology on X which is weaker than τ. Thus, if $\mathscr{B}$ is a base of neighborhoods of zero for τ then the family $\{\operatorname{conv} U \colon U \in \mathscr{B}\}$ is a base of neighborhoods of zero for $\mu(X)$ (see [10]).

3. Notations

Throughout the paper we use standard notations as in [9]. $\mathbf{C}$ will denote the complex field, $\mathbf{Z}_+$ and $\mathbf{R}_+$ the sets of all nonnegative integers and reals, respectively. $\mathbf{C}^n$, $\mathbf{R}_+^n$, $\mathbf{Z}_+^n$ will be the Cartesian products of n copies of $\mathbf{C}$, $\mathbf{R}_+$, and $\mathbf{Z}_+$, respectively. We will treat $\mathbf{R}_+^n$ as a subset in $\mathbf{C}^n$.

For $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n) \in \mathbf{C}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$ and $\lambda \in \mathbf{C}$ we denote

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j = \text{the standard inner product of } z \text{ and } w,$$

$$|z| = \langle z, z\rangle^{1/2} = \text{the norm of } z \text{ in } \mathbf{C}^n,$$

$$\alpha! = \alpha_1! \cdots \alpha_n!,$$

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad zw = (z_1 w_1, \dots, z_n w_n),$$

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \qquad \lambda z = (\lambda z_1, \dots, \lambda z_n).$$

$\mathbf{B} = \mathbf{B}_n = \{z \in \mathbf{C}^n \colon |z| < 1\}$ will denote the unit ball, $\mathbf{S} = \partial\mathbf{B}$ the unit sphere in $\mathbf{C}^n$, and $\mathbf{U}$ the unit disc in $\mathbf{C}$. For any region Ω in $\mathbf{C}^n$, $H(\Omega)$ is the space of all holomorphic functions on Ω while $H^\infty(\Omega)$ is the subspace of $H(\Omega)$ consisting of all bounded functions, which is equipped with the sup-norm $\|\cdot\|_\infty$.

Finally, if f is any function on $\mathbf{B}^n$ and $\zeta \in \mathbf{S}$, the slice *function* $(f)_\zeta$ on $\mathbf{U}$ is defined by $(f)_\zeta(\lambda) = f(\lambda\zeta)$.

4. **Lemma.** *For* $\zeta \in \mathbf{S}$, $0 < p < 1$ *and* $k = 1, 2 \ldots$ *define the functions* $v_{k,\zeta} \in H^\infty(\mathbf{B})$ *by* $\nu_{k,\zeta}(z) = \langle z, \zeta\rangle^k$, $z \in \mathbf{B}$. *Then*

(a) $|||\, v_{k,\zeta} \,|||_p \geq 1$ *for each* ζ *and* k,

(b) $\mu((LH)^p(\mathbf{B})) - \lim_{k\to\infty} v_{k,\zeta} = 0$ *for every* $\zeta \in \mathbf{S}$.

Proof. (a). Fix $k \in \mathbf{N}$ and $\zeta \in \mathbf{S}$. Let u be any pluriharmonic majorant of $|v_{k,\zeta}|^p$. The slice function $(u)_\zeta$ is harmonic on $\mathbf{U}$, so

$$u(0) = (u)_\zeta(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (u)_\zeta(\mathrm{r}e^{i\theta})\, d\theta \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |v_{k,\zeta}(\mathrm{r}e^{i\theta}\zeta)|^p\, d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\langle \mathrm{r}e^{i\theta}\zeta, \zeta\rangle^k|^p\, d\theta = r^{kp} \qquad \text{for all } r \in (0, 1).$$

Consequently, $u(0) \geq 1$ for every pluriharmonic majorant of $|v_{k,\zeta}|^p$.

