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AN ANALOGUE TO GLAUBERMAN'S ZJ-THEOREM

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ABSTRACT. Let P be a finite p-group, p an odd prime. Using certain versions of p-stability it is shown that there exists a nontrivial characteristic subgroup W in P that is normal in every finite p-stable group G satisfying $C_G(O_p(G)) \le O_p(G)$ and $P \in \mathrm{Syl}_p(G)$. Moreover, W contains every abelian subgroup of P normalized by W.

Let p be an odd prime and $H \neq 1$ a finite p-stable group (for a definition of p-stability see [3]) with $C_H(0_p(H)) \le 0_p(H)$. In [1] Glauberman has proven:

ZJ-Theorem. *Suppose that $S \in \mathrm{Syl}_p(H)$, then $Z(J(S))$ is normal in H.*

His theorem shows that there exists a fixed nontrivial characteristic subgroup of S, namely $Z(J(S))$, which is normal in every finite group H containing S as a Sylow p-subgroup and satisfying the above hypotheses. In this paper we want to prove an analogue to his theorem that does not use $Z(J(S))$.

Let p be a prime and S be a finite p-group. An *embedding* of S is a pair (τ, H) where H is a group and τ is a monomorphism from S into H. Two embeddings (τ_1, H_1) and (τ_2, H_2) are *equivalent*, if there exists an isomorphism φ from H_1 onto H_2 so that $\tau_1\varphi = \tau_2$. It is easy to see that this defines an equivalence relation on the class of all embeddings of S.

In the following two lemmas let $\mathscr{E}$ be a set of embeddings of S. By $W_{\mathscr{E}}(S)$ we denote the largest subgroup of S so that $W_{\mathscr{E}}(S)\tau$ is normal in H for every $(\tau, H) \in \mathscr{E}$.

Lemma 1. *Let $\alpha \in \mathrm{Aut}(S)$. Suppose that $(\alpha\tau, H)$ is equivalent to an element in $\mathscr{E}$ for every $(\tau, H) \in \mathscr{E}$. Then $W_{\mathscr{E}}(S)$ is α-invariant.*

Proof. Let $(\alpha\tau, H)$ be equivalent to $(\tilde{\tau}, \tilde{H}) \in \mathscr{E}$, i.e., there exists an isomorphism $\varphi: \tilde{H} \to H$ so that $\alpha\tau = \tilde{\tau}\varphi$. It follows that $W_{\mathscr{E}}(S)\alpha\tau = W_{\mathscr{E}}(S)\tilde{\tau}\varphi$, and so $(W_{\mathscr{E}}(S)\alpha)\tau$ is normal in H. Since this holds for every $(\tau, H) \in \mathscr{E}$ we get $W_{\mathscr{E}}(S)\alpha = W_{\mathscr{E}}(S)$.

Lemma 2. *Let $(\tau, H) \in \mathscr{E}$ and $\beta \in \mathrm{Aut}(H)$. Then (τ, H) is equivalent to $(\tau\beta, H)$.*

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The proof is obvious.

In this paper the following two versions of p-stability will be used.

Definition. Let X be a finite group and p an odd prime so that $C_X(O_p(X)) \leq O_p(X)$. Set $Z_0(N) = Z(O^p(O^{p'}(N))$ for every normal subgroup N of X.

Weak p-stability. X is weakly p-stable, if for every $x \in X$

(*) $[V, x, x] = 1$ implies $xC_X(V) \in O_p(X/C_X(V))$ for every abelian normal p-subgroup V of X and for $V = O_p(X)$.

Strong p-stability. X is strongly p-stable, if $N/Z_0(N)$ is weakly p-stable for every normal subgroup N of X.

An embedding (τ, H) of S is weakly (strongly) p-stable, if H is weakly (strongly) p-stable.

