CONVEX FUNCTIONS WITH RESTRICTED CURVATURE

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ABSTRACT. Given $0 < R_1 \leq R_2 \leq \infty$, we consider a class of normalized convex functions $f$ in the unit disk $D$, for which $\partial f(D)$ satisfies a Blaschke Rolling Theorem condition with radii $R_1$ and $R_2$. This class contains the convex functions of bounded type. We study the geometry of the image region $f(D)$ and various covering and distortion properties.

1. Introduction

Let $B(a, \delta)$ be the open disk of radius $\delta$ centered at $a$, and let $D = B(0, 1)$. As usual, $S$ denotes the class of normalized univalent functions in $D$, and we denote by $CV$ the subclass of $S$ consisting of convex functions. Assume $f \in CV$. For fixed $r \in (0, 1)$ the curve $C_r: w = f(re^{it}), 0 \leq t \leq 2\pi$, has radius of curvature at $f(z)$ given by

$$\rho(z) = \frac{|zf'(z)|}{\Re(1 + zf''(z)/f'(z))}.$$  

When clarity requires reference to the function $f$ we will write $\rho(f; z)$, and we adopt a similar convention when it is needed in notations which arise later. Let $\rho_M(r)$ and $\rho_m(r)$ denote the maximum and minimum, respectively, of $\rho(z)$ on $|z| = r$. Setting $n(z) = zf'(z)/|zf'(z)|$ (the unit exterior normal to $C_r$ at $f(z)$), Blaschke’s Rolling Theorem (e.g., see [3]) implies that

$$B(f(z) - \rho_m(r)n(z), \rho_M(r)) \subseteq f(B(0, r)) \subseteq B(f(z) - \rho_M(r)n(z), \rho_M(r))$$

for each $z$ with $|z| = r$.

Walsh [8] showed that either $\rho_M(r)/r$ is strictly increasing on $(0, 1)$ or $f(z) = z$. He also observed that $\rho_m(r)$ need not be monotone. Let

$$R_2(f) = \lim_{r \to 1} \rho_M(r)$$

and

$$R_1(f) = \lim_{r \to 1} \rho_m(r).$$

Since $\rho(z)/|z| \to |f'(0)| = 1$ as $z \to 0$, we see from Walsh’s result that $1 \leq \rho_M(r)/r \leq R_2(f)$, with equality if and only if $f(z) = z$. For $0 \leq R_1 \leq R_2 \leq \infty$
and $r_2 \geq 1$ let $CV(R_1, R_2) = \{ f \in CV : R_1 \leq R_1(f) \leq R_2(f) \leq R_2 \}$. These functions were studied by Goodman [1, 2], who called them convex functions of bounded type. Distortion and coefficient results for this class have been obtained by Wirths [9] and Mejia and Minda [4].

In this paper we consider a closely related class of functions whose definition is motivated by the Blaschke Rolling Theorem. Given $0 \leq R_1 \leq R_2 \leq \infty$ and $R_2 \geq 1$, let $CVG(R_1, R_2)$ be the class of functions $f$ in $S$ with the property that for each $\eta \in \partial f(D)$ there are open disks $D_1(\eta)$ and $D_2(\eta)$ of radius $R_1$ and $R_2$, respectively, such that $\eta \in \partial D_1(\eta) \cap \partial D_2(\eta)$ and

$$D_1(\eta) \subseteq f(D) \subseteq D_2(\eta).$$

If $R_1 = 0$ or $R_2 = \infty$ we interpret $D_1(\eta)$ or $D_2(\eta)$ to be the empty set or an open half-plane, respectively.

If $f$ is a bounded function in $CVG(R_1, R_2)$ and $R_1 > 0$, then $\partial f(D)$ is smooth in the sense that it has a continuously turning tangent. Yet we shall see that $\partial f(D)$ need not be a curve of class $C^2$ and $\rho(z)$ may not be continuous on $\overline{D}$. In §2 we show that $CV(R_1, R_2) \subseteq CVG(R_1, R_2)$. Next, §3 concerns characterizations of $CVG(R_1, R_2)$ in terms of the geometry of the regions $f(D)$. In §4 we obtain some partial results on the relationship between the largest disk centered at zero which is covered by $f(D)$ and the $\sup\{|f(z)| : z \in D\}$ when $f \in CVG(R_1, R_2)$. Finally, we study the order of growth of $|f'|$ and discuss some open questions.

