

PERSISTENCE DEFINITIONS AND THEIR CONNECTIONS

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ABSTRACT. We give various definitions of types of persistence of a dynamical system and establish a hierarchy among them by proving implications and demonstrating counterexamples. Under appropriate conditions, we show that several of the definitions are equivalent.

1. INTRODUCTION

The main purpose of this paper is to consider various forms of persistence (defined in the next section) in dynamical systems and to establish a hierarchy among them.

The work in this paper may be thought of as a continuation of work done in [2] (also see [3]). In [2], the concepts of weak persistence, persistence and uniform persistence were defined for the first time for dynamical systems in a locally compact metric space with respect to sets with boundary and nonempty interior. From the definitions it was clear that uniform persistence implies persistence, which in turn implies weak persistence. It was then shown that, under certain circumstances, weak persistence implies uniform persistence.

In this paper, we define two additional concepts of persistence, one which is weaker than weak persistence, denoted W -weak persistence, and has been used in applications, and the other which is a uniform version of weak persistence, denoted weak uniform persistence. Among other results, we prove that if the flow is dissipative or if the boundary is compact, then weak uniform persistence is equivalent to uniform persistence.

Previous work in persistence theory has typically dealt with the abstract theory, with applications, or with both. Criteria for the equivalence of various forms of persistence were considered in [2, 3]. Criteria for one or more forms

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of persistence to hold in general dynamical systems were given in [3, 4, 7, 10, 11, 13, 18, 19], in Lotka-Volterra systems in [10, 11, 13, 14], in infinite dimensional systems in [1, 9]. Persistence theory as applied to population survival or extinction was discussed in [5, 6, 12, 14, 15, 18]. There are also many papers dealing with persistence theory exclusively for discrete dynamical and semi-dynamical systems which are cited in the above references.

In a theoretical sense persistence definitions can be interpreted as a complete instability or as boundedness-like qualitative concepts (see [17, Chapter VI]). Our theorem about the equivalence of weak uniform persistence to uniform persistence is a modification of a theorem by V. A. Pliss [16] stating the equivalence of weak ultimate boundedness and ultimate boundedness for the case of ordinary differential equations with periodic coefficients.

The paper is organized as follows. In §2 we formulate the main definitions and notations. Section 3 is devoted to the connection between weak uniform persistence and uniform persistence. In §4 we present some counterexamples proving the independence of these persistence definitions. Finally in §5 we consider the special cases of autonomous and periodic Kolmogorov systems in R^n .

2. DEFINITIONS AND NOTATIONS

Here we briefly recall the definitions and notation introduced in [2], and define some new concepts as well.

Let ε be a locally compact metric space, and $E = \bar{E}$ (the closure of E) $\subset \varepsilon$. We consider the continuous flow $\mathcal{F} = (E, R, \pi)$, where $\pi: E \times R \rightarrow E$, $\pi(\pi(x, t), s) = \pi(x, t + s)$, $\forall x \in E$, $\forall s, t \in R$. To avoid trivialities assume that neither $\overset{\circ}{E}$ (the interior of E) nor ∂E (the boundary of E) is empty. We call the flow \mathcal{F} dissipative if $\forall x \in E$ the ω -limit set $\Lambda^+(x) \neq \emptyset$ and $\Omega(\mathcal{F}) = \bigcup_{x \in E} \Lambda^+(x)$ has compact closure. We shall say that

(i) \mathcal{F} is *weakly persistent* (WP) if $\forall x \in \overset{\circ}{E}$,

$$\limsup_{t \rightarrow +\infty} d(\pi(x, t), \partial E) > 0,$$

(ii) \mathcal{F} is *(strongly) persistent* (P) if $\forall x \in \overset{\circ}{E}$,

$$\liminf_{t \rightarrow +\infty} d(\pi(x, t), \partial E) > 0,$$

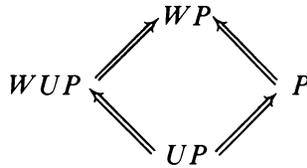
(iii) \mathcal{F} is *weakly uniformly persistent* (WUP) if $\exists \varepsilon_0 > 0 \ni \forall x \in \overset{\circ}{E}$,

$$\limsup_{t \rightarrow +\infty} (d(\pi(x, t)), \partial E) > \varepsilon_0.$$

(iv) \mathcal{F} is *uniformly persistent* (UP) if $\exists \varepsilon_0 > 0 \ni \forall x \in \overset{\circ}{E}$,

$$\liminf_{t \rightarrow +\infty} (d(\pi(x, t)), \partial E) > \varepsilon_0.$$

The following obvious relations are valid for these definitions:



