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SOME ABELIAN BANACH ALGEBRAS OF OPERATORS ON THE MATRICIAL II_1 FACTOR

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ABSTRACT. We show that, if G is an amenable discrete group, then the set of completely bounded [completely positive] multiplication operators on $VN(G)$ is maximal abelian and norm-1 complemented in various sets of bounded [positive] operators on $VN(G)$. Since there are many different amenable discrete groups G which generate the matricial II_1 factor R, this shows the set of completely bounded normal operators on R contains uncountably many non-isomorphic maximal abelian subalgebras each of which is complemented by a positivity-preserving projection of norm one. Our results are closely related to [16], which considers the case of group C^*-algebras of general locally compact groups.

1. INTRODUCTION

Let G be a discrete group and let $B(G)$ be the space of all coefficients of continuous unitary representations of G. Then $B(G)$ may be regarded as the dual space of the group C^*-algebra $C^*(G)$ and so has a natural norm defined on it. With this norm and pointwise product, $B(G)$ is a commutative Banach algebra. Let $A(G)$ denote the space of coefficients of the left regular representation λ of G on $l^2(G)$ and let $VN(G) = \lambda(G)''$, the group von Neumann algebra of G. $VN(G)$ is the dual space of $A(G)$, when the latter is equipped with its natural norm. The multiplier algebra $MA(G)$ is the space of continuous functions u on G such that $uv \in A(G)$ whenever $v \in A(G)$. If $u \in MA(G)$, let m_u be the corresponding bounded operator on $A(G)$ and let $M_u = m_u^*$. Then M_u is a normal (i.e., σ-weakly continuous) operator on $VN(G)$ satisfying $M_u\lambda(g) = u(g)\lambda(g)$ for all $g \in G$. Let $M_0A(G)$ denote the subalgebra of completely bounded multipliers with the norm $\|u\|_{M_0} = \|M_u\|_{cb}$.

If G is amenable, then $MA(G) = M_0A(G) = B(G)$ [1, Corollary 1.8] and there is therefore an isometric algebra isomorphism $u \to M_u$ of $B(G)$ into the algebra $CB_n(VN(G))$ of completely bounded normal maps on $VN(G)$. Let $M_G = \{M_u: u \in B(G)\}$ be the range of this isomorphism. The isomorphism is order-preserving in the sense that it sends the set of positive definite functions

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on G (denoted by $B_+(G)$) to completely positive maps in M_G (denoted M_G^+) [6]. Our main results can be summarized as follows.

Theorem. *If G is an amenable discrete group then M_G (whose elements are completely bounded) is a maximal abelian subalgebra of $B_n(VN(G))$, the algebra of all bounded normal operators on $VN(G)$. (See also* [16, Theorem 1].) *Moreover M_G is the range of a contractive projection P on $CB(VN(G))$, the completely bounded operators on $VN(G)$, equipped with the completely bounded norm. The projection P extends to an affine projection $\overline{P}$ from the set of all 2-positive maps on $VN(G)$ onto the set M_G^+ of completely positive multiplications. Finally M_G^+ is also maximal abelian in the cone of all positive maps on $VN(G)$.*

If G_1 and G_2 are nonisomorphic discrete groups then $B(G_1)$ and $B(G_2)$ are nonisomorphic as Banach algebras [15]. Also if G is any locally finite ICC group then $VN(G)$ is isomorphic to the matricial II_1 factor R [7]. (A group G is said to be locally finite if every finitely generated subgroup is finite.) According to [8, Corollary 6.12] there exist uncountably many nonisomorphic simple locally finite countable groups. Such groups are amenable and ICC, and the corresponding algebras M_G are nonisomorphic. We thus have the following immediate consequence of the theorem.

Corollary. *$CB_n(R)$ contains uncountably many nonisomorphic maximal abelian norm-1 complemented subalgebras of the form M_G, with G locally finite.*

The algebras M_G consist of *completely* bounded maps, so that in a sense $B_n(R)$ contains many "large" sets of completely bounded maps. Of course $B_n(R)$ also contains maximal abelian subalgebras which are not of the form M_G. For example, consider the abelian algebra generated by any element of $B_n(R)$ which is not completely bounded and a maximal abelian subalgebra containing it.

We have tried to follow the terminology of [1], [2], [4] wherever possible and we refer to these papers for proofs or precise references for results on multipliers and completely bounded maps. In [16], $CB(R)$ is referred to as the "dual algebra" of R and there the maximal abelian part of the above theorem is obtained in an essentially more general context.