2. Preliminaries

Each $f$ in $CV$ has a spherically continuous extension to $\overline{D}$. Moreover, there are at most two points on $\partial D$ at which $f$ takes the value infinity, and except for this occurrence, $f(\partial D)$ is a simple closed curve in the extended complex plane. Since $zf'$ is starlike, $\arg zf'(z)$ has a radial limit everywhere [6]. Thus, though $f(\partial D)$ may not be smooth, $n(\zeta)$ does exist as a radial limit for all $\zeta \in \partial D$. Let

$$\Phi(t) = \lim_{r \to 1} \arg n(re^{it}).$$

Since $\Phi$ is increasing [6], it has at most a countable number of discontinuities, each of which is a simple jump. Moreover, if $\Phi$ has a jump at $t$ of magnitude $-\pi$, $\alpha > 0$, and $f(e^{it})$ is finite, then $\alpha < 1$, and we shall say that $\partial f(D)$ has a corner at $f(e^{it})$ with interior angle $(1-\alpha)\pi$.

Lemma 1. If $f \in CV$ and $\partial f(D)$ has a corner at $f(e^{it_0})$, then $\rho(re^{it_0}) \to 0$ as $r \to 1$.

Proof. Assume $f \in CV$. Pommerenke [6, Lemma 1] showed that

$$zf'(z) = z \exp \left\{ (-1/\pi) \int_0^{2\pi} \log(1 - ze^{-it}) d\Phi(t) \right\},$$
so
\[ \rho(z) = |zf'(z)|/\left(1/2\pi\right) \int_0^{2\pi} \text{Re}\{(1 + ze^{-it})/(1 - ze^{-it})\} \, d\Phi(t). \]

Suppose \( \Phi \) has a jump of magnitude \( \alpha\pi \) at \( t_0 \). If \( \delta \in (0, \pi) \), then
\[ \left(1/2\pi\right) \int_0^{2\pi} \text{Re}\{(1 + re^{i(t_0-t)})/(1 - re^{i(t_0-t)})\} \, d\Phi(t) \]
\[ \geq (1/2\pi) \int_{t_0-\delta}^{t_0+\delta} \text{Re}\{(1 + re^{i(t_0-t)})/(1 - re^{i(t_0-t)})\} \, d\Phi(t), \]
and the quantity on the right decreases to \( \alpha(1 + r)/2(1 - r) \) as \( \delta \) decreases to zero. Thus, for \( z = re^{it_0} \),
\[ \text{Re}\{1 + zf''(z)/f'(z)\} \geq \alpha(1 + r)/2(1 - r). \]

Moreover, Pommerenke [6, p. 212] showed that if \( \eta > 0 \) and \( z = re^{it_0} \), then
\[ \ln |z f'(z)| < \ln r + (1/\pi)[\Phi(t_0 + \eta) - \Phi(t_0 - \eta)] \ln[(1 + r)/(1 - r)] + B, \]
where \( B = 2\pi/\sin^2(\eta) \). Given \( \epsilon > 0 \), fix \( \eta > 0 \) such that \( \Phi(t_0 + \eta) - \Phi(t_0 - \eta) < (\alpha + \epsilon)\pi \). It then follows from (2) and (3) that
\[ \alpha \rho(z) \leq 2re^{B\ln[(1 - r)/(1 + r)]^{1-\alpha-\epsilon}}. \]
Since \( \alpha < 1 \), we may choose \( \epsilon \) so that \( 1 - \alpha - \epsilon > 0 \), and the conclusion follows.

Let \( f_r(z) = f(rz)/r \), \( 0 < r < 1 \). Since \( \rho(f_r; z) \) is continuous on \( \overline{D} \), it is easy to see that \( R_1(f_r) \) and \( R_2(f_r) \) are the minimum and maximum, respectively, of \( \rho(f_r; z) \) on \( \partial D \). Then \( R_1(f_r) = \rho_m(r)/r \) and \( R_2(f_r) = \rho_M(r)/r \), so \( R_1(f) \) and \( R_2(f) \) are the limit inferior and limit, respectively, of \( R_1(f_r) \) and \( R_2(f_r) \), as \( r \to 1 \).

