PERIODIC SOLUTIONS OF SOME LIÉNARD EQUATIONS WITH SINGULARITIES

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Abstract. We consider the forced Liénard equation

\[ u'' + f(u)u' + g(t, u) = h(t) \]

together with the boundary conditions

\[ u(0) = u(T), \quad u'(0) = u'(T), \]

where \( g \) is continuous on \( \mathbb{R} \times (0, +\infty) \) and becomes infinite at \( u = 0 \). We consider classical solutions as well as generalized solutions that can go into the singularity \( u = 0 \). The method of approach uses upper and lower solutions and degree theory.

1. Introduction

Let \( h \in L^1(0, T) \), \( f \in C(\mathbb{R}, \mathbb{R}) \), and \( g: \mathbb{R} \times (0, +\infty) \to \mathbb{R} \) be a Carathéodory function, \( T \)-periodic in the first variable. We are interested in the existence of solutions to

\[
\begin{align*}
\tag{0}
\quad u'' + f(u)u' + g(t, u) = h(t), \\
\quad u(0) = u(T), \quad u'(0) = u'(T).
\end{align*}
\]

Throughout we assume that:

(A) Given real numbers \( 0 < a < b \), there exists a function \( k \in L^1(0, T) \) such that, if \( t \in [0, T] \) and \( a \leq u \leq b \), we have

\[ |g(t, u)| \leq k(t). \]

Moreover, our assumptions are such that \( u = 0 \) is a singularity of the "force" \( g \), which becomes unbounded as \( u \to 0 \).

The corresponding problem in the absence of dissipation has been studied by Lazer and Solimini [4]. We use some of the ideas of that paper to obtain existence results for (0). The paper is organized as follows: In §2 we state and prove a theorem on upper and lower solutions for the \( W^{2,1}(0, T) \) setting; such...
a result is a particular case of results obtained by Adje [1] in his doctoral dissertation. §3 deals with attractive forces; Theorem 1 provides classical solutions and Theorem 2 generalized solutions which allow collision with the singularity \( u = 0 \); the latter are related to a concept introduced by Bahri and Rabinowitz [2]. In §4 we study repulsive forces using degree theory. This approach is very similar to Habets and Sanchez [3] where a vector case is studied using a "strong force" condition. In the scalar case such a "strong force" assumption turns out to be unnecessary.

Throughout the paper we use \( \| \|_p \) to denote the usual norm in \( L^p(0, T) \) (\( 1 \leq p \leq \infty \)).

2. AN AUXILIARY RESULT INVOLVING LOWER AND UPPER SOLUTIONS

Let \( \alpha, \beta \) be two continuous functions in \([0, T]\) such that \( \alpha(t) \leq \beta(t) \) for all \( t \in [0, T] \) and set

\[ E = \{(t, x, y) : 0 < t < T, \alpha(t) < x < \beta(t), y \in \mathbb{R}\}. \]

Let \( f : E \to \mathbb{R} \) be a Carathéodory function, i.e. a function with the following properties: (i) for each \( x \in \mathbb{R} \) and \( y \in \mathbb{R} \) the function \( f(\cdot, x, y) \), with domain \( \{t \in [0, T] : (t, x, y) \in E\} \), is measurable; (ii) the functions \( f(\cdot, \alpha(\cdot), y) \) and \( f(\cdot, \beta(\cdot), y) \) are measurable for each \( y \in \mathbb{R} \); (iii) for almost every \( t \in [0, T] \), the function \( f(t, \cdot, \cdot) \), with domain \( \{(x, y) : \alpha(t) < x < \beta(t) \text{ and } y \in \mathbb{R}\} \) is continuous; (iv) there exists \( h \in L^1(0, T) \) such that, for all \((t, x)\) such that \((t, x, 0) \in E\),

\[ |f(t, x, 0)| \leq h(t). \]

Proposition 1. Let \( E \) and \( f \) be as above and assume in addition that there exists \( L \in L^1(0, T) \) such that for any \((t, x, y_1) \in E\) and \((t, x, y_2) \in E\) we have

\[ |f(t, x, y_1) - f(t, x, y_2)| \leq L(t)|y_1 - y_2|. \]

Suppose moreover that \( \alpha, \beta \in W^{2,1}(0, T) \) are such that:

