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OPEN RELATION THEOREM WITHOUT CLOSEDNESS ASSUMPTION

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ABSTRACT. It is shown that a convex relation such that the preimage of a bounded set has nonempty interior is lower semicontinuous throughout the algebraic interior of its domain. If, besides, the relation is closed-valued, then it is closed throughout the algebraic interior of its domain.

1. INTRODUCTION

A well-known theorem of convex analysis says that if Y is a topological vector space, $f: Y \to \overline{\mathbf{R}}$ is convex and there exists an open set Q_0 such that

$$\sup_{Q_0} f < +\infty, \tag{1.1}$$

then f is continuous on the algebraic interior of its domain $\operatorname{dom} f = \{y: f(y) < +\infty\}$ (e.g. [6]). We prove open relation and closed-graph theorems that are straightforward generalizations to convex relations of the above fact concerning convex functions. In particular, these theorems do not require graph-closedness assumptions.

Let us recall that the open relation theorem has been established (as an extension of the Banach open mapping [2]) by Ursescu [9] and, independently, by Robinson [7] under the provision of closed convex graph and refined by Borwein [3] who replaced closedness by convergent series closedness. (See also other developments in [4] by Borwein.)

Those open relation theorems specialize in another classical fact: if f is a lower semicontinuous convex function, Y is barreled and $\operatorname{int}(\operatorname{dom} f) \neq \varnothing$, then f is continuous throughout the algebraic interior of its domain (e.g. [6]). It is well known that this result may be reduced to that evoked at the very beginning by showing that $\operatorname{int}(\operatorname{dom} f) \neq \varnothing$ implies (1.1) provided that Y is a Baire space.

The relationship of our open relation theorem to the existing ones is analogous. Proofs of older theorems hinge on the fact that, for closed convex relations, surjectivity when conjugated with the Baire property implies a bounded-

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ness property. Our open relation theorem consists in the implication of openness by boundedness. But what seems to be entirely novel is that boundedness actually entails (graph-) closedness.

We divide the two mentioned facts into portions that will appear as consequences of the results of §3.

Recall that y belongs to the algebraic interior $\operatorname{int}_\alpha A$ of A if, for every $k \in Y$, there exists $t_0 > 0$ such that $y + tk \in A$ as $0 \le t \le t_0$.

1.1. **Corollary.** *If $f\colon Y \to \overline{\mathbf{R}}$ is convex and there exists an open set Q_0 such that* (1.1) *holds, then, for every $y \in \operatorname{int}_\alpha(\operatorname{dom} f)$, there is a neighborhood Q of y such that $\sup_Q f < +\infty$.*

1.2. **Corollary.** *If $f\colon Y \to \overline{\mathbf{R}}$ is convex and there exist a neighborhood D of the origin and $c_0 > 0$ such that $\sup_{y_0+D_0} f \le f(y_0)+c_0$, then there is a neighborhood Q of y_0, a neighborhood D of the origin and $c > 0$ such that, for each $y \in Q$ and every $0 \le t \le 1$,*

$$\sup_{y+tD} f \le f(y) + tc. \tag{1.2}$$

1.3. **Corollary.** *If there exists an open set Q_0 such that* (1.1) *holds, then f is upper semicontinuous throughout $\operatorname{int}_\alpha(\operatorname{dom} f)$.*

1.4. **Corollary.** *If there exists an open set Q_0 such that* (1.1) *holds, then f is lower semicontinuous throughout $\operatorname{int}_\alpha(\operatorname{dom} f)$.*

1.5. **Corollary.** *If f is lower semicontinuous and Y is barreled, then f is continuous on $\operatorname{int}(\operatorname{dom} f)$.*

2. Link between functions and relations

The link with the general theorems that will follow is through the epigraph of f, i.e. $\operatorname{epi} f = \{y, r) \in Y \times \mathbf{R} : f(y) \le r\}$. Of course, $\operatorname{epi} f$ is a relation between Y and $\mathbf{R}$, its inverse relation being the lower level relation $(\operatorname{epi} f)^- r = \{y\colon f(y) \le r\}$.

