UNIVERSAL MAPS AND SURJECTIVE CHARACTERIZATIONS
OF COMPLETELY METRIZABLE $LC^n$-SPACES

A. CHIGOGIDZE AND V. VALOV

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Abstract. We construct an $n$-dimensional completely metrizable $AE(n)$-space $P(n, \tau)$ of weight $\tau \geq \omega$ with the following property: for any at most $n$-dimensional completely metrizable space $Y$ of weight $\leq \tau$ the set of closed embeddings $Y \rightarrow P(n, \tau)$ is dense in the space $C(Y, P(n, \tau))$. It is also shown that completely metrizable $LC^n$-spaces of weight $\tau \geq \omega$ are precisely the $n$-invertible images of the Hilbert space $\ell_2(\tau)$.

Introduction

Let $\mathcal{Y}$ be a class of completely metrizable spaces. A space $X \in \mathcal{Y}$ is said to be strongly $\mathcal{Y}$-universal if for any space $Y \in \mathcal{Y}$ the set of closed embeddings $Y \rightarrow X$ is dense in the space $C(Y, X)$ of all continuous maps from $Y$ to $X$ endowed with the limitation topology (a stronger version of this notion under the same name was introduced in [BM]). This property is very important in the theory of manifolds modelled on certain model spaces. Let us recall the corresponding results:

(i) If $\mathcal{R}$ is the class of all metrizable compacta, then $X$ is homeomorphic to the Hilbert cube $Q$ iff $X$ is a strongly $\mathcal{R}$-universal $AE$-compactum $[T_1]$;

(ii) if $\mathcal{R}_n$ is the class of all at most $n$-dimensional metrizable compacta, then $X$ is homeomorphic to Menger's universal $n$-dimensional compactum $M_n^{2n+1}$ [E] iff $X$ is a strongly $\mathcal{R}_n$-universal $AE(n)$-compactum (for $n = 0$, [Br]; for $n \geq 1$, [B]);

(iii) if $\mathcal{M}_{\tau}$ is the class of all completely metrizable spaces of weight $\leq \tau$, $\tau \geq \omega$, then $X$ is homeomorphic to the Hilbert space $\ell_2(\tau)$ iff $X$ is a strongly $\mathcal{M}_\tau$-universal $AE$-space $[T_2]$.

(iv) if $\mathcal{M}_{0, \tau}$ is the class of all zero-dimensional completely metrizable spaces of weight $\leq \tau$, $\tau \geq \omega$, then $X$ is homeomorphic to the Baire space $B(\tau)$ iff $X$ is strongly $\mathcal{M}_{0, \tau}$-universal (for $\tau = \omega$ [AU]; for $\tau > \omega$, $\mathcal{M}_{0, \tau}$-universal).

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The main purpose of this paper is to show the existence of a strongly $\mathcal{M}_{n, \tau}$-universal $AE(n)$-space, where $\mathcal{M}_{n, \tau}$ is the class of all at most $n$-dimensional completely metrizable spaces of weight $\leq \tau$, $\tau \geq \omega$. Let us note that strongly $\mathcal{M}_{n, \omega}$-universal $AE(n)$-spaces were constructed by the first author in [C3]. There are many reasons to hope that the following problems have affirmative solutions:

**Problem.** Are any two strongly $\mathcal{M}_{n, \tau}$-universal $AE(n)$-spaces homeomorphic? In particular, is any strongly $\mathcal{M}_{n, \omega}$-universal $AE(n)$-space homeomorphic to Nobeling's universal $n$-dimensional space $N_{n}^{2n+1}$?

The second part of this paper is devoted to surjective characterizations of completely metrizable $LC^n$-spaces. Similar characterizations in the class of metrizable compacta were earlier obtained by Hoffman [H2] and Dranishnikov [D] (see also [C4], where the class of Polish spaces is considered).