(a) \[ \alpha''(t) \geq f(t, \alpha(t), \alpha'(t)) \quad \text{a.e. in } [0, T], \]
\[ \alpha(0) = \alpha(T), \quad \alpha'(0) \geq \alpha'(T); \]

(b) \[ \beta''(t) \leq f(t, \beta(t), \beta'(t)) \quad \text{a.e. in } [0, T], \]
\[ \beta(0) = \beta(T), \quad \beta'(0) \leq \beta'(T). \]

Then the periodic boundary value problem

\[ x''(t) = f(t, x(t), x'(t)), \]
\[ x(0) = x(T), \quad x'(0) = x'(T), \]

has at least one solution \( x(t) \) such that \( \alpha(t) \leq x(t) \leq \beta(t), \) for all \( t \in [0, T] \).

Remark. According to (a) and (b) we say that \( \alpha(t) \) is a lower solution, and \( \beta(t) \) is an upper solution, of problem (2). This proposition is a particular case of results obtained by Adje [1]. We give a complete proof for convenience of the reader.
Proof. We adapt the proof given by Mawhin [5, Theorem 1.1]. Using (iv) and (1) we fix a function $m \in L^1(0, T)$ such that for any $y$ with $|y| \leq 1$ we have
\begin{equation}
\max\{|f(t, \alpha(t), y)|, |f(t, \beta(t), y)|, h(t), L(t)\} < m(t).
\end{equation}
Consider the modified problem
\begin{equation}
x'' = F(t, x, x'),
x(0) = x(T), \quad x'(0) = x'(T),
\end{equation}
where $F: [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by
\begin{align*}
F(t, x, y) &= f(t, \alpha(t), y) + m(t)(x - \alpha(t)) \quad \text{if } x < \alpha(t), \\
&= f(t, x, y) \quad \text{if } \alpha(t) \leq x \leq \beta(t), \\
&= f(t, \beta(t), y) + m(t)(x - \beta(t)) \quad \text{if } x > \beta(t).
\end{align*}
First we show that any solution of (4) satisfies $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in [0, T]$ so that it is in fact a solution of (2). To see this we argue by contradiction. Suppose that $\max(x - \beta) = x(t_0) - \beta(t_0) > 0$. If $t_0 \in (0, T)$ this implies that the functions $x(t) - \beta(t)$ and
\begin{align*}
-|x'(t) - \beta'(t)| + x(t) - \beta(t)
\end{align*}
are strictly positive in some interval $(t_0 - \varepsilon, t_0 + \varepsilon)$. Thus we obtain a.e. in this interval
\begin{align*}
x''(t) &= f(t, \beta(t), x'(t)) + m(t)(x(t) - \beta(t)) \\
&\geq f(t, \beta(t), \beta'(t)) - m(t)|x'(t) - \beta'(t)| + m(t)(x(t) - \beta(t)) \\
&\geq f(t, \beta(t), \beta'(t)) \geq \beta''(t),
\end{align*}
where we have used (1) and the fact that $L(t) \leq m(t)$. This contradicts the fact that $x - \beta$ has a local maximum at $t_0$. If $t_0 = 0$, then since $x$ is $T$-periodic the maximum is attained at $T$ as well, and we obtain
\begin{align*}
x'(0) - \beta'(0) \leq 0 \leq x'(T) - \beta'(T)
\end{align*}
which, according to (b), implies $\beta'(0) = \beta'(T)$. Considering the $C^1$-extensions of $x(t)$ and $\beta(t)$ we reach a contradiction as above. Hence $x(t) \leq \beta(t)$ for all $t \in [0, T]$. Clearly, a similar argument applies to prove that $\alpha(t) \leq x(t)$ for all $t \in [0, T]$.

It remains to prove that (4) has at least one solution. To this end, consider the homotopy
\begin{equation}
x'' = (1 - \lambda)m(t)x + \lambda F(t, x, x'),
x(0) = x(T), \quad x'(0) = x'(T),
\end{equation}
where $0 \leq \lambda \leq 1$. By well-known results, it suffices as in [5] to obtain a priori estimates (in $C^1$-norm) for the possible solutions $x(t)$ of (5). In fact using (1) and (3) we can write
\begin{align*}
|x''(t)| &= (1 - \lambda)m(t)|x(t)| + \lambda[m(t)(1 + |x'(t)|) + m(t)|x(t)| + M] \\
&\leq m(t)(M + 1 + |x(t)| + |x'(t)|),
\end{align*}
where $M = \max\{\|\alpha\|_{\infty}, \|\beta\|_{\infty}\}$. The estimate on $\|x'\|_{\infty}$ will follow from Gronwall's inequality provided we can bound $\|x\|_{\infty}$. Let us prove that

