UNIQUE CONTINUATION THEOREMS  
FOR SOME PARABOLIC OPERATORS

J. D. McMICHAIL AND D. M. OBERLIN

(Communicated by Barbara L. Keyfitz)

Abstract. We prove a unique continuation theorem for a class of differential operators containing the heat operator.

Suppose \( j_1, \ldots, j_n \) and \( j \) are positive integers and let \( E \) be the differential operator on \( \mathbb{R}^n \) whose symbol is

\[
\sigma(\xi) = -\left( \sum_{l=1}^{n} \xi_l^{2j_l} \right)^{j} .
\]

The purpose of this paper is to prove a unique continuation theorem for the operator \( L = \partial / \partial t - E \) on \( \mathbb{R}^{n+1} \).

Theorem. Let \( r = \sum_{l=1}^{n} (2jj_l)^{-1} + 1 \). Suppose \( u \in C_c^\infty(\mathbb{R}^{n+1}) \) satisfies the differential inequality

\[
|Lu(x, t)| \leq |V(x, t)u(x, t)|
\]

for some \( V \in L'(\mathbb{R}^{n+1}) \). Then \( u \equiv 0 \).

If \( L \) is the heat operator \( \partial / \partial t - \Delta \), then \( r = n/2+1 \). In this case our theorem yields a conclusion analogous to [3, Corollary 1], although that result for the Schrödinger operator \( i\partial / \partial t + \Delta \) requires only that \( u \) vanish in a half-space. Our method of proof is also similar to that of [3] (and [2]) in that we deduce our unique continuation theorem from a Carleman estimate which is a consequence of a uniform Sobolev inequality. But while the Sobolev inequalities of [2] and [3] are based on restriction theorems for the Fourier transform, ours depends only on Young's inequality and an elementary estimate. This may reflect the fact that our operators \( L \) are "less singular" than those treated in [2] and [3]. Our main task, then, will be to establish the following result.

Received by the editors February 3, 1989.
Key words and phrases. Sobolev inequalities, unique continuation.

©1990 American Mathematical Society
0002-9939/90 $1.00 + $.25 per page

991
Lemma 1. With $L$ and $r$ as above, fix $p$ and $q$ with $1 < p < q < \infty$ and $1/p - 1/q = 1/r$. There is a constant $C = C(L, p)$ such that for any $\lambda < 0$ and any $u \in C_0^\infty(\mathbb{R}^{n+1})$ we have

\begin{equation}
\|u\|_q \leq C\|L - \lambda\|u\|_p.
\end{equation}

Combining the observation

\begin{equation}
e^{\lambda t}Lu = (L - \lambda)v \quad \text{if } v = e^{\lambda t}u
\end{equation}

with (2) yields a Carleman estimate

\begin{equation}
\|e^{\lambda t}u\|_q \leq C(p)\|e^{\lambda t}Lu\|_p, \quad u \in C_0^\infty(\mathbb{R}^{n+1}).
\end{equation}

Now let $W^p$ be the Sobolev space on $\mathbb{R}^{n+1}$ defined by the norm

\[ \|u\|_{W^p} = \left\| \left[ (\sqrt{1 + \tau^2} - \sigma(\xi)) \hat{u}(\xi, \tau) \right] \right\|_p. \]

A short argument similar to the proof of [2, Corollary 3.1] shows that (4) implies the following: if $1 < p < r$ and $u \in W^p$ satisfies (1) for some $V \in L^r(\mathbb{R}^{n+1})$, then $u \equiv 0$ if $u$ vanishes on a half-space $\{(x, t): t \geq t_0\}$. This clearly yields our theorem.