(b) For $\zeta \in \mathbf{S}$ and $f \in H(\mathbf{U})$ define the holomorphic function $J_\zeta f$ on $\mathbf{B}$ by $(J_\zeta f)(z) = f(\langle z, \zeta\rangle)$, $z \in \mathbf{B}$. Fix $\zeta \in \mathbf{U}$ and an arbitrary $f \in H^p(\mathbf{U})$. Let $h = \operatorname{Re} g$ be the smallest harmonic majorant of $|f|^p$, where $g \in H(\mathbf{U})$. Obviously, $u = \operatorname{Re} J_\zeta g$ is a pluriharmonic majorant of $|J_\zeta f|^p$ and $u(0) = h(0) = \|f\|_p$. Therefore, $|||\, J_\zeta f \,|||_p \leq \|f\|_p$ for every $f \in H^p(\mathbf{U})$, so J_ζ is a continuous linear operator from $H^p(\mathbf{U})$ into $(LH)^p(\mathbf{B})$. Consequently, J_ζ is continuous if we equip each H^p and $(LH)^p$ with its own Mackey topology. However, $v_{k,\zeta} = J_\zeta(\lambda^k)$, where λ^k is the standard monomial of degree k on $\mathbf{U}$, so for the proof of the lemma it is enough to show that $\mu(H^p) - \lim_{k\to\infty} \lambda^k = 0$.

It is well known that the Mackey topology of H^p coincides with the topology defined on H^p by the norm

$$\|f\| = \int_{\mathbf{U}} |f(\lambda)|(1 - |\lambda|)^{(1/p)-2}\, dA(\lambda),$$

where A is the area measure on $\mathbf{C}$ (see [2]). Now, the $\mu(H^p)$-convergence of λ^k to zero immediately follows from the dominated convergence theorem.

5. **Theorem.** *If* $n > 1$ *and* $0 < p < 1$, *then the Lumer's Hardy space* $(LH)^p(\mathbf{B}_n)$ *does not have the Orlicz-Pettis property.*

Proof. We will construct a sequence (f_j) in $H^\infty(\mathbf{B}_n)$ such that the series $\sum f_j$ is $\mu((LH)^p)$-subseries convergent but is not convergent in $(LH)^p$. This series will be almost the same as that used by W. Rudin [7] to build a copy of l^∞ in $(LH)^p$, $p \geq 1$.

Let $\{U_j\}$ be a sequence of pairwise disjoint balls in $\mathbf{R}^n_+$ with the centers $\zeta_j = (\zeta_{j1}, \ldots, \zeta_{jn})$, $|\zeta_j| = 1$, $j = 1, 2, \ldots$, such that

(a) $\zeta_{ji} \leq (2n)^{-1}$, $i = 2, \ldots, n$,

(b) $r_1^j \geq 2^{-1}$ for each $r = (r_1, \ldots, r_n) \in U_j$, and let

$$\varepsilon_j = \sup\left\{\sum_{i=1}^{n} r_i\zeta_{ji} : r \in \mathbf{R}_+^n \backslash U_j\,,\ |r| \le 1\right\},$$

$j = 1, 2, \ldots$. Obviously, $0 < \varepsilon_j < 1$, $j = 1, 2, \ldots$, so we can find a strictly increasing sequence $\{n_j\}$ of positive integers such that

(c) $n_j - j > 0$,
(d) $(2n)^j 2^{-n_j} \le 2^{-j}$,
(e) $n_j^j \varepsilon_j^{n_j - j} \le 2^{-j}$,

$j = 1, 2, \ldots$ ((e) is possible because $\lim_{x\to\infty} x^a b^x = 0$ for any $a > 0$ and $0 < b < 1$).

Let us define $f_j(z) = v_{n_j, \zeta_j}(z) = \langle z, \zeta_j\rangle^{n_j}$, $z \in \mathbf{B}_n$. By Lemma (a) the series $\sum f_j$ is not convergent in $(LH)^p(\mathbf{B}_n)$. We will show that this series is $\mu((LH)^p)$-convergent. For $k = 1, 2, \ldots$ and $j = k, k+1, \ldots$ we define the sets $A_{k,j} = \{\alpha \in \mathbf{Z}_+^n : \alpha_1 \ge k, |\alpha| = n_j\}$, $B_{k,j} = \{\alpha \in \mathbf{Z}_+^n : |\alpha| = n_j\}\backslash A_{k,j}$, and the functions

$$g_{k,j}(z) = \zeta_{j1}^k \sum_{\alpha\in A_{k,j}} \frac{n_j!}{\alpha!}(z_1\zeta_{j1})^{\alpha_1 - k}(z_2\zeta_{j2})^{\alpha_2}\cdots(z_n\zeta_{jn})^{\alpha_n},$$

$$h_{k,j}(z) = \sum_{\alpha\in B_{k,j}} \frac{n_j!}{\alpha!}(z\zeta_j)^{\alpha} \qquad (z \in \mathbf{B}_n).$$