**Theorem 1.** \( \overline{CV(R_1, R_2)} \subseteq CVG(R_1, R_2) \).

**Proof.** Assume \( f \in CV(R_1, R_2) \), and let \( \eta = f(\xi) \in \partial f(D) \). Applying Blaschke’s Rolling Theorem to \( f_r \), we have
\[ B(f_r(\xi) - R_1(f_r)n(\zeta), R_1(f_r)) \subseteq B(f_r(\xi) - R_2(f_r)n(\zeta), R_2(f_r)) \]
for each \( \zeta \in \partial D \). If \( w \in f(D) \), then \( w \in f_r(D) \) for sufficiently large \( r \). Letting \( r \) tend to 1, the right side of (4) yields
\[ w \in B(f(\xi) - R_1(f)n(\zeta), R_2(f)) \subseteq B(f(\xi) - R_2n(\zeta), R_2). \]
Setting \( D_2(\eta) = B(f(\xi) - R_2n(\zeta), R_2) \), we conclude (since \( f(D) \) is open) that \( f(D) \subseteq D_2(\eta) \). This establishes the right side of (1). If \( R_1 = 0 \), then the left side of (1) is clear, so assume \( R_1 > 0 \). Let \( w \in B(f(\xi) - R_1(f)n(\zeta), R_1(f)) \) and choose a sequence of values of \( r \) converging to 1 such that \( R_1(f_r) \to R_1(f) \). From (4) we have
\[ w \in B(f_r(\xi) - R_1(f_r)n(\zeta), R_1(f_r)) \subseteq f_r(D) \]
for all these values of \( r \) which are sufficiently near 1, and consequently \( w \in f(D) \). Thus,

\[
B(f(\zeta) - R_1 n(\zeta), R_1) \subseteq B(f(\zeta) - R_1(f)n(\xi), R_1(f)) \subseteq f(D),
\]

and \( D_1(\eta) = B(f(\zeta) - R_1 n(\zeta), R_1) \) satisfies the left side of (1).

Suppose \( \Omega \) is a bounded convex region. Given \( z \in \Omega \), let \( \phi_z \) denote the conformal mapping of \( D \) onto \( \Omega \) satisfying \( \phi_z(0) = z \) and \( \phi_z'(0) > 0 \). The hyperbolic metric for \( \Omega \) has density \( \lambda_{\Omega}(z) = 1/\phi_z'(0) \). The following lemma will be needed in subsequent sections and follows directly from results on \( \lambda_{\Omega} \) in [5, p. 474].

**Lemma 2.** Suppose \( \Omega \) is a bounded convex region that is symmetric about two distinct lines which intersect at \( z_0 \). If \( z_1 \in \Omega \setminus \{z_0\} \) and \( z_1 = (1-t)z_0 + tz_1 \), \( 0 < t < 1 \), then \( \phi_{z_1}'(0) < \phi_{z_0}'(0) < \phi_{z_1}'(0) \).

### 3. Geometry of \( f(D) \)

Assume \( 0 < R < \infty \), \( a, b \in C \) (the complex numbers), and \( |a - b| \leq 2R \). If \( |a - b| < 2R \), then there are two open disks \( \Delta_1 \) and \( \Delta_2 \) of radius \( R \) such that \( a, b \in \partial \Delta_1 \cap \partial \Delta_2 \), and we set \( E(a, b; R) = \Delta_1 \cap \Delta_2 \). If \( |a - b| = 2R \), set \( E(a, b; R) = B((a + b)/2, R) \). We shall say that a set \( A \subseteq C \) is \( R \)-convex if \( \text{diam}(A) \leq 2R \) and \( E(a, b; R) \subseteq A \) for all choices of \( a \) and \( b \) in \( A \). Assuming \( \text{diam}(A) \leq 2R \), we define the \( R \)-hull of \( A \) to be the smallest \( R \)-convex set containing \( A \) and denote it by \( \text{co}_R(A) \). In the limiting case when \( R = \infty \) we set \( E(a, b; \infty) = (a, b) \), which yields the usual notions of convexity and convex hull. Mejia and Minda [4] obtained the following result.

**Theorem A.** If \( 1 \leq R \leq \infty \) and \( f \in S \), then \( f \in CV(0, R) \) if and only if \( \text{diam}(f(D)) \leq 2R \) and \( E(a, b; R) \subseteq f(D) \) for all \( a, b \in f(D) \).

In the theorem which follows we present characterizations of \( CVG(R_1, R_2) \) in terms of the geometry of \( f(D) \). Suppose \( 0 < R_1 \leq R_2 \leq \infty \) and let \( a, b \in C \). If \( |a - b| \leq 2(R_2 - R_1) \), then \( \text{diam}(B(a, R_1) \cup B(b, R_1)) \leq 2R_2 \), and we set \( E(a, b; R_1, R_2) = \text{co}_{R_2}(B(a, R_1) \cup B(b, R_1)) \). If \( R_1 = 0 \) and \( R_2 > 0 \), we set \( E(a, b; 0, R_2) = E(a, b; R_2) \).