The proof of Lemma 1 depends on an elementary result given below as Lemma 2. To state it we require some notation. If $\beta = (\beta_1, \ldots, \beta_n)$ is an $n$-tuple of positive numbers, define

\[ |\beta| = \sum_{i=1}^n \beta_i \]

and, for $t > 0$ and $x \in \mathbb{R}^n$,

\[ t^\beta x = (x_1 t^{\beta_1}, \ldots, x_n t^{\beta_n}), \quad \frac{x}{t^\beta} = \left( \frac{1}{t} \right)^\beta x. \]

Lemma 2. Suppose $\beta$ is as above and $f \in L^{1+1/|\beta|}(\mathbb{R}^n)$. Define

\[ K(x, t) = \begin{cases} t^{-|\beta|}f\left( \frac{x}{t^\beta} \right) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases} \]

Then $K \in L^{1+1/|\beta|, \infty}(\mathbb{R}^{n+1})$.

Proof. The measure in $\mathbb{R}^n$ of the set

\[ \left\{ x: t^{-|\beta|}f\left( \frac{x}{t^\beta} \right) > s \right\} \]

is equal to

\[ |\{t^\beta y: |f(y)| > st^{|\beta|}\}| = t^{|\beta|}|\{ y: |f(y)| > st^{|\beta|}\}| = t^{|\beta|}D(st^{|\beta|}). \]

Thus the measure in $\mathbb{R}^{n+1}$ of the set $\{(x, t): |K(x, t)| > s\}$ is

\[ \int_0^\infty t^{|\beta|}D(st^{|\beta|}) dt = \|\beta\|^{-1}s^{-1-1/|\beta|} \int_0^\infty u^{1/|\beta|}D(u) du. \]
This last integral is finite if (and only if) \( f \in L^{1+1/|\beta|}(\mathbb{R}^n) \).

Proof of Lemma 1. Let \( r' \) be the conjugate index of \( r \). We will construct fundamental solutions \( K_\lambda \) for the operators \( L - \lambda \) on \( \mathbb{R}^{n+1} \) such that

\[
\|K_\lambda\|_{r', \infty} \leq C
\]

for some \( C = C(L) \) and all \( \lambda < 0 \). Since \( 1/p + 1/r' = 1/q + 1 \), it will follow from Young's convolution inequality for weak \( L^p \) (see the comment on [4, p. 121]) that

\[
\|K_\lambda \ast w\|_q \leq C(L, p)\|w\|_p
\]

for \( w \in C_0^\infty(\mathbb{R}^{n+1}) \). Taking \( w = (L - \lambda)u \) then yields (2). Now if \( K(x, t) \) is a fundamental solution for \( L \), then (3) shows that \( K_\lambda(x, t) = e^{i\lambda t}K(x, t) \) is a fundamental solution for \( L - \lambda \). If \( K \) vanishes for \( t < 0 \) and if \( \lambda < 0 \), it follows that

\[
\|K_\lambda\|_{r', \infty} \leq \|K\|_{r', \infty}.
\]

Thus it is sufficient to find a fundamental solution \( K \) for \( L \) which is in \( L^{r', \infty}(\mathbb{R}^{n+1}) \) and vanishes for \( t < 0 \). Let \( \beta = ((2j_1)^{-1}, \ldots, (2j_n)^{-1}) \) and let \( f \) be the rapidly decreasing function on \( \mathbb{R}^n \) satisfying \( \hat{f}(\xi) = e^{s(\xi)} \). Define \( K \) as in Lemma 2. Since \( 1 + 1/|\beta| = r' \), it follows from Lemma 2 that \( K \in L^{r', \infty}(\mathbb{R}^{n+1}) \). The equation \( \sigma(t^\beta \xi) = t\sigma(\xi) \) and a change of variables show that, for \( t > 0 \),

\[
\hat{K}(\cdot, t)(\xi) = e^{t\sigma(\xi)}, \quad \xi \in \mathbb{R}^n.
\]

It follows from this and an argument analogous to [1, Proof of Theorem 4.6, p. 195] that \( K \) is a fundamental solution for \( L \) on \( \mathbb{R}^{n+1} \).

REFERENCES


DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306