1. Preliminaries

All spaces considered are metrizable and maps continuous. By dimension $\dim$ we mean covering dimension. A metrizable space $X$ is an absolute (neighborhood) extensor in dimension $n$ (briefly, $X \in A(N)E(n)$) if for any at most $n$-dimensional metrizable space $Y$ and any closed subspace $A$ of it each map $f: A \to X$ can be extended to the whole of $Y$ (respectively, to a neighborhood of $A$ in $Y$). It is well known that for $n > 0$, $X \in A(N)E(n)$ iff $X \in LC^{n-1} \cap C^{n-1}$ (respectively, $X \in LC^{n-1}$). Note also that any metrizable space, and hence any completely metrizable space, is an $AE(0)$. (The argument can be made as follows: Let $\dim Y = 0$ and $A \subset Y$ be closed. Let $X$ be metrizable and $f: A \to X$ be a map. It is well known that there is a retraction $r: Y \to A$. Then $f \circ r$ is the desired extension of $f$.) The notion of $n$-soft map between compacta was introduced by Schepin [S]. Later Chigogidze [C1] extended it to the class of all Tychonov spaces. Below we use the following definition of this notion: a map $f: X \to Y$ between metrizable spaces is called $n$-soft if for any at most $n$-dimensional paracompact space $Z$, any closed subspace $A$ of it and any two maps $g: Z \to Y$, $h: A \to X$ with $f \circ h = g|A$, there exists a map $k: Z \to X$ such that $f \circ k = g$ and $k|A = h$.

2. Universal maps

**Lemma 2.1.** Let $f: X \to Y$ be an $n$-soft map between metrizable spaces. Suppose $\dim X \leq n$ and $Y$ is a complete absolute extensor for the class of all metrizable spaces. Then $X$ is complete.

**Proof.** Consider the Stone–Čech compactification $\beta X$ of $X$. Denote by $Z$ the space obtained from $\beta X$ by means of making the points of $\beta X - X$ isolated. This space is shown to be paracompact in the proof of [P, Lemma 2].

**Claim.** $\dim Z \leq n$. Since $Z$ is normal it suffices to extend to $Z$ an arbitrary map $g: F \to S^n$ from a closed subset $F$ of $Z$ into the $n$-dimensional sphere.
The case \( F \subseteq Z - X \) is trivial. Suppose now that \( F \cap X \neq \emptyset \). Since \( \dim X \leq n \), there exists an extension \( g_1 : F \cup X \to S^n \) of \( g \). Observe that \( F \cup X \) is closed in \( Z \). Hence we can extend \( g_1 \) to a map \( g_2 \) from \( Z \) into the \( (n+1) \)-dimensional disk \( B^{n+1} \). Put \( H = g_2^{-1}(B^{n+1} - \{b\}) \), where \( b \in B^{n+1} - S^n \). Fix a retraction \( r : (B^{n+1} - \{b\}) \to S^n \). Clearly, \( H \) is clopen in \( Z \); so there exists a map \( g_3 : Z \to S^n \) extending the composition \( r \circ g_2 : H \to S^n \). Obviously, \( g_3|F = g \).

The claim is proved.

Since \( Y \) is metrizable and is an AE for metrizable spaces, then \( Y \) is an absolute retract for metrizable spaces. Further, complete metrizable implies Čech complete. Being metrizable, \( Y \) is a paracompact \( p \)-space, so using [P, Fact 6 and Corollary 1(b)], we conclude that \( Y \) is an absolute extensor for the class of collectionwise normal spaces. Take a map \( h : Z \to Y \) such that \( h|X = f \). It follows by the \( n \)-softness of \( f \) that there exists a retraction from \( Z \) onto \( X \). Now, by arguments of Przymusinski ([P, the proof of Lemma 2]), \( X \) is complete.

**Lemma 2.2** (for \( \tau = \omega \), [C2]). Let \( 0 \leq n < \omega \leq \tau \). Then there exist an \( n \)-dimensional completely metrizable space \( X \) of weight \( \tau \) and an \( n \)-soft map \( f : X \to \ell_2(\tau) \).