**Theorem 2.** If \( 0 \leq R_1 \leq R_2 \leq \infty \), \( R_2 \geq 1 \), and \( f \in S \), then the following are equivalent:

(i) \( f \in CVG(R_1, R_2) \);
(ii) \( f(D) \) is the intersection of open disks of radius \( R_2 \) and, in the case \( R_1 > 0 \), the union of open disks of radius \( R_1 \);
(iii) \( \forall u, v \in f(D) \exists a, b \in f(D) \) such that \( u, v \in E(a, b; R_1, R_2) \subseteq f(D) \);
(iv) \( f \in CV \), and for each \( \zeta \in \partial D \) for which \( f(\zeta) \) is finite,

\[
f(D) \subseteq B(f(\zeta) - R_2 n(\zeta), R_2),
\]

and in the case \( R_1 > 0 \), \( B(f(\zeta) - R_1 n(\zeta), R_1) \subseteq f(D) \).
Remark. If $R_2 = \infty$, then an open disk of radius $R_2$ is to be interpreted as an open half-plane, and $B(f(\zeta) - R_2 n(\zeta), R_2)$ is to be interpreted as the limiting half-plane as $R_2 \to \infty$.

Proof. (i) $\Rightarrow$ (ii). Assuming $f \in CVG(R_1, R_2)$, we assert that $f(D) = \cap \{D_2(\eta): \eta \in \partial f(D)\}$. Certainly $f(D) \subseteq \cap \{D_2(\eta): \eta \in \partial f(D)\}$. To obtain the opposite containment, suppose $w \notin f(D)$. Fix $w_0 \in f(D)$ and choose $\eta \in [w_0, w] \cap \partial f(D)$. Then $f(D) \subseteq D_2(\eta)$ implies $w \notin D_2(\eta)$, and thus $w \notin \cap \{D_2(\eta): \eta \in \partial f(D)\}$. Now assume $R_1 > 0$ and let $w \in f(D)$. If $\text{dist}(w, \partial f(D)) \geq R_1$, let $D(w) = B(w, R_1)$. If $\text{dist}(w, \partial f(D)) < R_1$, choose $\eta \in \partial (f(D))$ such that $|\eta - w| = \text{dist}(w, \partial f(D))$ and let $D(w) = D_1(\eta)$. Then $f(D) = \bigcup\{D(w): w \in f(D)\}$, and (ii) is established.

(ii) $\Rightarrow$ (iii). Let $u, v \in f(D)$. Suppose first that $R_1 = 0$ and let $\{D_\alpha: \alpha \in A\}$ be a collection of open disks of radius $R_2$ such that $f(D) = \cap \{D_\alpha: \alpha \in A\}$. Since $f(D)$ is convex (being the intersection of open disks), $[u, v] \subseteq f(D)$. Choose $a$ and $b$ on the line determined by $u$ and $v$ such that $[u, v] \subseteq (a, b) \subseteq f(D)$. For each $\alpha \in A$, $(a, b) \subseteq D_\alpha$, and hence $E(a, b; R_2) \subseteq D_\alpha$. Thus, $u, v \in E(a, b; R_2) \subseteq f(D)$. Now assume that $R_1 > 0$, in which case we assume in addition that $f(D)$ is the union of open disks of radius $R_1$. Choose open disks $D_u$ and $D_v$ of radius $R_1$ with centers $a$ and $b$, respectively, such that $u \in D_u \subseteq f(D)$ and $v \in D_v \subseteq f(D)$. Since $f(D)$ lies in a disk of radius $R_2$, $\text{diam}(D_u \cup D_v) \leq 2R_2$. We claim that $E(a, b; R_1, R_2) = \text{co}_R(D_u \cup D_v) \subseteq f(D)$. Indeed, since $CVG(R_1, R_2) \subseteq CVG(0, R_2)$, we may apply the earlier part of the proof to conclude that $E(a', b'; R_2) \subseteq f(D)$ for all $a' \in D_u$ and $b' \in D_v$, and the conclusion follows.

(iii) $\Rightarrow$ (iv). Since $E(a, b; R_1, R_2)$ is a convex set, (iii) implies that $f(D)$ is convex. It is implicit in (iii) that $\text{diam}(f(D)) \leq 2R_2$. Fix $\zeta \in \partial D$ such that $f(\zeta) \neq \infty$, and consider $w \in f(D)$. Suppose first that $R_1 > 0$. For sufficiently small $\varepsilon > 0$, $f(\zeta - \varepsilon n(\zeta)) \in f(D)$, so there are points $a_\varepsilon$ and $b_\varepsilon$ such that $w, f(\zeta - \varepsilon n(\zeta)) \in E(a_\varepsilon, b_\varepsilon; R_1, R_2) \subseteq f(D)$. Let $\varepsilon$ tend to zero through a sequence of values for which $a_\varepsilon$ and $b_\varepsilon$ are convergent, say to $a$ and $b$, respectively. Then $w \in E(a, b; R_1, R_2)$, and $f(\zeta) \in \partial E(a, b; R_1, R_2)$. Since $R_1 > 0$, $\partial f(D)$ has a tangent at $f(\zeta)$, and it follows from the shape of $E(a, b; R_1, R_2)$ that

$$B(f(\zeta) - R_1 n(\zeta), R_1) \subseteq E(a, b; R_1, R_2) \subseteq B(f(\zeta) - R_2 n(\zeta), R_2).$$