Let Y, Z be topological spaces. A relation $\Omega \subset Y \times Z$ is called *lower semicontinuous at* y_0 if, for each open subset Q of z such that $\Omega y_0 \cap Q \ne \varnothing$, there exists a neighborhood P of y_0 for which $P \subset \Omega^- Q$. By definition, Ω^- is *open at* y_0 whenever Ω is lower semicontinuous at y_0. It is said to be *closed at* y_0 if, for every $z \notin \Omega y_0$, there exist neighborhoods P of y_0 and Q of z such that $Q \cap \Omega P = \varnothing$. Recall that

(2.A) A function f is upper semicontinuous at y_0 if and only if $\operatorname{epi} f$ is lower semicontinuous at y_0.

(2.B) A function f is lower semicontinuous at y_0 if and only if $\operatorname{epi} f$ is closed at y_0.

Now let Y be a topological vector space

(2.C) A function f fulfills (1.2) if and only if there exist a ball B in $\mathbf{R}$ and a neighborhood D of the origin in Y such that

$$y + tD \subset (\operatorname{epi} f)^{-}(f(y) + tB).$$

(2.D) A function f fulfills (1.1) if and only if there exists a bounded subset H_0 of $\mathbf{R}$ such that $\operatorname{int}(\operatorname{epi} f)^{-} H_0 \neq \varnothing$.

Finally, recall that

(2.E) $$\operatorname{dom} f = (\operatorname{epi} f)^{-}\mathbf{R}.$$

3. Open and closed relation theorems

Let X, Y be topological vector spaces and let Δ be a convex subset of $X \times Y$.

3.1. **Proposition.** *Suppose that there is a bounded subset H_0 of X such that $\operatorname{int} \Delta H_0 \neq \varnothing$. Then, for every $y \in \operatorname{int}_\alpha \Delta X$, there exists a bounded set H such that $y \in \operatorname{int} \Delta H$.*

Proof. Let $y_0 \in Y$ and D be a neighborhood of the origin of Y such that $y_0 + D \subset \Delta H$. Since $y \in \operatorname{int}_\alpha \Delta X$, there exists $y_1 \in \Delta X$ and $0 \leq \alpha < 1$ such that $y = (1-\alpha)y_0 + \alpha y_1$. Let $x_1 \in \Delta^{-} y_1$ and $H = (1-\alpha)H_0 + \alpha x_1$. Then, by the convexity of Δ,

$$\begin{aligned} y + (1-\alpha)D &= (1-\alpha)(y_0 + D) + \alpha y_1 \\ &\subset (1-\alpha)\Delta H_0 + \alpha \Delta x_1 \\ &\subset \Delta[(1-\alpha)H_0 + \alpha x_1] = \Delta H, \end{aligned}$$

what completes the proof because H is bounded. □

3.2. **Theorem** [3]. *Let $y_0 \in \Delta x_0$ and let V and W be balanced neighborhoods of 0 in Y and X, respectively, for which*

(3.1) $$y_0 + W \subset \Delta(x_0 + V),$$

then, for each $\varepsilon \geq 0$ and balanced neighborhoods B and D of 0 in X and Y, respectively, such that $V + \varepsilon V \subset B$ and $D + \varepsilon D \subset W$, the formula

(3.2) $$y + tD \subset \Delta(x + tB)$$

holds for every $(x, y) \in (x_0 + \varepsilon V) \times (y_0 + \varepsilon D) \cap \Delta$ and $0 \leq t \leq 1$.

Proof. Choose $\varepsilon \geq 0$. Let D and B satisfy the assumptions. Then, for $x \in x_0 + \varepsilon V$ and $y \in y_0 + \varepsilon D$, we have (in view of (3.1))

(3.3) $$y + D \subset y_0 + \varepsilon D + D \subset y_0 + W \subset \Delta(x_0 + V) \subset \Delta(x + \varepsilon V + V) \subset \Delta(x + B).$$

Let $0 < t < 1$. Since $y \in \Delta x$, by (3.1) and by the convexity of Δ, we have

$$\begin{aligned} y + tD &= (1-t)y + t(y + D) \\ &\subset (1-t)\Delta x + t\Delta(x + B) \\ &\subset \Delta[(1-t)x + t(x + B)] = \Delta(x + tB). \qquad \square \end{aligned}$$

3.3. **Theorem.** *If there exists a bounded subset* H_0 *of* X *such that* $\operatorname{int}\Delta H_0 \neq \varnothing$, *then* Δ^- *is lower semicontinuous on* $\operatorname{int}_\alpha \Delta X$.