\begin{equation}
\|x\|_{\infty} \leq M + 1.
\end{equation}

Suppose that this is not the case. Then there exists $t_0 \in [0, T]$ such that

\begin{equation}
\|x\|_{\infty} = |x(t_0)| > M + 1.
\end{equation}

If $x(t_0) > 0$, we can fix an interval $(t_0 - \epsilon, t_0 + \epsilon)$ where $|x'(t)| < 1$ and $x(t) > \beta(t) + 1$ so that a.e. in this interval we have

\[x''(t) = (1 - \lambda)m(t)x(t) + \lambda[f(t, \beta(t), x'(t)) + m(t)(x(t) - \beta(t))],\]

where according to (3) the bracket is larger than $-m(t) + m(t) = 0$. This shows that, for any $\lambda \in [0, 1]$, we have $x''(t) > 0$ a.e. in $(t_0 - \epsilon, t_0 + \epsilon)$. Hence $x(t_0)$ cannot be a maximum. A similar argument applies if $x(t_0) < 0$. This contradiction proves (6). Therefore the proof is complete.

Remark. If the function $f$ is continuous, the estimate on $x'(t)$ may be obtained via a Nagumo condition (cf. [5]). Here we use instead assumption (1) which, in some sense, is less general, but is adequate to our purpose. More general conditions of this type may be found in [1].

3. Attractive forces

"Solutions" of (0) are always understood as (strictly) positive functions such that (0) is satisfied in the usual $W^{2,1}(0, T)$-sense.

Let us introduce the following assumptions:

$(G_1)$ \(\lim_{u \to 0}^+ g(t, u) = +\infty\) uniformly in $t \in [0, T]$,

$(G_2)$ there exist $R > 0$ and $g_0 \in L^1(0, T)$ such that $g(t, u) \leq g_0(t)$ if $u \geq R$ and $t \in [0, T]$.

Theorem 1. Assume (A), $(G_1)$, $(G_2)$ hold and in addition

\begin{equation}
\int_0^T g_0(t) \, dt \leq \int_0^T h(t) \, dt,
\end{equation}

and there exists $M \in \mathbb{R}$ such that, for all $t \in [0, T]$,

\begin{equation}
h(t) \leq M.
\end{equation}

Then the problem (0) has at least one solution.

By a $(T$-periodic) generalized solution of (0) we mean a continuous function $u(t)$ such that: (i) $u(0) = u(T)$, (ii) $u(t) \geq 0$ for all $t \in [0, T]$, (iii) the set $u^{-1}(0)$ has zero measure; (iv) given any component $J$ of $[0, T]\setminus u^{-1}(0)$, we have $u \in W^{2,1}(J)$ and $u(t)$ satisfies equation (0) in the $W^{2,1}(J)$-sense. The use of this broader concept of "solution" is useful if we wish to discard the assumption (8).
Theorem 2. Assume (A), (G₁), (G₂) and
\[ \int_0^T g_0(t) \, dt < \int_0^T h(t) \, dt. \]  
In addition, let there exist \( \varepsilon > 0 \) and \( B \in L^1(0, T) \) such that, for all \( t \in [0, T] \) and \( u \in (0, \varepsilon) \), we have
\[ u g(t, u) \leq B(t). \]  
Then problem (0) has a generalized solution.