Thus, $B(f(\zeta) - R_1 n(\zeta), R_1) \subseteq f(D)$, and $w \in B(f(\zeta) - R_2 n(\zeta), R_2)$ for all $w \in f(D)$. Since $f(D)$ is open we conclude that $f(D) \subseteq B(f(\zeta) - R_2 n(\zeta), R_2)$. Now consider the case $R_1 = 0$. The preceding argument yields $f(D) \subseteq B(f(\zeta) - R_2 n(\zeta), R_2)$ when $\partial f(D)$ has a tangent at $f(\zeta)$, and this occurs for all but a countable number of points $\zeta \in \partial D$. Suppose $\partial f(D)$ has a corner at $f(\zeta)$, $\zeta = e^{i\theta}$, and let $n(\zeta^+)$ and $n(\zeta^-)$ denote the limits of $n(e^{it})$ as $t$ tends to $\theta$ from the right and left, respectively. From the previous case we
conclude that
\[ f(D) \subseteq B(f(\zeta) - R_2n(\zeta^+), R_2) \cap B(f(\zeta) - R_2n(\zeta^-), R_2). \]

Since, \( \text{arg } n(\zeta^-) \leq \text{arg } n(\zeta) \leq \text{arg } n(\zeta^+) \), the right side of Equation (5) lies in
\[ B(f(\zeta) - R_2n(\zeta), R_2). \]

Finally, it is clear that (iv) implies (i).

The following result occurs in the proof of (iii) implies (iv) in Theorem 2, and we record it here for emphasis.

**Corollary.** Suppose \( f \in \text{CVG}(0, R) \) and \( \partial f(D) \) has a corner at \( f(\zeta), \zeta = e^{it_0} \). Then
\[ f(D) \subseteq B(f(\zeta) - R_2n(\zeta^+), R_2) \cap B(f(\zeta) - R_2n(\zeta^-), R_2), \]
where \( n(\zeta^+) \) and \( n(\zeta^-) \) denote the limits of \( n(e^{it}) \) as \( t \to t_0 \) from the right and left, respectively.

### 4. Covering properties

Given \( f \) in \( \text{CV} \), let \( d(f) \) and \( M(f) \) be the minimum and maximum, respectively, of \( |f(z)| \) on \( |z| = 1 \). In general, \( 1/2 \leq d(f) \leq M(f) \leq \infty \).

Goodman [1] considered the problems of finding \( \inf d(f) \) and \( \sup M(f) \) for \( f \) in \( \text{CV}(0, R) \), and he found the following function and its rotations to be extremal for both problems.

**Example.** If \( 1 \leq R < \infty \), let \( k_R(z) = z/(1 - z\sqrt{1 - 1/R^2}) \), and if \( R = \infty \), let \( k_\infty(z) = z/(1 - z) \). Then \( k_R(D) \) is \( B(\sqrt{R^2 - R}, R) \) or \( \{ z: \text{Re } z > -1/2 \} \), depending on whether \( R < \infty \) or \( R = \infty \), respectively. To avoid the need to distinguish between the two cases, \( R < \infty \) and \( R = \infty \), we shall agree, in the case \( R = \infty \), to interpret \( B(\sqrt{R^2 - R}, R) \) to be \( \{ z: \text{Re } z > 1/2 \} \) and quantities such as \( R - \sqrt{R^2 - R} \) to be their limits (in this case \( 1/2 \)) as \( R \to \infty \). Then for \( 1 \leq R \leq \infty \) we have \( R_1(k_R) = R = R_2(k_R) \), \( d(k_R) = R - \sqrt{R^2 - R} \), and \( M(k_R) = R + \sqrt{R^2 - R} \).

The conclusions of the following theorem were obtained by Goodman [1] for the class \( \text{CV}(R_1, R_2) \). The proof for \( \text{CVG}(R_1, R_2) \) is virtually the same and will not be repeated.

**Theorem B.** If \( f \in \text{CVG}(R_1, R_2) \), then
\[ d(f) \geq R_2 - \sqrt{R^2 - R_2} \]
and
\[ M(f) \leq R_2 + \sqrt{R^2 - R_2}. \]
Equality occurs in (6) or (7) if and only if $f$ is a rotation of $kR_2$. Moreover, if $1 \leq R_1 \leq \infty$, then

$$d(f) \leq R_1 - \sqrt{R_2^2 - R_1^2},$$

with equality if and only if $f$ is a rotation of $kR_1$.