A nonempty invariant set for a flow which is the maximal invariant set in some neighbourhood of itself is called an *isolated invariant set*. Let $\partial\mathcal{F}$ be the restriction of \mathcal{F} to ∂E (assumed invariant under \mathcal{F}). The flow $\partial\mathcal{F}$ is *isolated* if there exists a finite covering $\mathcal{M} = \{M_i\}$ of $\Omega(\partial\mathcal{F})$ by pairwise disjoint compact isolated invariant sets M_1, \dots, M_k for $\partial\mathcal{F}$ such that each M_i is also isolated for \mathcal{F} . Let M, N be isolated invariant sets for \mathcal{F} . We shall say that M is *chained* to N ($M \rightarrow N$) if there exists $x \notin M \cup N$ such that the α and ω -limit sets of x belong correspondingly to M and N . A chain of isolated invariant sets $M_1 \rightarrow M_2 \rightarrow \dots \rightarrow M_k$ is a *cycle* if $M_1 = M_k$. Finally $\partial\mathcal{F}$ is called *acyclic* if, for some isolated covering \mathcal{M} of $\Omega(\partial\mathcal{F})$, no subset of $\mathcal{M} = \{M_i\}$ forms a cycle.

3. CRITERIA FOR THE EQUIVALENCE OF WUP AND UP

In the previous section, we have introduced a new concept, namely weak uniform persistence. As mentioned there, $UP \Rightarrow WUP$. Here we state two theorems showing when $WUP \Rightarrow UP$.

Theorem 1. *Let \mathcal{F} be a continuous, dissipative flow on E with metric d . Let $\partial\mathcal{F}$ be the restriction of \mathcal{F} to ∂E (assumed invariant under \mathcal{F}). Then $WUP \Rightarrow UP$.*

Proof. Suppose not. Then there exist sequences $\{\tilde{x}_k\}_{k=1}^\infty, \{e_k\}_{k=1}^\infty, \tilde{x}_k \in \overset{\circ}{E}, 0 < e_k \in R \ni \lim_{k \rightarrow +\infty} e_k = 0$, and such that

$$(1) \quad \liminf_{t \rightarrow +\infty} d(\pi(\tilde{x}_k, t), \partial E) < e_k.$$

One can assume $e_k < \varepsilon_0 \ \forall k$, where ε_0 is from the definition of WUP .

From WUP it follows that there exists a sequence $\{\tau_k\}_{k=1}^\infty, \tau_k > 0$ such that

$$(2) \quad d(\pi(\tilde{x}_k, \tau_k), \partial E) > \varepsilon_0.$$

As $\overline{\Omega(\mathcal{F})}$, the closure of the union of all w -limit sets, is compact and E is locally compact, we can find an open set G such that $\overline{\Omega(\mathcal{F})} \subset G$ and \overline{G} is compact. Assume that τ_k is chosen large enough in the sense that if $t \geq \tau_k$, then $\pi(\tilde{x}_k, t) \in \overline{G}$.

From (1) it can be seen that there exists a sequence $\{\tilde{t}_k\}_{k=1}^\infty, \tau_k < \tilde{t}_k$ and that

$$(3) \quad d(\pi(\tilde{x}_k, \tilde{t}_k), \partial E) < e_k.$$

Since \mathcal{F} is continuous there exists a sequence $\{\tilde{\theta}_k\}$ such that $\tilde{\theta}_k \in [\tau_k, \tilde{t}_k]$ and

$$(4) \quad d(\pi(\tilde{x}_k, \tilde{\theta}_k), \partial E) = \varepsilon_0,$$

and, for $\tilde{\theta}_k < t < \tilde{t}_k$,

$$(5) \quad d(\pi(\tilde{x}_k, t), \partial E) < \varepsilon_0.$$

Let $x_k = \pi(\tilde{x}_k, \tilde{\theta}_k)$. From the choice of τ_k and $\tilde{\theta}_k$ it is clear that $x_k \in \bar{G}$, which is a compact set, so we can assume that $\{x_k\}_{k=1}^\infty$ is a convergent sequence (otherwise we would choose a convergent subsequence). Let $\lim_{k \rightarrow +\infty} x_k = x' \in \bar{G}$. It follows from (4) that $d(x', \partial E) = \varepsilon_0$, so $x' \in \overset{\circ}{E}$.