Proof of Theorem 1. By condition (G₁) and (8) we can choose a constant \( \alpha > 0 \) sufficiently small so that for all \( t \in [0, T] \)
\[ g(t, \alpha) \geq M, \]  
where \( M \) is the upper bound for \( h(t) \) in (8). Clearly, \( \alpha(t) \equiv \alpha \) is a lower solution of (0). To construct an upper solution, set, for each \( b \in L^1(0, T) \),
\[ b = \frac{1}{T} \int_0^T b(t) \, dt. \]  
Fix a constant \( C = T\|\varphi\|_1 + R + \alpha \), where \( \varphi(t) = (g_0(t) - h(t)) - (\overline{g}_0 - \overline{h}) \). We claim that there exists a solution \( \beta_0 \in W^{2,1}(0, T) \) of the following problem:
\[ \beta_0'' + f(\beta_0 + C)\beta_0' + \varphi(t) = 0, \]
\[ \beta_0(0) = \beta_0(T), \quad \beta_0'(0) = \beta_0'(T), \]
\[ \beta_0 = 0. \]
According to well-known results of degree theory, this follows from considering the homotopy
\[ \beta'' + \lambda f(\beta + C)\beta' + \varphi(t) = 0, \]
\[ \beta(0) = \beta(T), \quad \beta'(0) = \beta'(T), \]
\[ \beta = 0 \]
\((0 \leq \lambda \leq 1)\), if we can show that solutions of (12) are bounded, say, in \( H^1(0, T) \). Now multiplying (12) by \( \beta \) and integrating we get
\[ \|\beta'\|_2^2 = \int_0^T \varphi \beta \leq \|\varphi\|_1 \|\beta\|_\infty \leq \sqrt{T} \|\varphi\|_1 \|\beta'\|_2, \]
so that \( \|\beta'\|_2 \leq \sqrt{T} \|\varphi\|_1 \) and finally \( \|\beta\|_\infty \leq T \|\varphi\|_1 \). Thus the claim is proved and moreover we know that (11) has a solution \( \beta_0 \) such that \( \|\beta_0\|_\infty \leq T \|\varphi\|_1 \). Setting \( \beta = \beta_0 + C \), it follows that
\[ \beta'' + f(\beta)\beta' + \varphi = 0 \]
and \( \beta \) is a \( T \)-periodic function such that \( \beta(t) \geq R + \alpha \). In particular, using (G₂) and (7) we obtain
\[ g(t, \beta(t)) - h(t) \leq g_0(t) - h(t) = \varphi(t) + \overline{g}_0 - \overline{h} \leq \varphi(t) \]
so that, by virtue of (13), $\beta$ is the desired upper solution. Clearly, equation (0) and the functions $\alpha$ and $\beta$ satisfy all the requirements in the hypothesis of Proposition 1, so that the proof is complete.

**Remark.** Condition $(G_1)$ can be weakened in such a way that we are still able to construct a lower solution of (0). In fact replace $(G_1)$ by the following assumption:

$$(G_1)'.$$ There exists $r > 0$ such that for all $u \in (0, r)$ we have $g(t, u) \geq g_0(t)$, and either

$$g_0 \geq h(t) \quad (t \in [0, T]) \quad \text{and} \quad \|g_0 - g_0\|_1 < r/2T,$$

or

$$g_0 \geq \bar{h} \quad \text{and} \quad \|g_0 - h - (\bar{g}_0 - \bar{h})\|_1 < r/2T.$$

It is easy to show (by an argument similar to one used in the above proof) that there exists a solution $\alpha(t)$ of

$$\alpha'' + f(\alpha)\alpha' + g_0 - \bar{g}_0 = 0,$$

$$\bar{\alpha} = \frac{r}{2}, \quad \alpha(0) = \alpha(T), \quad \alpha'(0) = \alpha'(T),$$

satisfying in addition $\|\alpha\|_2 \leq T^{1/2}\|g_0 - \bar{g}_0\|_1$ and $\|\alpha - r/2\|_\infty \leq T\|g_0 - \bar{g}_0\|_1$. It follows from $(G_1)'$ that $\alpha$ takes values in $(0, r)$ and hence it is a lower solution of (0).

**Proof of Theorem 2.** For each $n \in \mathbb{N}$, let $h_n = \inf(h, n)$. From Theorem 1 it follows that the truncated problem

$$(14)\ u''_n+f(u_n)u'_n + g(t, u_n) = h_n(t),$$

$$u_n(0) = u_n(T), \quad u'_n(0) = u'_n(T),$$

has a solution $u_n \in W^{2,1}(0, T)$ which is strictly positive. To obtain estimates on $u_n$ we first remark that, for $n$ sufficiently large, we cannot have $u_n(t) \geq R$ for all $t \in [0, T]$, since

$$\int_0^T g_0(t) \geq \int_0^T g(t, u_n(t)) \, dt = \int_0^T h_n(t) \, dt$$

and this would contradict (9). Hence there exists $t_n$ such that $u_n(t_n) \leq R$. On the other hand

$$\|u'_n\|_2^2 = \int_0^T [g(t, u_n)u_n - h(t)u_n] \, dt$$

which, on the basis of (A), $(G_2)$ and (10) implies

$$\|u'_n\|_2^2 \leq c_1\|u_n\|_\infty + c_2 \leq c_1(R + \sqrt{T}\|u'_n\|_2) + c_2,$$

where $c_1, c_2$ are positive constants independent of $n$. Thus for some subsequence, still labelled $(u_n)$, we have