We now look at the effect on $M(f)$ when $f \in \text{CVG}(R_1, R_2)$ and $f$ covers the disk $B(0, d)$, where $R_2 - \sqrt{R_2^2 - R_1^2} < d < 1$. We shall need to also assume that $R_1 \leq d$. The case $R_2 = \infty$ is somewhat special in that functions in $\text{CVG}(R_1, \infty)$ need not be bounded, and we begin by discussing some of the features of this situation.

Given $r > 0$, let $g_r$ be the conformal mapping of $D$ onto $\text{co}(B(0, r) \cup (0, \infty))$ such that $g_r(0) = 0$ and $g_r'(0) > 0$. If $0 < r_1 < r_2$, then $g_r_i$ is properly subordinate to $g_r$, and consequently $g_r'(0)$ is a strictly increasing function of $r$. If $r \leq \pi/4$, then $g_r < (1/2) \log[(1 + z)/(1 - z)]$ and $g_r'(0) < 1$, whereas $r \geq 1$ implies $z < g_r$ and $g_r'(0) > 1$. Thus, there is a unique $r^* \in (\pi/4, 1)$ such that $g^* = g_{r^*} \in \text{CV}$.

**Lemma 3.** Assume $f \in \text{CV}$. If $d(f) > r^*$, then $f$ is bounded. If $d(f) = r^*$ and $f$ is not bounded, then $f$ is a rotation of $g^*$.

**Proof.** Suppose $d(f) > r^*$ and $f$ is not bounded. Choose $\{w_n\}_{n=1}^{\infty} \subseteq f(D)$ such that $w_n \to \infty$ and $\arg w_n \to \beta$. By a rotation we need only consider $\beta = 0$, in which case $\text{co}(B(0, d(f)) \cup (0, \infty)) \subseteq f(D)$. Thus, $g^* < f$, and, since both functions are normalized, we conclude that $g^* = f$ and $d(f) = r^*$.

Now, suppose $0 \leq R_1 \leq R_2 < \infty$, $R_2 > 1$, $R_2 - \sqrt{R_2^2 - R_1^2} < d < 1$, and $R_1 \leq d$. Let $a \in [R_1 - d, 2R_2 - R_1 - d]$ and let $U_a$ be $B(0, d) \cup B(a, R_1)$ or $B(0, d) \cup (0, a)$ depending on whether $R_1 > 0$ or $R_1 = 0$. Then $\text{diam } U_a \leq 2R_2$, and there is a unique conformal map $\varphi_a$ of $D$ onto $\text{co}_{R_1}(U_a)$ such that $\varphi_a(0) = 0$ and $\varphi_a'(0) > 0$. If $a \in [R_1 - d, d - R_1]$, then $\varphi_a(D) = B(0, d)$ and $\varphi_a'(0) = d < 1$. Since $R_1 \leq d$, the regions $\varphi_a(D)$ expand as $a$ increases, and consequently $\varphi_a'(0)$ is a strictly increasing function of $a$ on $[d - R_1, 2R_2 - R_1 - d]$. If $a = 2R_2 - R_1 - d$, then $\varphi_a(D) = B(R_2 - d, R_2)$, and since $d > R_2 - \sqrt{R_2^2 - R_1^2}$, Lemma 2 implies that $\varphi_a'(0) = k_{R_2}'(0) = 1$. Thus, there is a unique $a \in [d - R_1, 2R_2 - R_1 - d]$ for which $\varphi_a'(0) = 1$. This normalized function, denoted hereafter by $h_d$, is in $\text{CVG}(R_1, R_2)$.

Now, consider $R_2 = \infty$. In this case we choose $a \in (R_1 - d, \infty)$ and define $\varphi_a$ as before. As long as $d > r^*$, $\lim_{a \to \infty} \varphi_a'(0) = g_d'(0) > 1$, and again there will be a unique value of $a$ for which $\varphi_a'(0) = 1$. This normalized function will also be denoted by $h_d$.

**Theorem 3.** Assume $0 \leq R_1 \leq d$, $R_2 > 1$, and $R_2 - \sqrt{R_2^2 - R_1^2} < d < 1$. Let $f \in \text{CVG}(R_1, R_2)$ and suppose $d(f) \geq d$. If $R_2 < \infty$, then $M(f) \leq M(h_d)$,
with equality if and only if \( f \) is a rotation of \( h_d \). If \( R_2 = \infty \), then the same conclusions hold as long as \( d > r^* \).