Let $t_k = \tilde{t}_k - \tilde{\theta}_k$. From the group property of π and (3), (5), it follows that

$$(6) \quad d(\pi(x_k, t_k), \partial E) < e_k (< \varepsilon_0)$$

and, if $0 < t \leq t_k$,

$$(7) \quad d(\pi(x_k, t), \partial E) < \varepsilon_0.$$

Now consider the orbit with initial point x' . It follows from *WUP* that there exists $\tau' > 0$ such that

$$(8) \quad d(\pi(x', \tau'), \partial E) > \varepsilon_0.$$

Since $\overset{\circ}{E}$ is invariant under \mathcal{F} we can find $0 < m < \varepsilon_0$ such that

$$(9) \quad d(\pi(x', t), \partial E) > m, \quad 0 \leq t \leq \tau'.$$

It follows from the continuity of π , $\lim_{k \rightarrow +\infty} x_k = x'$, and (8), (9) that if k is sufficiently large then

$$(10) \quad d(\pi(x_k, \tau'), \partial E) > \varepsilon_0$$

and

$$(11) \quad d(\pi(x_k, t), \partial E) > m (> e_k), \quad 0 \leq t \leq \tau'.$$

Inequalities (10), (11) contradict (6), (7). In fact if $\tau' < t_k$, then (10) contradicts (7), if $\tau' \geq t_k$ then (11) contradicts (6). These contradictions prove the theorem! \square

Theorem 2. *In Theorem 1, if the dissipativity hypothesis is replaced by a hypothesis that ∂E is compact, then again $WUP \Rightarrow UP$.*

Proof. The existence of a convergent sequence $\{x_k\} \subset \bar{H} = \{x | x \in E, d(x, \partial E) \leq \varepsilon_0\}$ follows from the compactness of ∂E . Using H instead of G , the proof follows as in Theorem 1. \square

Corollary 3. *If there exists $\varepsilon_0 > 0 \ni$ for all $x \in \overset{\circ}{E}, \exists \tau(x) > 0 \ni d(\pi(x, \tau(x)), \partial E) > \varepsilon_0$ then \mathcal{F} is weakly uniformly persistent with constant ε_0 .*

Proof. Suppose not. Then there is an $\tilde{x} \in \overset{\circ}{E} \ni \limsup_{t \rightarrow +\infty} d(\pi(\tilde{x}, t), \partial E) = \varepsilon_1 < \varepsilon_0$. But then, for all sufficiently large $t \geq T, d(\pi(\tilde{x}, t), \partial E) < \varepsilon_0$. Let

$x = x(\tilde{x}, T)$. From the group property of π and the previous inequality it follows that

$$d(\pi(x, t), \partial E) < \varepsilon_0 \quad \forall t \geq 0,$$

which contradicts the assumption of the statement.

Remark 4. Without additional assumptions, *WUP* is not equivalent to *UP*. To show this, let $E = R_+^2$, and \mathcal{F} the flow generated by the simplest Lotka-Volterra system $\dot{x} = x(a - by)$, $\dot{y} = y(-c + dx)$, $a, b, c, d > 0$. All interior orbits are closed around the fixed point $(\frac{a}{b}, \frac{c}{d})$, and, for any ε_0 , $0 < \varepsilon_0 < \min(\frac{a}{b}, \frac{c}{d})$, \mathcal{F} is *WUP*, but not *UP*.

4. EQUIVALENCES AND NONEQUIVALENCES OF PERSISTENCE DEFINITIONS

As already noted the following implications hold: $UP \Rightarrow WUP \Rightarrow WP$ and $UP \Rightarrow P \Rightarrow WP$. In the previous section we have shown that if the flow is dissipative, then $WUP \Rightarrow UP$, i.e., *UP* and *WUP* are equivalent. In this case one has $WUP = UP \Rightarrow P \Rightarrow WP$. Here we show that, even with dissipativity, $P \not\Rightarrow UP$ and $WP \not\Rightarrow P$. However, if certain additional assumptions (acyclicity, isolatedness of $\partial\mathcal{F}$) are made, then we note (see [2]) that $WP \Rightarrow UP$ so that all definitions are equivalent.

Example 5 ($WP \not\Rightarrow P$). A counterexample demonstrating the trajectories of a dissipative, weakly persistent, but not persistent, system can be seen in Figure 1. The detailed description of the system is described below (see Figure 1).