Lemma 3. *Let X be strongly p-stable and let N be a normal subgroup of X. Then N/Y is weakly p-stable for every normal subgroup Y of N in $Z_0(N)$.*

Proof. Set $\overline{N} = N/Z_0(N)$ and $\widetilde{N} = N/Y$, and let V be a normal p-subgroup of N so that $\overline{V}$ is abelian or $\overline{V} = O_p(\overline{N})$. Suppose that $[V, y, y] \leq Y$ for some p-element y in N. Since $\overline{N}$ is weakly p-stable we get that $[\overline{V}, O^p(\langle \overline{y}^{\overline{N}} \rangle)] = 1$ and thus also $[V, O^p(\langle y^N \rangle)] = 1$, i.e., $\widetilde{y} \in O_p(\widetilde{N}/C_{\widetilde{N}}(\widetilde{V}))$.

Remark. The p-stability defined in [3] implies weak p-stability (for groups X satisfying $C_X(O_p(X)) \leq O_p(X)$). On the other hand, there are examples of weakly p-stable groups that are not p-stable.

Lemma 4. *Let p be an odd prime and S be a p-group. Suppose that $\mathscr{E}$ is a set of weakly p-stable embeddings of S so that $S\tau \in \mathrm{Syl}_p(H)$ for $(\tau, H) \in \mathscr{E}$. Then $Z(S) \leq W_{\mathscr{E}}(S)$. In particular $W_{\mathscr{E}}(S) \neq 1$ if $S \neq 1$.*

Proof. Let $\mathscr{E} = \{(\tau_i, H_i) \mid i \in I\}$ and let G be the amalgamated product of H_i, $i \in I$, over S (for the definition see [4]). As usual we identify the groups H_i and S with the corresponding subgroups of G, i.e., $G = \langle H_i \mid i \in I \rangle$ and $S = \bigcap_{i \in I} H_i$. Then $W_{\mathscr{E}}(S)$ is the largest subgroup of S that is normal in G.

Let Γ be the coset graph of G with respect to the subgroups H_i, $i \in I$, and S, i.e., the vertices of Γ are the cosets $H_i x$, $i \in I$, and Sx for $x \in G$, and two vertices Ax and By are adjacent, if and only if $Ax \neq By$ and $Ax \subseteq By$ or $By \subseteq Ax$. Then G operates on Γ by right multiplication. We identify Γ with its vertex set and use the following notation for $\delta \in \Gamma$.

$d(\ ,\ )$: the usual distance metric on Γ,

$\Delta(\delta)$: the set of vertices adjacent to δ,

G_δ: the stabilizer of δ in G,

$Q_\delta = O_p(G_\delta)$,

$Z_\delta = \langle Z(T) \mid T \in \mathrm{Syl}_p(G_\delta) \rangle$.

(1) $O_p(G_\delta / C_{G_\delta}(Z_\delta)) = 1$.

This follows from the definition of Z_δ and the Frattini argument.

The next properties are immediate consequences of the definition of Γ (see also [2]).

(2) Γ is connected.

(3) G_δ is conjugate in G to S or some H_i, $i \in I$.

(4) $G_\delta \cap G_\lambda \in \mathrm{Syl}_p(G_\delta)$ for $\lambda \in \Delta(\delta)$.

(5) $Q_\delta = \bigcap_{\lambda \in \Delta(\delta)} (G_\delta \cap G_\lambda)$.

(6) $Z_\delta \leq Z(Q_\delta)$.

(7) G_δ is transitive on $\Delta(\delta)$, or G_δ is conjugate to S and $G_\delta = Q_\delta$.

Since $W_{\mathscr{G}}(S)$ is the largest normal subgroup of G contained in S we conclude from (5) that $W_{\mathscr{G}}(S)$ is the kernel of the operation of G on Γ.

We now assume that $Z(S) \nleq W_{\mathscr{G}}(S)$. Then (5) yields:

(8) There exist $\alpha, \alpha' \in \Gamma$ so that $Z_\alpha \nleq Q_{\alpha'}$.

Let α and α' be as in (8) so that $b := d(\alpha, \alpha')$ is minimal, i.e., $Z_\alpha \leq Q_\delta$ for every $\delta \in \Gamma$ with $d(\alpha, \delta) < b$; in particular $Z_\alpha \leq G_{\alpha'}$ by (5).