**Proof.** In either case, choose \( \zeta \in \partial D \) such that \(|f(\zeta)| = M(f)| \). By a rotation it suffices to consider \( f(\zeta) > 0 \), in which case \( n(\zeta) = 1 \). Then,

\[
B(f(\zeta) - R_1, R_1) \subseteq f(D),
\]

and consequently \( \co_{R_2}(B(0, d) \cup B(f(\zeta) - R_1, R_1)) \subseteq f(D) \). If \( f(\zeta) \geq M(h_d) \), then \( h_d \) is subordinate to \( f \). But, \( f'(0) = 1 = h'_d(0) \), so \( f = h_d \).

If \( R_2 = \infty \), \( 1/2 \leq d \leq r^* \), and \( f \in \text{CVG}(R_1, R_2) \), then \( f(D) \) may not be bounded, but there is a restriction on how large a sector \( f \) may cover. For each \( \theta \in (0, \pi] \), let \( \sigma_\theta \) be the conformal map of \( D \) onto \( \co(B(0, d) \cup \{z : |\arg z| < \theta/2\}) \) such that \( \sigma_\theta(0) = 0 \) and \( \sigma'_\theta(0) > 0 \). Let \( \sigma_0 = g_d \). Then \( \sigma'_\theta(0) \) is a strictly increasing function of \( \theta \) on \([0, \pi]\). Moreover, \( d \geq 1/2 \) implies \( z/(1-z) < g_\pi \) and \( g_\pi(0) \geq 1 \), with equality only when \( d = 1/2 \). Also, \( d \leq r^* \) implies \( \sigma_0 < g^* \) and \( \sigma'_0(0) \leq 1 \), with equality only when \( d = r^* \). Thus, there is a unique \( \theta(d) \) in \([0, \pi]\) such that \( \sigma'_\theta(d)(0) = 1 \), and we denote this function by \( s_d \).

**Theorem 4.** Suppose \( 1/2 \leq d < r^* \), \( f \in \text{CVG}(R_1, \infty) \), \( R_1 \leq d \), and \( d(f) \geq d \). If \( f(D) \) contains an open sector of angle \( \theta(d) \), then \( f \) is a rotation of \( s_d \).

**Proof.** Suppose \( f(D) \) contains an open sector \( \Sigma \) with vertex angle \( \theta(d) \). By a rotation we may assume that \( \Sigma \) is symmetric with respect to the real axis and contains \((0, \infty)\), in which case \( s_d < f \). Thus, \( f = s_d \).

### 5. Growth of \(|f'|\)

The following result, due to Pommerenke [6, Theorem 1], shows that for convex functions the growth of \( M(r, f') = \max\{|f'(z)| : |z| = r\} \) is governed by the magnitude of the largest jump of \( \Phi \).

**Theorem C.** If \( f \in \text{CV} \) and \( \alpha \pi = \max\{\Phi(t^+) - \Phi(t^-) : 0 \leq t \leq 2\pi\} \), then

\[
M(r, f') \geq 1/4(1-r)^\alpha
\]

and

\[
\alpha = \lim_{r \to 1} \log M(r, f') / \log(1/(1-r)) .
\]

If \( f \in \text{CVG}(R_1, R_2) \), \( f(D) \) is bounded, and \( R_1 > 0 \), then \( \Phi \) is continuous and \( \alpha = 0 \). Hence we consider the case \( R_1 = 0, \ R_2 = R \in [1, \infty) \), and proceed to determine the largest possible jump of \( \Phi \). First we discuss the function in \( \text{CVG}(0, R) \) which exhibits the extremal behavior.

**Example.** Suppose \( \alpha \in [0, 1) \), let \( \phi(z) = (1+z)/(1-z) \), and consider \( f_\alpha(z) = \phi^{-1}(\phi(z)^{-1})/(1-\alpha) \). Elementary calculations show that \( f(D) = E(a, b; R) \), where \( a = -(1-\alpha)^{-1} \), \( b = (1-\alpha)^{-1} \), and

\[
R = 1/(1-\alpha) \sin[(1-\alpha)\pi/2].
\]
Thus, \( f \in \text{CVG}(0, R) \). For each \( R \in [1, \infty) \) there is a unique \( \alpha = \alpha(R) \in [0, 1) \) determined implicitly by (11), and we set \( f_R = f_{\alpha(R)} \). The function \( \Phi(f_R, t) \) is continuous on \( (0, \pi) \cup (\pi, 2\pi) \) and has a jump discontinuity of magnitude \( \alpha(R)\pi \) at 0 and at \( \pi \).