Let $E = R_+^2$, \mathcal{F} be the flow generated by the system

$$(12) \quad \dot{x}_i = x_i f_i(x_1, x_2), \quad i = 1, 2.$$

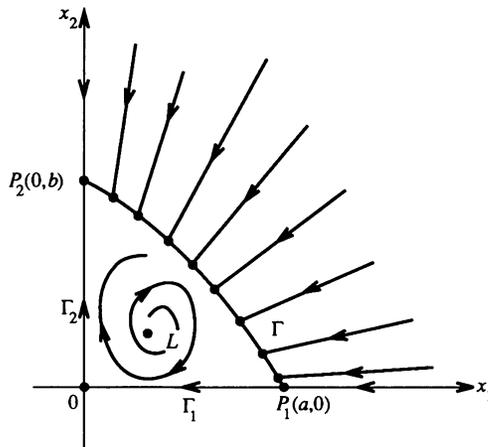


FIGURE 1.

To obtain the required $f_i(x_1, x_2)$, we first consider the system

$$(13) \quad \dot{x}_i = x_i g_i(x_1, x_2),$$

which has the following properties:

- (i) There are four equilibria in \mathbf{R}_+^2 , $O(0, 0)$, $P_1(a, 0)$, $P_2(0, b)$, $L(l_1, l_2)$, $a, b, l_1, l_2 > 0$.
- (ii) O, P_1, P_2 are saddle points connected by separatrices as follows: Γ_1 denotes the orbit along the x_1 axis from O to P_1 , Γ_2 denotes the orbit along the x_2 axis from P_2 to O , Γ denotes the orbit lying in $\overset{\circ}{\mathbf{R}}_+^2$ from P_1 to P_2 . We assume that the curve Γ can be parametrized by the relation $x_2 = \varphi(x_1)$, $0 < x_1 < a$. We then smoothly extend φ on \mathbf{R}^+ so that $\varphi(x_1) < 0$ for $x_1 > a$. We denote the open region bounded by $\bar{\Gamma}_1, \bar{\Gamma}_2$ and $\bar{\Gamma}$ as D , i.e.,

$$D = \{(x_1, x_2) | 0 < x_1 < a, 0 < x_2 < \varphi(x_1)\}.$$

- (iii) L is an unstable focus lying in D .
- (iv) All solutions with initial values lying in $D \setminus \{L\}$ have L as their α -limit sets and $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}$ as their ω -limit sets.

Note that such a system can be constructed (see [8, pp. 405–409]).

We now define

$$f_i(x_1, x_2) = \begin{cases} (x_2 - \varphi(x_1))^2 g_i(x_1, x_2), & (x_1, x_2) \in \bar{D} \\ -(x_2 - \varphi(x_1))^2, & (x_1, x_2) \in \mathbf{R}_+^2 \setminus \bar{D}. \end{cases}$$

Then system (12) has the properties that $x_2 = \varphi(x_1)$ is a curve of equilibria and solutions initiating in $\mathbf{R}_+^2 \setminus \bar{D}$ approach $\bar{\Gamma}$. Hence (12) is dissipative (in fact trajectories initiating in $\overset{\circ}{\mathbf{R}}_+^2 \setminus D$ approach Γ). Trajectories initiating in D have the same properties as described for system (13), and hence system (12) weakly persists, but does not persist.

Example 6 ($P \not\Rightarrow UP$). If, in the previous example, one replaces the unstable focus L by a center and all orbits in the interior of $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}$ are closed, then a persistent but not uniformly persistent dissipative system is obtained.

Remark 7. If dissipativity does not hold, then clearly all four definitions are independent. However, if ∂E is compact, then, under the same assumptions, i.e., the flow $\partial \mathcal{F}$ is isolated and acyclic (except for dissipativity) as in [2], $WP \Leftrightarrow P \Leftrightarrow UP$ holds. The proof of this is analagous to the proof in [2].

5. KOLMOGOROV SYSTEMS

In this section we consider autonomous Kolmogorov systems of the form

$$(14) \quad \dot{x}_i = x_i f_i(x_1, \dots, x_n), \quad i = 1, \dots, n,$$

as well as nonautonomous, periodic Kolmogorov systems

$$(15) \quad \dot{x}_i = x_i g_i(t, x_1, \dots, x_n) = x_i g_i(t + T, x_1, \dots, x_n), \quad i = 1, \dots, n, \quad T > 0.$$

For the flow described by (14), $E = \mathbf{R}_+^n$ and ∂E is invariant under this flow. Also, the definitions WP , P , WUP , UP may be written in coordinate form as follows for solutions with initial values in $\overset{\circ}{\mathbf{R}}_+^n$:

- (i) (14) is WP if $\limsup_{t \rightarrow \infty} \min_{1 \leq i \leq n} \{x_i(t)\} > 0$;
- (ii) (14) is P if $\liminf_{t \rightarrow \infty} x_i(t) > 0, \forall i = 1, \dots, n$;
- (iii) (14) is WUP if $\exists \varepsilon_0 > 0 \ni \limsup_{t \rightarrow \infty} \min\{x_i(t)\} > \varepsilon_0$;
- (iv) (14) is UP if $\exists \varepsilon_0 > 0 \ni \liminf_{t \rightarrow \infty} x_i(t) > \varepsilon_0 \forall i = 1, \dots, n$.