We will show that $g_{k,j}$, $h_{k,j} \in H^\infty(\mathbf{B}_n)$ and

$$(*) \qquad |g_{k,j}(z)| \le \begin{cases} 2 & \text{if } z \in V_j\,, \\ 2^{-j} & \text{if } z \in \mathbf{B}_n\backslash V_j\,, \end{cases}$$

$$\|h_{k,j}\|_\infty \le 2^{-j},$$

for all k, j, where $V_j = \mathscr{G}^{-1}(U_j) \cap \mathbf{B}_n$ and $\mathscr{G}$ is the function acting from $\mathbf{C}^n$ into $\mathbf{R}_+^n$ defined by $\mathscr{G}(z_1, \ldots, z_n) = (|z_1|, \ldots, |z_n|)$. Indeed, for $z \in V_j$ we have

$$|g_{k,j}(z)| \le |z_1|^{-k} \sum_{|\alpha| = n_j} \frac{n_j!}{\alpha!}|z_1\zeta_{j1}|^{\alpha_1}\cdots|z_n\zeta_{jn}|^{\alpha_n}$$

$$\le |z_1|^{-j}\left(\sum_{i=1}^{n} |z_i|\zeta_{ji}\right)^{n_j} \le |z_1|^{-j} \le 2 \quad \text{(by (b))}$$

and

$$\begin{aligned}|h_{k,j}(z)| &\leq \sum_{\alpha\in\mathbf{B}_{k,j}} \frac{n_j!}{\alpha!}|\zeta_{j2}|^{\alpha_2}\cdots|\zeta_{jn}|^{\alpha_n} \\ &\leq \sum_{\alpha\in\mathbf{B}_{k,j}} \frac{n_j!}{\alpha!}(2n)^{k-n_j} \qquad \text{(by (a))} \\ &\leq (2n)^{j-n_j}\sum_{|\alpha|=n_j}\frac{n_j!}{\alpha!} \\ &\leq (2n)^{j-n_j}n^{n_j} \leq 2^{-j} \qquad \text{(by (d))}.\end{aligned}$$

Suppose now that $z \in \mathbf{B}_n \backslash V_j$. Then,

$$\begin{aligned}|g_{k,j}(z)| &\leq n_j^k \sum_{\alpha\in A_{k,j}} \frac{(n_j-k)!}{(\alpha_1-k)!\alpha_2!\cdots\alpha_n}!|z_1\zeta_{j1}|^{\alpha_1-k}|z_2\zeta_{j2}|^{\alpha_2}\cdots|z_n\zeta_{jn}|^{\alpha_n} \\ &\leq n_j^j \sum_{\beta\in\mathbf{Z}_+^n,\,|\beta|=n_j-k} \frac{(n_j-k)!}{\beta!}|z_1\zeta_{j1}|^{\beta_1}\cdots|z_n\zeta_{jn}|^{\beta_n} \\ &\leq n_j^j\left(\sum_{i=1}^n |z_i|\zeta_{ji}\right)^{n_j-j} \leq n_j^j\varepsilon_j^{n_j-j} \leq 2^{-j} \qquad \text{(by (e))}\end{aligned}$$

and

$$\begin{aligned}|h_{k,j}(z)| &\leq \sum_{|\alpha|=n_j} \frac{n_j!}{\alpha!}|z_1\zeta_{j1}|^{\alpha_1}\cdots|z_n\zeta_{jn}|^{\alpha_n} \\ &\leq \left(\sum_{i=1}^n |z_i|\zeta_{ji}\right)^{n_j} \leq \varepsilon_j^{n_j} \leq 2^{-j} \qquad \text{(by (e))}.\end{aligned}$$

Now, define

$$g_k(z) = \sum_{j=k}^{\infty} g_{k,j}(z) \quad \text{and} \quad h_k(z) = \sum_{j=k}^{\infty} h_{k,j}(z)$$

for $z \in \mathbf{B}_n$. Obviously, $\|h_k\|_\infty \leq 2^{1-k}$ (see $(*)$), so $\lim_{k\to\infty} |||\, h_k \,|||_p = 0$. Consequently, $(\mu((LH)^p) - \lim_{k\to\infty} h_k = 0$. Moreover, since no two sets V_j intersect, $|g_k(z)| \leq 2 + \sum_{j=1}^\infty 2^{-j} = 3$ for each $z \in \mathbf{B}_n$.