Proof. Let $y_0 \in \operatorname{int}_\alpha \Delta X$. By virtue of Proposition 3.1, there exists a bounded set H and a balanced neighborhood W of 0 such that $y_0 + W \subset \Delta H$. Let $x_0 \in \Delta^- y_0$ and V be an arbitrary balanced neighborhood of the origin of X. Then there exists $\lambda \geq 1$ such that $x_0 + \lambda V \supset H$, that is, $y_0 + W \subset \Delta(x_0 + \lambda V)$. In view of Theorem 3.2 (applied in the special case of $\varepsilon = 0$), we get that $y_0 + (1/\lambda)W \subset \Delta(x_0 + V)$, that is Δ^- is lower semicontinuous at (y_0, x_0). □

3.4. **Theorem.** *If* $\Delta^- y_0$ *is closed and there is a bounded subset* H_0 *of* X *such that* $y_0 \in \operatorname{int}\Delta H_0$, *then* Δ^- *is closed at* y_0.

Proof. Let $x \notin \Delta^- y_0$. Then we may find a balanced neighborhood B of the origin in X such that

$$(x + 3B) \cap \Delta^- y_0 = \varnothing. \tag{3.4}$$

By the convexity of Δ, for each $k \in Y$ and every $0 \leq t \leq 1$,

$$t\Delta^-(y_0 + k) + \Delta^-(y_0 - tk) \subset (1 + t)\Delta^- y_0. \tag{3.5}$$

Let D be a balanced neighborhood of the origin in Y such that $y_0 + D \subset \Delta H_0$, that is, for each $k \in D$, $\Delta^-(y_0 + k) \cap H_0 \neq \varnothing$. Since H_0 is bounded, there exists such a t, $0 < t < 1$, that $tH_0 \subset B$ and besides $tx \in B$, so that

$$x + 2B \subset (1 + t)(x + 3B). \tag{3.6}$$

As a consequence of the former inclusion, $0 \in t\Delta^-(y_0 + k) + B$ and, in view of (3.5), for each $k \in D$,

$$\Delta^-(y_0 - tk) \subset (1 + t)\Delta^- y_0 + B. \tag{3.7}$$

By (3.4) and (3.6), $(1 + t)\Delta^- y_0 + B$ is disjoint from $x + B$ and thus, by (3.7), $\Delta^-(y_0 - tk) \cap (x + B) = \varnothing$ for each $k \in D$, proving the closedness of Δ^- at y_0. □

The boundedness assumption used in Theorems 3.2, 3.3, and 3.4 is a consequence of surjectivity in the case when one can apply a Baire argument. We quote

3.5. **Theorem** [3]. *If* X *is a Fréchet space* (*i.e. completely metrizable and locally convex*), Y *is barreled and* Δ *is closed, then* Δ^- *is lower semicontinuous throughout* $\operatorname{int}_\alpha \Delta X$.

3.6. *Remark.* The proof in [3] uses the Baire argument and the completeness of X to establish (3.1) for each $y_0 \in \operatorname{int}_\alpha \Delta X$ and $x_0 \in \Delta^- y_0$.

Actually, Borwein proves Theorem 3.5 under the assumption that Δ is *convergent-series closed*, i.e. if for $n \geq 1$, $z_n \in \Delta$, $\lambda_n \geq 0$, $\sum_{n=1}^\infty \lambda_n = 1$ and there exists $z = \sum_{n=1}^\infty \lambda_n z_n$ then $z \in \Delta$ (which is weaker than that of being convex and closed). □

In view of Theorems 3.4, 3.5 and the fact that the ball is bounded in a normed space we get

3.7. **Theorem.** *Let X be a Banach space, Y a barreled space and Δ convergent-series closed and such that $\Delta^- y$ is closed for each $y \in Y$. Then Δ^- is closed throughout* $\operatorname{int}_\alpha \Delta X$.