In addition to the above, one can define for system (14) a weaker form of weak persistence, WWP , as follows:

- (v) (14) is WWP if $\limsup_{t \rightarrow \infty} x_i(t) > 0 \forall i = 1, \dots, n$.

We note that in several papers in the literature (see e.g. [5]), what was defined as WP was actually WWP .

Clearly $WP \Rightarrow WWP$. The next example shows that the reverse is not true, even if (14) is dissipative.

Example 8. Consider the nontransitive competition model described in [15, 18],

$$(16) \quad \begin{aligned} \dot{x}_1 &= x_1(1 - x_1 - \alpha x_2 - \beta x_3) \\ \dot{x}_2 &= x_2(1 - \beta x_1 - x_2 - \alpha x_3) \\ \dot{x}_3 &= x_3(1 - \alpha x_1 - \beta x_2 - x_3), \end{aligned}$$

where $\alpha + \beta > 2, \alpha < 1$. There are five equilibria, $O(0, 0, 0), P_1(1, 0, 0), P_2(0, 1, 0), P_3(0, 0, 1), L(l, l, l)$, where $l = (1 + \alpha + \beta)^{-1}$. $P_i, i = 1, 2, 3$ are hyperbolic saddle points such that P_2 and P_1 are connected by a separatrix orbit Γ_{21} in the $x_1 - x_2$ plane and directed from P_2 to P_1 . Γ_{21} can be parametrized by a relation of the form $x_2 = \varphi_{21}(x_1), x_3 = 0, 0 < x_1 < 1$. Here we extend φ_{21} in a smooth manner in \mathbf{R}^+ so that $\varphi_{21}(x_1) < 0$ for $x_1 > 1$. Similarly there exists Γ_{13} defining $x_1 = \varphi_{13}(x_3), x_2 = 0$, and Γ_{32} defining $x_3 = \varphi_{32}(x_2), x_1 = 0$. As was shown in [15], every orbit with positive initial conditions (except for two singular orbits which tend to L and L itself) has $\Gamma = \bar{\Gamma}_{21} \cup \bar{\Gamma}_{13} \cup \bar{\Gamma}_{32}$ as its ω -limit set. Hence this is an example of a system which exhibits WWP , but not WP .

We now define a new system by multiplying the right hand sides of (16) by

$$[x_3^2 + (x_2 - \varphi_{21}(x_1))^2][x_2^2 + (x_1 - \varphi_{13}(x_3))^2][x_1^2 + (x_3 - \varphi_{32}(x_2))^2].$$

For this new system all points lying on Γ are equilibria. All other orbits together with their orientations are the same as for orbits of (16). We note that for this example, the system remains WWP , but not WP , and all the conditions of the main theorem of [2], i.e., dissipativity, isolatedness, acyclicity except WP (replaced by WWP) are satisfied. The points of Γ were fixed to obtain the acyclic condition. So, from WWP under these conditions, it does not follow that $UP \Leftrightarrow P \Leftrightarrow WUP \Leftrightarrow WP$.

Finally, we consider system (15). This system may be transformed into an autonomous system (but not of Kolmogorov type) by introducing the coordinate

$\theta = t$. Then we have the transformed system

$$(17) \quad \begin{aligned} x'_i &= x_i f_i(\theta, x_1, \dots, x_n), & i = 1, \dots, n \\ \theta' &= 1, \end{aligned}$$

which defines a flow on the cylindrical phase space $E = S^1 \times \mathbf{R}_+^n$, where points $(\theta, x_1, \dots, x_n)$ and $(\theta+T, x_1, \dots, x_n)$ coincide. Here $\partial E = S^1 \times \partial \mathbf{R}_+^n$. The persistence definitions are the same as for system (14) (there is no condition for θ). We note that in this case dissipativity follows whenever there exists a $\rho > 0$ such that, for all solutions, $\limsup_{t \rightarrow \infty} \|x(t)\| < \rho$ (see [16]).

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