Since every normal p-subgroup of $G_{\alpha'}$ is in $Q_{\alpha'}$ we have that $\langle Z_\alpha^{G_{\alpha'}} \rangle$ is non-abelian. We get:

(9) There exist $\lambda, \lambda' \in \Gamma$ so that $[Z_\lambda, Z_{\lambda'}] \neq 1$.

Let λ and λ' be as in (9) so that $r := d(\lambda, \lambda')$ is minimal, i.e., $[Z_\lambda, Z_\delta] = 1$ for every $\delta \in \Gamma$ with $d(\lambda, \delta) < r$. Let

$$\Lambda_r = \{(\lambda, \lambda') \mid d(\lambda, \lambda') = r \text{ and } [Z_\lambda, Z_{\lambda'}] \neq 1\}.$$

(10) $b > r/2$.

Assume that $b \leq r/2$. Note that $[Q_{\alpha'}, Z_\alpha] \leq \langle Z_\alpha^{Q_{\alpha'}} \rangle$ and that by (5) $d(\alpha, \alpha^q) < r$ for $q \in Q_{\alpha'}$. It follows from the minimality of r that $\langle Z_\alpha^{Q_{\alpha'}} \rangle$ is abelian, i.e., $[Q_{\alpha'}, Z_\alpha, Z_\alpha] = 1$. Since $G_{\alpha'}$ is weakly p-stable by (3) and $C_{G_{\alpha'}}(Q_{\alpha'}) \leq Q_{\alpha'}$ we get $Z_\alpha \leq Q_{\alpha'}$, a contradiction.

Now let $(\lambda, \lambda') \in \Lambda_r$ and $(\lambda_0, \cdots, \lambda_r)$ be a path of length r with $\lambda_0 = \lambda$ and $\lambda_r = \lambda'$. Note that $Z_\lambda \leq G_{\lambda_i}$ for $i \leq b$.

(11) Let $b \leq s$ and $V = \langle Z_{\lambda'}^{G_{\lambda_s}} \rangle$. Suppose that $Z_\lambda \leq G_{\lambda_i}$ for $0 \leq i \leq s$. Then the following hold:

(a) V is an abelian normal subgroup of G_{λ_s} and $[V, Z_\lambda, Z_\lambda] = 1$.

(b) If $s \neq r$, then there exists $(\lambda, \lambda'') \in \Lambda_r$ and a path $(\widetilde{\lambda}_0, \cdots, \widetilde{\lambda}_r)$ of length r with $\widetilde{\lambda}_0 = \lambda$ and $\widetilde{\lambda}_r = \lambda''$ and $Z_\lambda \leq G_{\widetilde{\lambda}_i}$ for $0 \leq i \leq s+1$.

Note that $Z_{\lambda'}$ is abelian by (6) and $d(\lambda_s, \lambda') \leq r - b < r/2 < b$ by (10). Hence $Z_{\lambda'} \leq Q_{\lambda_s}$ and $d(\lambda', \lambda'^g) < r$ for $g \in G_{\lambda_s}$, and V is abelian.

Now let $W = \langle Z_\lambda^V \rangle$. Since V fixes λ_{r-b} we get $d(\lambda, \lambda^v) \leq 2(r-b) < r$ for $v \in V$, and W is abelian. This implies $[V, Z_\lambda, Z_\lambda] = 1$, and (a) holds.

Assume now that $s \neq r$. Let N be the inverse image of $O_p(G_{\lambda_s}/C_{G_{\lambda_s}}(V))$ in G_{λ_s}. Since G_{λ_s} is weakly p-stable by (3) we get from (a) that $Z_\lambda \leq N$.

Let $T = G_{\lambda_{s-1}} \cap N$ and $\widetilde{T} = G_{\lambda_{s+1}} \cap N$. By (4) T, $\widetilde{T} \in \mathrm{Syl}_p(N)$ and $Z_\lambda \leq T$. Moreover, we have $N = TC_N(V)$. Hence, there is $c \in C_N(V)$ so that $\widetilde{T}^c = T$, i.e., $Z_\lambda \leq G_{\lambda^c_{s+1}}$. Since $Z_{\lambda'} = Z^c_{\lambda'} = Z_{\lambda'^c}$ claim (b) follows for $\lambda'' = \lambda'^c$ and the path $(\lambda_0, \cdots, \lambda_s, \lambda^c_{s+1}, \cdots, \lambda^c_r)$.