**Proof.** By [C2, Theorem 5], we can suppose that \( \tau > \omega \). Let \( d_1 \) be any metric on \( \ell_2(\tau) \). Fix a completely zero-dimensional (with respect to \( d_1 \)) surjection \( g : \ell_2(\tau) \to Y \), where \( Y \) is a separable metrizable space [AP]. By an application of [C2, Theorem 5.1] (see also the remarks following it) there exist an at most \( n \)-dimensional separable space \( Z \) and an \( n \)-soft map \( h : Z \to Y \). Let \( X \) be a fibered product (pullback) of the spaces \( \ell_2(\tau) \) and \( Z \) with respect to the maps \( g \) and \( h \). Denote by \( f : X \to \ell_2(\tau) \) and \( p : X \to Z \) the corresponding canonical projections. Let \( d_2 \) be any metric on \( Z \). Clearly, the map \( p \) is completely zero-dimensional with respect to the metric \( d = (d_1^2 + d_2^2)^{1/2} \) [AP, Chapter 6, §3, Lemma 4]. Hence \( \dim X \leq \dim Z \leq n \). It is easy to see that \( \omega(X) = \tau \). Observe also that the \( n \)-softness of \( h \) implies the \( n \)-softness of \( f \). Since \( \ell_2(\tau) \) contains a copy of the \( n \)-dimensional cube \( I^n \) and \( f \) is an \( n \)-soft map, the space \( X \) contains a copy of \( I^n \) too. Thus \( \dim X = n \). By Lemma 2.1, \( X \) is completely metrizable.

**Corollary 2.3** (for \( \tau = \omega \), [C3]). Let \( 0 \leq n < \omega \leq \tau \). Then for every (completely) metrizable space \( Y \) of weight \( \tau \) there exist an at most \( n \)-dimensional (completely) metrizable space \( Z \) of weight \( \tau \) and an \( n \)-soft map \( g : Z \to Y \).

**Proof.** Embed \( Y \) into \( \ell_2(\tau) \) as a (closed) subspace and consider the map \( f : X \to \ell_2(\tau) \) from Lemma 2.2. Put \( Z = f^{-1}(Y) \) and \( g = f|Z \).

**Definition 2.4** (for \( \tau = \omega \), [C3]). Let \( 0 \leq n < \omega \leq \tau \). A map \( f : X \to Y \) is said to be \( (n, \tau) \)-full if for any map \( g : Z \to Y \) from any at most \( n \)-dimensional completely metrizable space \( Z \) of weight \( \leq \tau \) there exists a closed embedding \( h : Z \to X \) such that \( f \circ h = g \).
**Definition 2.5** (for $\tau = \omega$, [C$_3$]). Let $0 < n < \omega \leq \tau$. A map $f: X \to Y$ is called strongly $(n, \tau)$-universal if for any open cover $\mathcal{U}$ of $X$, any at most $n$-dimensional completely metrizable space $Z$ of weight $\leq \tau$ and any map $g: Z \to X$ there exists a closed embedding $h: Z \to X$ $\mathcal{U}$-close to $g$ with $f \circ h = f \circ g$. We shall also say that a space $X$ is strongly $(n, \tau)$-universal if the constant map $X \to *$ is strongly $(n, \tau)$-universal in the above sense.

**Lemma 2.6** (for $\tau = \omega$, [C$_3$]). Let $0 < n < \omega \leq \tau$ and $S = \{X_k, p^{k+1}_k, \omega\}$ be an inverse sequence consisting of completely metrizable spaces $X_k$ of weight $\leq \tau$ and $n$-soft, $(n, \tau)$-full projections $p^{k+1}_k$. Then the limit projection $p_0: X \to X_0$, where $X = \lim S$, is strongly $(n, \tau)$-universal.

**Proof.** Equip $X$ with the metric $d\{x_k, y_k\} = \max_k d_k(x_k, y_k)$, where $d_k$ is a metric for $X_k$ with $d_k \leq 2^{-k}$, $k \in \omega$. If suffices to show that, given a completely metrizable space $Y$ with $\dim Y \leq n$ and $\omega(Y) \leq \tau$ and maps $f: Y \to X$, $\alpha: X \to (0, 1)$, there is a closed embedding $g: Y \to X$ with $p_0 \circ g = p_0 \circ f$ and $d(f(y), g(y)) \leq \alpha(f(y))$ for each $y \in Y$.