**Theorem 5.** If \( 1 \leq R < \infty \) and \( f \in \text{CVG}(0, R) \), then \( \Phi(t^+) - \Phi(t^-) \leq \alpha(R)\pi \) for all \( t \), and equality occurs for some \( t \) if and only if \( f \) is a rotation of \( f_R \).

**Proof.** Fix \( \zeta = e^{it} \) and assume \( \Phi(t^+) - \Phi(t^-) = \beta\pi > 0 \). By the corollary to Theorem 2,

\[
\Omega = \{ f(D) \subseteq \Omega = B(f(\zeta) - Rn(\zeta^+), R) \cap B(f(\zeta) - Rn(\zeta^-), R) \}.
\]

Note that \( \Omega = E(f(\zeta), b; R) \) for an appropriate choice of \( b \in C \), and the interior angle of \( \Omega \) at \( f(\zeta) \) is \( (1 - \beta)\pi \). There exist \( \mu, \nu \in C, |\mu| = 1 \), such that \( G = (\mu f_R + \nu)(D) = E(f(\zeta), c; R) \), where \( c \) lies on the ray from \( f(\zeta) \) through \( b \). Now suppose \( \beta > \alpha(R) \). Then \( |c - f(\zeta)| > |b - f(\zeta)| \) and \( f(D) \subseteq \Omega \subseteq G \). Let \( g \) be the conformal map of \( D \) onto \( G \) such that \( g(0) = f(0) \) and \( g'(0) > 0 \). Then \( \beta > \alpha(R) \) implies \( f \) is properly subordinate to \( g \), and consequently \( 1 = f'(0) < g'(0) \). But Lemma 2 gives \( g'(0) \leq [(\mu f_R + \nu)'(0)]^2 = 1 \), a contradiction. Thus, \( \beta \leq \alpha(R) \), and the conclusion follows. If \( \beta = \alpha(R) \), then the same considerations yield \( f(D) = G \) and \( f(0) = 0 = (\mu f_R + \nu)(0) \), so \( f \) is a rotation of \( f_R \).

**Corollary.** If \( 1 \leq R < \infty \) and \( f \in \text{CVG}(0, R) \), then

\[
M(r, f') = O((1-r)^{-\alpha(R)}).
\]

**Proof.** Suppose \( f \in \text{CVG}(0, R) \) and \( f \neq f_R \). By the previous theorem, \( \beta = \max\{\Phi(t^+) - \Phi(t^-) : 0 \leq t \leq 2\pi\}/\pi < \alpha(R) \). If \( \epsilon \) is chosen so that \( \beta + \epsilon < \alpha(R) \), then (10) gives \( M(r, f') = O((1-r)^{-(\beta+\epsilon)}) = o((1-r)^{-\alpha(R)}) \). Now, consider \( f_R \). A brief calculation yields

\[
f_R'(z) = 4/[((1+z)^{1-\alpha(R)} + (1-z)^{1-\alpha(R)})^2((1-z^2)^{\alpha(R)})].
\]

The quantity \((1+z)^{1-\alpha(R)} + (1-z)^{1-\alpha(R)}\) has a positive minimum modulus \( m \) on \( \overline{D} \), so \( |f_R'(z)| \leq 4/m^2((1-|z|^2)^{\alpha(R)}) \), and hence \( M(r, f_R') = O((1-r)^{-\alpha(R)}) \).

### 6. Comments

It follows readily from classical results in differential geometry (e.g., see [7]) that if \( f \in \text{CVG}(R_1, R_2) \) and \( f \) is analytic on \( \overline{D} \), then \( f \in \text{CV}(R_1, R_2) \), but it remains to be determined whether or not \( \text{CVG}(R_1, R_2) = \text{CV}(R_1, R_2) \). Mejia and Minda [4] have shown that this is the case when \( R_1 = 0 \), and we expect that it is also true for \( R_1 > 0 \).

Very little seems to be known about the behavior of \( \rho_m(r) \) as \( r \to 1 \). Although Walsh noted that it need not be monotone, it would be interesting to know if \( \rho_m(r) \) has a limit as \( r \to 1 \), or if, perhaps, it has at most a finite number of local extrema on \((0, 1)\).
The results in Theorem 3 are incomplete due to the assumption $R_1 \leq d$. If $R_1 > d$, then for an appropriate choice of $a \in [R_1 - d, 2R_2 - R_1 - d]$ there is a normalized conformal mapping of $D$ onto $\text{co}_{R_2}(B(R_1 - d, R_1) \cup B(a, R_1))$, and this may be the extremal function. However, in this case we do not get the subordination used in the proof of Theorem 3 because when $f$ is rotated so that $M(f) = f(\zeta) > 0$, the point on $\partial f(D)$ nearest to $z = 0$ may not be $-d$.

REFERENCES