We now derive a final contradiction. According to (11)(b) we can choose $(\lambda, \lambda') \in \Lambda_r$ and $(\lambda_0, \cdots, \lambda_r)$ so that $\lambda_0 = \lambda$, $\lambda_r = \lambda'$ and $Z_\lambda \leq G_{\lambda_i}$ for $i = 0, \cdots, r$. Now we get from (11)(a) for $s = r$ that $[Z_{\lambda'}, Z_\lambda, Z_\lambda] = 1$. Since $G_{\lambda'}$ is weakly p-stable we conclude

$$Z_\lambda C_{G_{\lambda'}}(Z_{\lambda'})/C_{G_{\lambda'}}(Z_{\lambda'}) \leq O_p(G_{\lambda'}/C_{G_{\lambda'}}(Z_{\lambda'})).$$

But now (1) implies $Z_\lambda \leq C_{G_{\lambda'}}(Z_{\lambda'})$, which contradicts the definition of Λ_r.

Theorem 1. *Let p be an odd prime and S a p-group. Then there exists a characteristic subgroup $W(S)$ of S so that $Z(S) \leq W(S)$, and $W(S)$ is normal in every weakly p-stable group H with $S \in \mathrm{Syl}_p(H)$.*

Proof. Let $\mathscr{E}$ be a maximal set of pairwise nonequivalent weakly p-stable embeddings of S so that $S\tau \in \mathrm{Syl}_p(H)$ for $(\tau, H) \in \mathscr{E}$. Note that $\mathscr{E}$ exists since by condition $C_H(O_p(H)) \leq O_p(H)$ there are only finitely many nonequivalent weakly p-stable embeddings $(\tilde{\tau}, \widetilde{H})$ of S with $S\tilde{\tau} \in \mathrm{Syl}_p(\widetilde{H})$. Moreover, every weakly p-stable embedding $(\tilde{\tau}, \widetilde{H})$ of S with $S\tilde{\tau} \in \mathrm{Syl}_p(\widetilde{H})$ is equivalent to some element in $\mathscr{E}$. Hence, by Lemma 1 $W_{\mathscr{E}}(S)$ is a characteristic subgroup of S and by Lemma 4 $Z(S) \leq W_{\mathscr{E}}(S)$. Thus, Theorem 1 holds for $W(S) := W_{\mathscr{E}}(S)$.

In the next theorem we denote by $W(S)$ the largest characteristic subgroup of S for which Theorem 1 holds.

Theorem 2. *Let p be an odd prime and S a p-group. Then there exists a characteristic subgroup $W^*(S)$ of S so that*

(i) $Z(S) \leq W(S) \leq W^*(S)$,

(ii) *$W^*(S)$ is normal in every strongly p-stable group H with $S \in \mathrm{Syl}_p(H)$,*

(iii) $C_S(W^*(S)) \leq W^*(S)$.

(iv) *If $A \leq S$ and $[W^*(S), A, A] = 1$, then $A \leq W^*(S)$. In particular, every normal abelian subgroup of S is contained in $W^*(S)$.*

Proof. Let $\mathscr{E}$ be a maximal set of pairwise nonequivalent strongly p-stable embeddings of S so that $S\tau \in \mathrm{Syl}_p(H)$ for $(\tau, H) \in \mathscr{E}$. As in Theorem 1 $\mathscr{E}$ is finite and every strongly p-stable embedding $(\tilde{\tau}, \widetilde{H})$ of S with $S\tilde{\tau} \in \mathrm{Syl}_p(\widetilde{H})$ is equivalent to some element in $\mathscr{E}$.

Set $W^*(S) = W_{\mathscr{E}}(S)$. Note that H is weakly p-stable for every $(\tau, H) \in \mathscr{E}$. Hence, $W^*(S)$ is characteristic subgroup of S by Lemma 1, and Theorem 1 gives (i) and (ii).