For each $k \in \omega$ fix a closed embedding $i_{k+1}: X_{k+1} \to \ell_2(\tau)$. Since $\ell_2(\tau) \in AE$ there is a map $h: (\ell_2(\tau))^2 \times [0, \infty) \to \ell_2(\tau)$ such that $h(a, b, t) = a$ for $t \leq 1$ and $h(a, b, t) = b$ for $t \geq 2$. By Corollary 2.3, for each $k \in \omega$ there is an $n$-soft map $q_{k+1}: Z_{k+1} \to X_k \times \ell_2(\tau)$, where $Z_{k+1}$ is a completely metrizable space with $\dim Z_{k+1} \leq n$ and $\omega(Z_{k+1}) = \tau$. The $n$-softness of the projection $p^{k+1}_k$ implies the existence of a $r_{k+1}: Z_{k+1} \to X_k$ with $p^{k+1}_k \circ r_{k+1} = \pi_k \circ q_{k+1}$ and $r_{k+1}(A_{k+1}) = (p^{k+1}_k \Delta i_{k+1})^{-1} \circ q_{k+1}(A_{k+1})$, where

$$A_{k+1} = q_{k+1}^{-1}((p^{k+1}_k \Delta i_{k+1})(X_k))$$

and

$$\pi_k: X_k \times \ell_2(\tau) \to X_k$$

denotes the natural projection. Put $g_0 = p_0 \circ f$. By our assumption, the projection $p_0^1$ of the spectrum $S$ is $(n, \tau)$-full. Hence there exists a closed embedding $j_1: Y \to X_1$ such that $p_0^1 \circ j_1 = g_0$. Consider now the map

$$g_0 \Delta h(i_1 \circ p_1 \circ f \Delta i_1 \circ j_1 \Delta 2\alpha \circ f): Y \to X_0 \times \ell_2(\tau).$$

Since $\dim Y \leq n$ and the map $q_1$ is $n$-soft there is a map $s_1: Y \to Z_1$ such that $g_0 \Delta h(i_1 \circ p_1 \circ f \Delta i_1 \circ j_1 \Delta 2\alpha \circ f) = q_1 \circ s_1$. We define a map $g_1: Y \to X_1$ by the formula $g_1 = r_1 \circ s_1$. Note that $p_0^1 \circ g_1 = g_0$. If $y \in Y$ and $\alpha(f(y)) \leq 2^{-1}$, then $g_1(y) = p_1(f(y))$. (Use the fact that $g_1(y) = r_1(s_1(y))$, show that $s_1(y) \in A_1$, and then use the above formula for $q_1 \circ s_1$.)

Let us construct a map $g_2: Y \to X_2$. By our assumption, the projection $p_1^2$ is $(n, \tau)$-full. Hence there is a closed embedding $j_2: Y \to X_2$ such that $p_1^2 \circ j_2 = g_1$. Since $\dim Y \leq n$ and the map $q_2$ is $n$-soft there is a map $s_2: Y \to Z_2$ such that $q_2 \circ s_2 = g_1 \Delta h(i_2 \circ p_2 \circ f \Delta i_2 \circ j_2 \Delta 2\alpha \circ f)$. Put $g_2 = r_2 \circ s_2$. As above, $p_2^2 \circ g_2 = g_1$. Note that if $y \in Y$ and $\alpha(f(y)) \leq 2^{-2}$, then $g_2(y) = p_2(f(y))$; if $\alpha(f(y)) \geq 2^{-1}$, then $g_2(y) = j_2(y)$. 

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Let us suppose that for each $i$, $2 \leq i \leq k$, we have already constructed maps $g_i : Y \to X_i$ and closed embeddings $j_i : Y \to X_i$ satisfying the following conditions:

1. $p_i^{i-1} \circ g_i = g_{i-1}$;
2. $p_i^{i-1} \circ j_i = g_{i-1}$;
3. If $y \in Y$ and $\alpha(f(y)) \leq 2^{-i}$, then $g_i(y) = p_i(f(y))$;
4. If $y \in Y$ and $\alpha(f(y)) \leq 2^{-i+1}$, then $g_i(y) = j_i(y)$.