Now, for each $k = 1, 2, \ldots$ we define the multiplication operator M_k: $(LH)^p \to (LH)^p$ by $(M_k f)(z) = f(z)g_k(z)$, $z \in \mathbf{B}$. Obviously, $|||\, M_k f \,|||_p \leq 3 \,|||\, f \,|||_p$ for each $f \in (LH)^p$ and k, so the family $\mathscr{M} = \{M_k : k = 1, 2, \ldots\}$ is equicontinuous. This implies that $\mathscr{M}$ is equicontinuous if we equip $(LH)^p$ with its Mackey topology.

We are ready to prove that the series $\sum f_j$ is $\mu((LH)^p)$-convergent. Since the sets V_j, $j = 1, 2, \ldots$, are pairwise disjoint, $|f_j(z)| \leq 2^{-j}$ for $z \in \mathbf{B}_n \backslash V_j$

(see (e)), and $\|f_j\|_\infty \le 1$, so the series $\sum f_j$ is pointwise convergent and its sum f belongs to $H^\infty(\mathbf{B}_n) \subset (LH)^p(\mathbf{B}_n)$. Moreover, we have

$$\begin{aligned} f(z) - \sum_{j=1}^{k-1} f_j(z) &= \sum_{j=k}^{\infty} \langle z, \zeta_j\rangle^{n_j} = \sum_{j=k}^{\infty} \sum_{|\alpha|=n_j} \frac{n_j!}{\alpha!}(z\zeta)^\alpha \\ &= z_1^k \sum_{j=k}^{\infty} \zeta_{j1}^k \sum_{\alpha\in A_{k,j}} \frac{n_j!}{\alpha!}(z_1\zeta_{j1})^{\alpha_1-k}(z_2\zeta_{j2})^{\alpha_2}\cdots(z_n\zeta_{jn})^{\alpha_n} \\ &\quad + \sum_{j=k}^{\infty} \sum_{\alpha\in B_{k,j}} \frac{n_j!}{\alpha!}(z\zeta_j)^\alpha \\ &= v_{k,e_1}(z)g_k(z) + h_k(z) \qquad \text{for every } z \in \mathbf{B}, \end{aligned}$$

where e_1 is the first unit vector in $\mathbf{C}^n$. Consequently,

$$f - \sum_{j=1}^{k-1} f_j = M_k(v_{k,e_1}) + h_k \underset{k\to\infty}{\longrightarrow} 0 \quad (\mu((LH)^p)),$$

since the operators $\{M_k\}$ are $\mu((LH)^p)$-equicontinuous and both sequences $\{v_{k,e_1}\}$ and $\{h_k\}$ tend to zero in the Mackey topology of $(LH)^p$. The series $\sum f_j$ is $\mu((LH)^p)$-convergent to f.

Finally, let $\sum f_{i_j}$ be any subseries of $\sum f_j$. The proof that this subseries is $\mu((LH)^p)$-convergent is quite the same as the above one for the entire series $\sum f_j$. It is enough to replace j by i_j everywhere except the sums $\sum_{j=1}^{k-1}$ and $\sum_{j=k}^{\infty}$. The proof is finished.

6. *Remark.* Let $\{f_j\}$ be as in the proof of the theorem. It is easily seen that for each sequence $\gamma = \{\gamma_j\} \in l^\infty$ the series $\sum \gamma_j f_j$ converges pointwise to a function $f_\gamma \in H^\infty(\mathbf{B})$ such that $|||\, f_\gamma \,|||_p \le 2\|\gamma\|_\infty$ (see [8]). Therefore, the linear operator $T\colon \gamma \to f_\gamma$ is continuous. Since $Te_k = v_{n_k,\zeta_k}$, where e_k is the kth unit vector in l^∞, does not tend to zero in $(LH)^p(\mathbf{B}_n)$, so there is an infinite subset M of $\mathbf{N}$ such that $T_{|l^\infty(M)}$ is an isomorphism of $l^\infty(M) \approx l^\infty$ into $(LH)^p(\mathbf{B}_n)$, where $l^\infty(M)$ is the subspace of l^∞ consisting of all sequences with supports contained in M (see [1]).

The proof of this remark is due to Lech Drewnowski.

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