For $(\tau, H) \in \mathscr{E}$ we define:

$$S_1 = C_S(W^*(S)), \qquad H_1 = C_H(W^*(S)), \qquad \tau_1 = \tau|_{S_1}.$$

Then $\mathscr{E}_1 = \{(\tau_1, H_1) \mid (\tau, H) \in \mathscr{E}\}$ is a set of embeddings of S_1, and since H_1 is normal in H and H is strongly p-stable, all these embeddings are weakly p-stable by Lemma 3.

Set $W_*(S_1) = W_{\mathscr{E}_1}(S_1)$. By Lemma 4 we have $Z(S_1) \le W_*(S_1)$. Moreover, $H = H_1 N_H(S_1\tau_1)$ for $(\tau_1, H_1) \in \mathscr{E}_1$, and Lemma 2 together with Lemma 1 shows that $W_*(S_1)\tau_1$ is normal in H. By the maximality of $W^*(S)$ we get

$$W_*(S_1) \le W^*(S) \text{ and } W_*(S_1) = Z(S_1);$$

in particular $Z(S_1)\tau_1 = Z(H_1)$ for $(\tau_1, H_1) \in \mathscr{E}_1$.

For $(\tau_1, H_1) \in \mathscr{E}_1$ we now define:

$$\overline{S}_1 = S_1/Z(S_1) \text{ and } \overline{H}_1 = H_1/Z(S_1)\tau_1,$$

and $\overline{\tau}_1$ is the monomorphism into $\overline{H}_1$ induced by τ_1. Then $\overline{\mathscr{E}}_1 = \{(\overline{\tau}_1, \overline{H}_1) \mid (\tau_1, H_1) \in \mathscr{E}_1\}$ is a set of embeddings of $\overline{S}_1$, and all these embeddings are weakly p-stable by Lemma 3.

Set $W_*(\overline{S}_1) = W_{\overline{\mathscr{E}}_1}(\overline{S}_1)$ and let $\widetilde{W}_*(S_1)$ be the inverse image of $W_*(\overline{S}_1)$ in S_1. Note that by Lemma 4 $Z(\overline{S}_1) \le W_*(\overline{S}_1)$. It follows that $\widetilde{W}_*(S_1)\tau_1$ is normal in H_1 for $(\tau_1, H_1) \in \mathscr{E}_1$ and thus $Z(S_1) \le \widetilde{W}_*(S_1) \le W_*(S_1) = Z(S_1)$, i.e.,

$$W_*(S_1) = \widetilde{W}_*(S_1) \text{ and } Z(\overline{S}_1) = W_*(\overline{S}_1) = 1.$$

This implies

$$W_*(S_1) = Z(S_1) = S_1 = H_1$$

and $C_S(W^*(S)) = S_1 = W_*(S_1) \le W^*(S)$. This proves (iii).

Suppose that $[W^*(S), A, A] = 1$, but $A \nleq W^*(S)$ for some $A \le S$. As in the proof of Lemma 4 let G be the amalgamated product of the H_i's over S where $(\tau_i, H_i) \in \mathscr{E}$. Then $W^*(S)$ is the kernel of the operation of G on the corresponding coset graph. Since $A \nleq W^*(S)$ there exists a G-conjugate A^* of A in S and $(\tau, H) \in \mathscr{E}$ so that $A^* \nleq O_p(H)$.

Set $H_0 = \langle A^{*H} \rangle$. Then strong p-stability gives $[W^*(S), O^p(H_0)] = 1$, and (iii) implies that $O^p(H_0) = 1$, i.e., $A^* \le O_p(H)$, a contradiction.

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References

1. G. Glauberman, *A characteristic subgroup of a p-stable group*, Canad. J. Math. **20** (1968), 1101–1135.
2. D. Goldschmidt, *Automorphisms of trivalent graphs*, Ann. of Math. **111** (1980), 377–406.
3. D. Gorenstein, *Finite groups*, Harper & Row, New York.
4. A. G. Kurosh, *Theory of groups*, Chelsea, New York, 1955.

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