4. An application

We shall see how our Theorem 3.4 may be applied to generalize the closed image lemma and to simplify the proof of necessary optimality conditions with weakened assumptions of Alexeev, Tikhomirov and Fomin [1].

4.1. **Lemma** [1]. *Let X, Y, Z be Banach spaces, $A: X \to Y$, $B: X \to Z$ linear continuous mappings such that AX is closed and BA^-0 is closed. Then $(A \times B)X$ is closed.*

Note that, thanks to the linearity of A and B, BA^-0 is closed if and only if BA^-y is closed for every $y \in Y$.

Here is the announced generalization: Let X, Y be Banach spaces and Z a normed space. Let $A: X \rightrightarrows Y$ and $B: X \rightrightarrows Z$.

4.2. **Theorem.** *If A is a linear and closed mapping such that AX is closed and B is a convex lower semicontinuous relation such that, for each $y \in Y$, BA^-y is closed, then $(A \times B)X$ is closed.*

Proof. One checks that $(A \times B)X = BA^- \subset Y \times Z$. As AX is a closed subspace of a Banach space, in view of Theorem 3.5, $A^-: AX \rightrightarrows X$ is lower semicontinuous on AX. Consequently $BA^-: AX \rightrightarrows Z$ is a lower semicontinuous closed-valued relation. Since Z is normed there is a bounded open set H_0 such that $AB^-H_0 = (BA^-)^-H_0$ is open in AX. By virtue of Theorem 3.4, BA^- is closed. □

Consider the problem

$$\begin{aligned} &\min f_0(x), \\ &f_i(x) \le 0, \quad i = 1, 2, \dots, n \\ &F(x) = 0, \end{aligned} \tag{4.1}$$

where f_i, $i = 0, 1, \dots, n$ are continuously differentiable functions on a Banach space X and F a continuously differentiable map from X to a Banach space Y.

An application of Theorem 4.2 enables us to provide a direct short proof of the following theorem without using the reduction procedure of [1].

4.3. **Theorem** [1]. *If x_0 is a solution of* (4.1) *and $F'(x_0)X$ is closed, then there exist $\lambda_0, \lambda_1, \dots, \lambda_n \ge 0$ and $\xi \in Y'$, not all null, such that*

$$\sum \lambda_i f_i'(x_0) + F'(x_0)^* \xi = 0 \tag{4.2}$$

and such that $\lambda_i f_i(x_0) = 0$ for $i = 1, 2, \dots, n$.

Here $F'(x_0)^*$ denotes the adjoint operator of $F'(x_0)$.

Proof. By a standard procedure one gets the slackness condition: $\lambda_i f_i(x_0) = 0$. Define

$$A = F'(x_0) \quad \text{and} \quad Bh = (f'_0(x_0)h, f'_1(x_0)h, \dots, f_n'(x_0)h) + \mathbf{R}_+^{n+1}.$$

If x_0 is a solution of (4.1), then by [5, Proposition 3.3] and [8, Theorem 1], 0 is a boundary point of $(A \times B)X$, which is a closed convex cone (by virtue of Theorem 4.2) included in $AX \times \mathbf{R}^{n+1}$ and whose linear part has finite codimension in $AX^- \times \mathbf{R}^{n+1}$. By the Hahn–Banach theorem, there exist $\lambda_0, \lambda_1, \dots, \lambda_n$ and $\xi \in Y'$ not simultaneously null such that

$$0 = \sup\left\{ -\sum_{i=0}^{n} \lambda_i r_i + \langle y, \xi \rangle : (y, r_0, \dots, r_n) \in (A \times B)X \right\}$$

which amounts to (4.2) and to $\lambda_0, \lambda_1, \dots, \lambda_n \geq 0$. □

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Added in proof. Professor J.-P. Penot was kind to draw my attention to a related work: B. Ricceri, *Remarks on multifunctions with convex graph*, Arch. Math. **52** (1989), 519–520, where graph-closedness is derived from lower semicontinuity with closed values.

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