2. The case of bounded maps

We begin by considering bounded and completely bounded maps. The proofs are relatively easy because they rely on results of de Cannière and Haagerup. Unless the contrary is explicitly stated, G will denote an *amenable* discrete group. The notation of the introduction is used without comment. Although our first result is included in [16, Theorem 1] we give a simple proof for the special case at hand.

Proposition 1. *M_G is a maximal abelian subalgebra of $B_n(VN(G))$.*

Proof. Suppose that the bounded normal operator T on $VN(G)$ commutes with M_u for each $u \in B(G)$. Fix an element $h \in G$. The operator $T(\lambda(h))$ can be expressed as convolution on $l^2(G)$ by some function $\alpha \in l^2(G)$ [7, Theorem 6.7.2]. We can therefore write

$$T(\lambda(h)) = \sum_{g \in G} \alpha(g)\lambda(g).$$

For each $u \in B(G)$, we have

$$\begin{aligned} 0 &= (M_u T - TM_u)\lambda(h) \\ &= \sum_g (u(g)\alpha(g)\lambda(g) - u(h)\alpha(g)\lambda(g)) \\ &= \sum_g (u(g) - u(h))\alpha(g)\lambda(g). \end{aligned}$$

Since $B(G)$ separates points of G, we have $\alpha(g) = 0$ for $g \neq h$. Thus $T(\lambda(h)) = \alpha(h)\lambda(h)$ for $h \in G$. It follows that $\alpha \in MA(G)$ [1, Proposition 1.2] and that $T = M_\alpha$. Since G is amenable $MA(G) = B(G)$, so $T \in M_G$, as required. □

If T is a bounded linear map on $VN(G)$, let u_T be the complex-valued function on G defined by

$$u_T(g) = \operatorname{tr}(T(\lambda(g))\lambda(g)^*),$$

where tr is the canonical trace on $VN(G)$ defined by $\operatorname{tr}(x) = \langle x\delta, \delta\rangle$, where δ is the characteristic function of $\{e\}$. The fact that M_G is complemented in $CB(VN(G))$ is based upon the following.

Proposition 2. *If* $T \in CB(VN(G))$ *then* $u_T \in B(G)$ *and* $\|u_T\| \leq \|T\|_{cb}$.

Proof. We may suppose that $\|T\|_{cb} = 1$. By [9, Theorem 7.4], there exists a Hilbert space K, a $*$ homomorphism $\pi: VN(G) \to B(K)$ and isometries U, $V: l^2(G) \to K$ such that $T(a) = U^*\pi(a)V$ for all $a \in VN(G)$. Now for all g, $h \in G$, if $x = \lambda(g)$, $y = \lambda(h)$, we have

$$\begin{aligned} u_T(h^{-1}g) &= \langle T(y^*x)x^*y\delta, \delta\rangle \\ &= \langle yU^*\pi(y^*x)Vx^*\delta, \delta\rangle \\ &= \langle yU^*\pi(y^*)\pi(x)Vx^*\delta, \delta\rangle \\ &= \langle \pi(x)Vx^*\delta, \pi(y)Uy^*\delta\rangle_K \\ &= \langle \xi(g), \eta(h)\rangle_K, \end{aligned}$$

where ξ, $\eta: G \to K$ satisfy $\|\xi\|_\infty \leq 1$ and $\|\eta\|_\infty \leq 1$. It follows from a result of Haagerup [3, Theorem 6.10; 5, Theorem 2.2] that $u_T \in M_0A(G) = B(G)$, and $\|u_T\| \leq 1$. □

Observe now that $u_{M_u} = u$ for each $u \in B(G)$, since

$$u_{M_U}(g) = \operatorname{tr}((u(g)\lambda(g))\lambda(g)^*) = \operatorname{tr}(u(g)1) = u(g).$$

We therefore obtain

Corollary 3. *The map P defined on $CB(VN(G))$ by $P(T) = M_{u_T}$ is a contractive projection from $CB(VN(G))$ onto M_G, with respect to the completely bounded norm.* □

3. The case of positive maps

The first thing to notice is that, since G is amenable, any positive multiplier on G is automatically completely positive, i.e., belongs to M_G^+ [1, Corollary 4.4]. Exactly the same argument as in Proposition 1 therefore shows the following.

Proposition 4. *M_G^+ is maximal abelian in the cone of all positive normal maps on $VN(G)$.*

We now show that if $T: VN(G) \to VN(G)$ is 2-positive then the function u_T is positive definite. In order to do this we need a slight extension of a result of Størmer [11].

Lemma 5. *Let T be a 2-positive contraction from a C^*-algebra A into an injective von Neumann algebra N acting on a Hilbert space H. Then for each $\xi \in H$, there exists a completely positive contraction $T_\xi: A \to N$ such that*

$$T(a)\xi = T_\xi(a)\xi$$

for each $a \in A$. (i.e., T is "locally completely positive.")