Let us construct a map $g_{k+1} : Y \to X_{k+1}$ and a closed embedding $j_{k+1} : Y \to X_{k+1}$ with the desired properties. Choose an arbitrary closed embedding $j_{k+1} : Y \to X_{k+1}$ with $p_{k+1} \circ j_{k+1} = g_k$ (the existence of $j_{k+1}$ follows from the $(n, \tau)$-fullness of $p_{k+1}$). As above there is a map $s_{k+1} : Y \to Z_{k+1}$ such that $q_{k+1} \circ s_{k+1} = g_k \Delta h(i_{k+1}) \circ p_{k+1} \circ f \Delta i_{k+1} \circ j_{k+1} \Delta 2^{k+1} \alpha \circ f)$. We put $g_{k+1} = r_{k+1} \circ s_{k+1}$. The verification of the conditions $(1)_{k+1} - (4)_{k+1}$ is left to the reader.

It follows from the conditions $(1)_k$, $k \in \omega$, that the diagonal product $g = \Delta g_k : k \in \omega$ maps $Y$ into $X$ and satisfies the following equalities: $p_k \circ g = g_k$, $k \in \omega$. In particular, we have $p_0 \circ g = g_0 = p_0 \circ f$.

Let $y \in Y$. If $\alpha(f(y)) \in [2^{-k-1}, 2^{-k}]$, then $2^i \alpha(f(y)) \leq 1$ for $i \leq k$; hence

$$p_i(f(y)) = g_i(y) \quad \text{for } i \leq k$$

and

$$d(f(y)g(y)) = \max\{d_i(p_i(f(y)), g_i(y)) : i = k + 1, k + 2, \ldots\} \leq 2^{-k-1} \leq \alpha(f(y)).$$

In order to show that $g$ is a closed embedding it suffices to use the corresponding arguments from the proof of [C2, Lemma 7.11].

**Theorem 2.7** (for $\tau = \omega$, [C3]). Let $0 \leq n < \omega \leq \tau$. Then there exist an $n$-dimensional completely metrizable space $P(n, \tau)$ of weight $\tau$ and a strongly $(n, \tau)$-universal $n$-soft map $f(n, \tau) : P(n, \tau) \to \ell_2(\tau)$.

**Proof.** Put $X_0 = l_2(\tau)$. By Corollary 2.3, there exists at most an $n$-dimensional completely metrizable space $X_{k+1}$ of weight $\tau$ and an $n$-soft map $h_{k+1} : X_k \to X_k \times \ell_2(\tau)$. Put $p_{k+1} = \pi_k \circ h_{k+1}$, where $\pi_k : X_k \times \ell_2(\tau) \to X_k$ is the natural projection, $k \in \omega$. So we get an inverse sequence $S = \{X_k, p_{k+1}\}$ consisting of at most $n$-dimensional completely metrizable spaces of weight $\tau$ and $(n, \tau)$-full $n$-soft projections. Put $P(n, \tau) = \lim S$ and $f(n, \tau) = p_0$. By Lemma 2.6, the map $f(n, \tau)$ is strongly $(n, \tau)$-universal. Obviously, this map is $n$-soft. Since $\dim X_k \leq n$ for each $k \geq 1$, we have $\dim P(n, \tau) \leq n$ [N]. The inverse inequality $\dim P(n, \tau) \geq n$ follows from the strong $(n, \tau)$-universality of $f(n, \tau)$. Finally observe that $P(n, \tau)$ is a complete metrizable space of weight $\tau$. This completes the proof.
Corollary 2.8 (for \( \tau = \omega \), \([C_3]\)). Let \( 0 \leq n < \omega \leq \tau \). The space \( P(n, \tau) \) is an \( n \)-dimensional strongly \((n, \tau)\)-universal completely metrizable \( AE(n) \)-space of weight \( \tau \).

Proof. \( P(n, \tau) \) is an \( AE(n) \)-space as an \( n \)-soft preimage of \( \ell_2(\tau) \).

Remark 2.9. The space \( P(n, \tau) \) has the following property: for every open subspace \( U \) of \( P(n, \tau) \) and any at most \( n \)-dimensional completely metrizable space \( X \) of weight \( \leq \tau \) there exists an embedding \( h: X \to U \) such that \( h(X) \) is closed in \( P(n, \tau) \). Indeed, consider a constant map \( g: X \to \{ p \} \), where \( p \) is an arbitrary point of \( U \), and the open cover \( \mathcal{U} = \{ U, P(n, \tau) - \{ p \} \} \) of \( P(n, \tau) \). By Definition 2.5, there is a closed embedding \( h: X \to P(n, \tau) \) \( \mathcal{U} \)-close to \( g \). Clearly, \( h(X) \subseteq U \). In particular, we have \( \omega(U) = \tau \) for every open subset \( U \) of \( P(n, \tau) \). Hence, \( P(0, \tau) \) is homeomorphic to the Baire space \( B(\tau) \) (see [St]).