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(0, T) \quad \text{and uniformly in } [0, T].$$
Of course, \( u(t) \geq 0 \) for all \( t \in [0, T] \). Set
\[
\tilde{g}(t) = g(t, u(t)) \quad \text{if} \quad u(t) > 0,
\]
\[
= +\infty \quad \text{if} \quad u(t) = 0.
\]

Our hypothesis (A) on \( g \) and (G1) show that the sequence of functions \( g(t, u_n(t)) \) is bounded below by some integrable function. Then using Fatou's lemma we have
\[
\int_0^T \tilde{g}(t) \, dt \leq \lim \int_0^T g(t, u_n(t)) \, dt = \lim \int_0^T h_n(t) \, dt = \int_0^T h(t) \, dt,
\]
showing that \( \tilde{g}(t) \) is finite almost everywhere in \( (0, T) \). This proves that the set \( u^{-1}(0) \) has zero measure. Now let \( J \) be any component of \( [0, T] \setminus u^{-1}(0) \).

Taking an arbitrary test function \( \varphi \in C_0^\infty(J) \), we obtain from (14)
\[
\int_0^T [-u_n' \varphi' + f(u_n)u_n' \varphi + g(t, u_n)\varphi] \, dt = \int_0^T h_n \varphi
\]
and, since \( g(t, u_n(t)) \to g(t, u(t)) \) in \( L^1(\text{supp} \varphi) \),
\[
\int_0^T [-u' \varphi' + f(u)u' \varphi + g(t, u(t))\varphi] \, dt = \int_0^T h \varphi.
\]
But we know that \( g(t, u(t)) = \tilde{g}(t) \) is \( L^1(0, T) \) so that \( u \in W^{2,1}(J) \) and the result follows.

### 4. Repelling forces

In this section we consider the problem
\[
\begin{align*}
\quad u'' + f(u)u' + g(u) = h(t), \\
u(0) = u(T), \quad u'(0) = u'(T),
\end{align*}
\]
where \( f \in C(\mathbb{R}, \mathbb{R}) \) and \( g \in C((0, +\infty), \mathbb{R}) \). We introduce the following assumptions:

(F1) There exists \( a > 0 \) such that, for every \( u > 0 \),
\[ f(u) \geq a \quad \text{[respectively] \quad f(u) \leq -a}. \]

(G3) There exists \( R > 1 \) such that, for every \( u > R \),
\[ g(u) > \bar{h}. \]

(G4) There exists \( r < 1 \) such that, for every \( u \in (0, r) \),
\[ g(u) < \bar{h}. \]

(G5) \[ \int_0^1 g(u) \, du = -\infty. \]
Theorem 3. Assume $h \in L^2(0, T)$ and let $f$ and $g$ be real continuous functions in $\mathbb{R}$ and in $(0, +\infty)$, respectively, satisfying (F1), (G3), (G4) and (G5). Then the problem (15) has at least one solution.

Proof. Consider the homotopy

$$u'' + f(u)u' + g_\lambda(u) = h_\lambda(u),$$

$$u(0) = u(T), \quad u'(0) = u'(T),$$

(16)

where $g_\lambda(u) = (1 - \lambda)\left(h + 1 - u^{-1}\right) + \lambda g(u)$, $h_\lambda(t) = (1 - \lambda)h + \lambda h(t)$. Note that $g_\lambda$ satisfies (G3), (G4) and (G5) uniformly in $\lambda \in [0, 1]$ and that $h_\lambda = h$.