Proof. Since T satisfies the Schwarz inequality $T(a)^*T(a) \le T(a^*a)$ [2], it follows from [11, Theorem 7.4] that there exists a completely positive map $T_\xi: A \to B(H)$ with the above property. Composing T_ξ with the conditional expectation from $B(H)$ onto N gives the required map. □

Proposition 6. *Let $T: VN(G) \to VN(G)$ be a 2-positive map (where, as usual, G is amenable discrete). Then the function u_T is positive definite on G.*

Proof. We may assume that T is contractive. Since $VN(G)$ is an injective von Neumann algebra, there exists, by Lemma 5, a completely positive map $S: VN(G) \to VN(G)$ such that $T(a)\delta = S(a)\delta$ for all $a \in A$.

Let $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ and $g_1, \ldots, g_n \in G$. Then

$$\begin{aligned}
\sum_{ij} \bar{\alpha}_i \alpha_j u_T(g_i^{-1} g_j) &= \sum_{ij} \bar{\alpha}_i \alpha_j \langle T(\lambda(g_i^{-1} g_j))\lambda(g_j^{-1} g_i)\delta, \delta\rangle \\
&= \sum_{ij} \bar{\alpha}_i \alpha_j \langle T(v_i^* v_j) v_j^* v_i \delta, \delta\rangle \qquad \text{where } v_i = \lambda(g_i) \\
&= \sum_{ij} \bar{\alpha}_i \alpha_j \langle v_j^* v_i T(v_i^* v_j)\delta, \delta\rangle \\
&= \sum_{ij} \bar{\alpha}_i \alpha_j \langle v_j^* v_i S(v_i^* v_j)\delta, \delta\rangle \\
&= \sum_{ij} \langle S(v_i^* v_j)\alpha_j v_j^* \delta, \alpha_i v_i^* \delta\rangle \\
&\ge 0,
\end{aligned}$$

since S is completely positive. This shows that u_T is positive definite. □

Proposition 7. *The map* $\overline{P}$ *defined on the set of all* 2*-positive maps* $T: VN(G) \to VN(G)$ *by* $\overline{P}(T) = M_{u_T}$ *is a norm-decreasing affine projection onto* M_G^+.

Proof. $\overline{P}$ is contractive since if $\|T\| \leq 1$ then $\|T(1)\| \leq 1$, so $|u_T(e)| = |\operatorname{tr}(T(\lambda(e))\lambda(e)^*)| = |\operatorname{tr}(T(1))| \leq 1$. Hence $\|M_{u_T}\|_{cb} = \|M_{u_T}\| = \|u_T\| = |u_T(e)| \leq 1$. The remaining assertions are clear. □

Remark. Proposition 6 may fail if G is not amenable. For take $G = \mathbf{F}_2$, the free group on two generators. According to [1, Corollary 4.8] there exists a function $u: G \to \mathbb{C}$ which is not positive definite but has the property that $T = M_u$ is 2-positive. Thus $u_T = u \notin B^+(G)$.

It follows from a result of Tits [14], that this example works for any nonamenable linear group. It may well be that the validity of Proposition 6 characterizes amenable (discrete) groups.

However, even if G is not amenable, the conclusions of Propositions 6 and 7 remain valid if we consider only *completely positive* maps T, by (a simplification of) the same arguments.

Example. Proposition 6 may fail if T is merely a positive contraction, even if G is a finite group. In fact take $G = \Sigma_3$, the group of permutations of three symbols $\{1, 2, 3\}$. Thus $VN(G) = \mathbf{C} \oplus \mathbf{C} \oplus M_2\mathbf{C}$.

Let T be the $*$ anti-automorphism of $VN(G)$ defined by $T(\lambda(g)) = \lambda(g^{-1})$. Then

$$u_T(g) = \operatorname{tr}(\lambda(g^{-2})) = \begin{cases} 1 & \text{if } g^2 = e, \\ 0 & \text{otherwise.} \end{cases}$$

Let g_1 and g_2 be the transpositions $g_1 = (1\ 2)$, $g_2 = (2\ 3)$, and let $g_3 = g_1 g_2$. Clearly $g_1^2 = g_2^2 = e$ and $g_1 g_2 \neq g_2 g_1$. Thus $(g_1^{-1} g_2)^2 = (g_1 g_2)^2 \neq e$ and $(g_1^{-1} g_3)^2 = e = (g_2^{-1} g_3)^2$. Hence

$$[u_T(g_i^{-1} g_j)] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

which is not positive definite, so that $u_T \notin B^+(G)$.

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