Remark 2.10. A. Wasko proved, in [W], that for every \( n \geq 0 \) and every \( \tau \geq \omega \) there exists an \( n \)-dimensional completely metrizable space \( X_{n, \tau} \) of weight \( \tau \) such that every at most \( n \)-dimensional completely metrizable space of weight \( \leq \tau \) is embedded in \( X_{n, \tau} \) as a closed subset (for \( \tau = \omega \) this was proved earlier by the first author, see [C1, Corollary 3]). Recently E. Pol, [Po], strengthened this result of A. Wasko by proving that for every \( n \)-dimensional completely metrizable space \( X \) of weight \( \leq \tau \) the set of all embeddings of \( X \) onto a closed subset of \( S(\tau)^\omega \) contained in \( K_n(\tau) \) is residual in the space \( C(X, S(\tau)^\omega) \). Here \( S(\tau) \) is the \( \tau \)-star-space and \( K_n(\tau) \) denotes Nagata's universal \( n \)-dimensional space.

3. Surjective characterizations of \( LC^n \)-spaces

Definition 3.1 [H2]. A space \( X \) is said to be in the class \( AE(n, m) \), where \( 0 \leq n \leq m \leq \infty \), if for any metrizable space \( Z \) with \( \dim Z \leq m \), any closed subspace \( A \) of it with \( \dim A \leq n \), any map \( f: A \to X \) can be extended to the whole of \( Z \). If \( f \) can be extended only to a neighborhood of \( A \) in \( Z \), we get a definition of the class \( ANE(n, m) \).

Lemma 3.2. For every \( n \in \omega \) the following equalities are true in the class of all metrizable spaces: \( A(N)E(n + 1) = A(N)E(n, n + 1) \).

In the class of all metrizable compacta this lemma was proved by Dranishnikov (see [D, Lemma 3.1]). The same proof remains valid in the general case.

Lemma 3.3. If \( 0 \leq n < m \leq \infty \), then the following equalities are true in the class of all metrizable spaces: \( A(N)E(n + 1) = A(N)E(n, m) \).

Proof. Since \( n + 1 \leq m \), then \( A(N)E(n, m) \subseteq A(N)E(n, n + 1) \); thus, \( A(N)E(n, m) \subseteq A(N)E(n + 1) \) follows from Lemma 3.2. Suppose \( X \) is a metrizable \( AE(n + 1) \)-space. Let \( Z \) be a metrizable space with \( \dim Z \leq m \), \( A \) a closed subspace of \( Z \) with \( \dim A \leq n \) and \( f \) a map from \( A \) to \( X \). Take a metrizable \( AE \)-space, say \( Y \), of dimension \( \leq n + 1 \) containing \( A \) as a closed
subspace ([K], see also [C,] for Polish spaces and [Ts] for completely metrizable spaces). Now choose a map \( k : Y \to X \) such that \( k|A = f \). This is possible because, by Lemma 3.2, \( X \in AE(n, n + 1) \). Since \( Y \in AE \), there exists a map \( g : Z \to Y \) such that \( g|A = id \). Then \( k \circ g \) is an extension of \( f \). The inclusion \( ANE(n + 1) \subset ANE(n, m) \) follows from the same arguments.

**Definition 3.4 [H,].** A map \( f : X \to Y \) is called \( n \)-invertible if for any at most \( n \)-dimensional metrizable space \( Z \) and any map \( g : Z \to Y \) there exists a map \( h : Z \to X \) such that \( g = f \circ h \).