First we obtain an $L^2(0, T)$-bound for the derivative of any solution $u(t)$ of (16). Indeed, multiplication by $u'$ and integration yields

$$a\|\dot{u}\|_2^2 \leq \pm \int_0^T f(u)u'^2 = \pm \int_0^T h_\lambda u' \leq \|h\|_2 \|u'\|_2.$$

Next, it follows easily from (G3) that there exists a $t_0$ such that

$$u(t_0) < R;$$

this, together with what has just been shown gives an upper bound for all solutions $u(t)$ of (16):

$$u(t) \leq u(t_0) + \int_{t_0}^t |u'| < R + T^{1/2}\|h\|_2 a^{-1} \equiv M.$$

Now we are going to obtain a uniform bound on $u'(t)$. To this purpose, extend $u$ as a $T$-periodic function and let $u(t_1)$ be any extremum of $u$. For $t \geq t_1$ it turns out that

$$u'(t) = \int_{t_1}^t u'' = -\int_{t_1}^t f(u)u' - \int_{t_1}^t g_\lambda(u) + \int_{t_1}^t h_\lambda.$$

Since $\|u\|_\infty < M$, (G4) implies that $g_\lambda(u)$ is bounded above. Hence the right-hand side is bounded from below, say

$$u'(t) > -M_1.$$

Similarly, for $t \leq t_1$,

$$u'(t) = \int_t^{t_1} f(u)u' + \int_t^{t_1} g_\lambda(u) + \int_t^{t_1} h_\lambda \leq M_1.$$

This shows that

(17) \quad $\|u'\|_\infty < M_1$.

On the basis of (G4) we can choose $t_2$ such that $u(t_2) > r$. Multiplying (16) by $u'$ and integrating in an interval $[t_2, t]$ we get

$$\frac{u'(t)^2}{2} - \frac{u'(t_2)^2}{2} + \int_{t_2}^t f(u)u'^2 + \int_{t_2}^t g_\lambda(u)u' = \int_{t_2}^t h_\lambda u'.$$

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so that, using the bounds on \(u(t)\), \(\|u\|_\infty\) and (17), we see that
\[
\int_{t_2}^t g_\lambda(u)u' = \lambda \int_{u(t_2)}^{u(t)} g(u) \, du + (1 - \lambda) \left[ \ln \frac{u(t_2)}{u(t)} + (\delta + 1)(u(t) - u(t_2)) \right]
\]
is bounded. Assumption (G5) then shows that there exists \(\epsilon > 0\) such that, for any such solution \(u(t)\),
\[
u(t) > \epsilon, \quad t \in [0, T].
\]
The estimates thus obtained enable us to apply coincidence degree to the homotopy (16). Define the operators
\[
L: D(L) \subset H^2(0, T) \rightarrow L^2(0, T),
\]
\[
N: [0, 1] \times \Omega \rightarrow L^2(0, T),
\]
where
\[
D(L) = \{u \in H^2(0, T): u(0) = u(T), u'(0) = u'(T)\},
\]
\[
Lu = u'' \quad \text{if} \quad u \in D(L),
\]
\[
\Omega = \{u \in C^1([0, T]): \epsilon < u(t) < M, \|u\|_\infty < M_1\},
\]
\[
N(\lambda, u) = \lambda u' - f(u)u - g_\lambda(u).
\]
From well-known properties of the coincidence degree [5] one has
\[
d_L(L - N(\lambda, \cdot), \Omega) = d_L(L - N(0, \cdot), \Omega) = \pm dB(\lambda - g_0(u), D) = \pm 1.
\]
Here \(dB\) denotes the Brower degree, \(D = \{\epsilon, M\}\), and we have used the fact that, by virtue of (F1), the only solution of (16) for \(\lambda = 0\) is the constant solution \(u = 1\). This shows that (16) is solvable for \(\lambda = 1\) and so the proof is complete.

It is easy to see that all the estimates in the above proof remains valid if we replace (F1) by

(G6). There exists a constant \(C > 0\) such that, for every \(t \in [0, T]\) and every \(u > 0\),
\[
g(u) - h(t) \leq C.
\]
The only difference is that the bound on \(\|u'\|_2\) now follows from the inequalities
\[
\int_0^T u'^2 \leq \int_0^T (g(u) - h)u \leq k\|u\|_1
\]
and
\[
u(t) \leq R + T^{1/2}\|u'\|_2
\]
( obtained as in the above proof, using (G3)). Hence we can state:

**Theorem 4.** Assume \(h \in L^2(0, T)\) and let \(f, g\) be real continuous functions in \(R\) and in \((0, +\infty)\), respectively, satisfying (G3), (G4), (G5) and (G6). Then the problem (15) has at least one solution.
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