Obviously, every \( n \)-invertible map is a surjection. For a metrizable compactum \( X \) it is known \([D], [H,]\) that \( X \) is an \( AE(n+1) \) iff \( X \) is an \( n \)-invertible image of \( Q \). It is also true \([D]\) that the class of all metrizable \( AE(n) \)-compacta coincides with the class of \( n \)-invertible images of the \( n \)-dimensional universal Menger compactum \( m_n^{2n+1} \). The first author \([C,]\) gave similar characterizations of \( AE(n+1) \) and \( AE(n) \) in the class of Polish spaces. In the case of metrizable spaces of uncountable weight we know only the following facts: (i) the classical characterization of completely metrizable spaces of weight \( \tau \) as open images of the Baire space \( B(\tau) \); (ii) a metrizable space \( X \) of weight \( \tau \) is \( \check{C}ech \)-complete iff \( X \) is a 0-invertible image of \( B(\tau) \) \([V]\).

Below we give similar characterizations of metrizable \( AE(n) \)-spaces of arbitrary weight.

**Definition 3.5 \([C,]\).** A map \( f : X \to Y \) is said to be inductively \( n \)-soft if there exists a closed subspace \( Z \) of \( X \) such that the restriction \( f|Z : Z \to Y \) is \( n \)-soft.

**Theorem 3.6.** Let \( X \) be a metrizable space of weight \( \tau \geq \omega \). Then for every \( n \in \omega \) the following conditions are equivalent:

(i) \( X \in AE(n + 1) \) (respectively, \( X \in ANE(n + 1) \));

(ii) \( X \) is an inductively \( n \)-soft image of an \( AE \) (respectively, of an \( ANE \));

(iii) \( X \) is an \( n \)-invertible image of an \( AE \) (respectively, of an \( ANE \));

**Proof.** We shall prove only the global variant. The local one follows from the same arguments.

(i) \( \to \) (ii). By Corollary 2.3, there exist an at most \( n \)-dimensional metrizable space \( Y \) of weight \( \tau \) and an \( n \)-soft map \( g : Y \to X \). Embed \( Y \) into a metrizable \( AE \)-space \( Z \) as a closed subspace. By Lemma 3.3, there exists an extension \( h : Z \to X \) of \( g \). Clearly, \( h \) is inductively \( n \)-soft.

(ii) \( \to \) (iii). This implication is trivial, because any \( n \)-soft map is \( n \)-invertible.

(iii) \( \to \) (i). Let \( Z \) be an \( AE \)-space and \( f : Z \to X \) be an \( n \)-invertible map. In view of Lemma 3.2 it suffices to show that \( X \in AE(n, n + 1) \). Let \( B \) be any at most \( (n + 1) \)-dimensional metrizable space, \( A \) a closed subspace of \( B \) with \( \dim A \leq n \) and \( g \) a map from \( A \) to \( X \). Since \( f \) is \( n \)-invertible, there exists a map \( h : A \to Z \) such that \( f \circ h = g \). Take any extension \( k : B \to Z \) of \( h \). Then the map \( f \circ k \) is an extension of \( g \).
Let us consider the proof of Theorem 3.6. If $X$ is a complete metrizable space of weight $\tau$, then the space $Y$ is also complete, so we can suppose that $Z$ is the space $\ell_2(\tau)$. Thus, the following theorem is true.

**Theorem 3.7.** Let $X$ be a completely metrizable space of weight $\tau \geq \omega$. Then for every $n \in \omega$ the following conditions are equivalent:

(i) $X \in \text{AE}(n+1)$ (respectively, $X \in \text{ANE}(n+1)$);
(ii) $X$ is an inductively $n$-soft image of $\ell_2(\tau)$ (respectively, of an open subspace of $\ell_2(\tau)$);
(iii) $X$ is an $n$-invertible image of $\ell_2(\tau)$ (respectively, of an open subset of $\ell_2(\tau)$).

The proof of the following result is analogous to the proof of Theorem 3.6.

**Theorem 3.8.** Let $X$ be a completely metrizable space of weight $\tau \geq \omega$. Then for every $n \in \omega$ the following conditions are equivalent:

(i) $X \in \text{AE}(n)$ (respectively, $X \in \text{ANE}(n)$);
(ii) $X$ is an inductively $n$-soft image of $P(n, \tau)$ (respectively, of an open subset of $P(n, \tau)$);
(iii) $X$ is an $n$-invertible image of $P(n, \tau)$ (respectively, of an open subset of $P(n, \tau)$).

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Department of Mathematics, Moskow State University, Moskow 119899, USSR

Department of Mathematics, Sofia State University, Sofia 1126, Bulgaria