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A CHARACTERIZATION OF SUMS OF $2n$TH POWERS OF GLOBAL MEROMORPHIC FUNCTIONS

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ABSTRACT. Let M be a real analytic manifold. In this note we prove

Theorem. *Let X be a compact analytic set of M and Σ its singular locus. Then, a meromorphic function h on X is a sum of $2n$-th powers of meromorphic functions if and only if, for every analytic curve $\sigma\colon (-\varepsilon, \varepsilon) \to X$ not contained in Σ, it holds $h \circ \sigma = at^m + \cdots$, with $a > 0$ and $2n$ dividing m.*

INTRODUCTION

Let M be a real analytic manifold. The goal of this note is to prove the following

Theorem. *Let X be a compact analytic set of M and Σ its singular locus. Then, a meromorphic function h on X is a sum of $2n$-th powers of meromorphic functions if and only if, for every analytic curve $\sigma\colon (-\varepsilon, \varepsilon) \to X$ not contained in Σ, it holds $h \circ \sigma = at^m + \cdots$, with $a > 0$ and $2n$ dividing m.*

(We allow a slight abuse of notation when either $h \circ \sigma = 0$ or $h \circ \sigma$ is not defined.)

This includes the solution of Hilbert's 17th Problem given in [Rz]. Indeed, for $n = 1$, the theorem above becomes

Corollary. *A meromorphic function on X is a sum of squares of meromorphic functions if and only if it is positive semidefinite off Σ.*

On the other hand, this theorem implies that h being a sum of $2n$th powers is a local question, which depends only on the germs h_x for x a limit of regular points.

Note that the singular locus has to be considered. Take, for instance, the "stereographic closure" of Whitney's umbrella:

$$X = \{(t, x, y, z) \in \mathbf{R}^4 \colon t^2 + x^2 + y^2 + z^2 = 1,\ (1-t)x^2 - zy^2 = 0\}.$$

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This set is a compact analytic set, and $h = z/(1-t) = (x/y)^2$ is a sum of squares of meromorphic functions on X (actually a square), although it is negative at the singular point $(0, 0, 0, -1)$.

Our proof is based on Becker's theory of $2n$th powers in formally real fields and uses two special features of the compact real analytic sets: their completeness with respect to real valuations, and the Artin–Lang specialization property. In addition, it also uses a real going-down for completions of excellent rings and Hironaka's resolution of singularities.

Results similar to our theorem have been obtained for rational functions over real algebraic varieties [Br-Sch, K-P, Sch], once the discussion of $2n$th powers was started by E. Becker [B1, B2]. In the analytic setting, the same theorem has been proved for X nonsingular of dimension 2 by W. Kucharz, [K].

1. Preliminaries

Let M, X, and Σ be as in the introduction. We shall review here several facts from the theory of real analytic sets. All of them can be found in [B-W], [C], [F], [T].

(1.1) **Localizations.** Let $\mathcal{O}$ stand for the sheaf of germs of analytic functions of M, and $\mathcal{O}(M)$ for its ring of global sections; $\mathcal{O}(M)$ is the ring of global analytic functions on M. Then the ideal of X,

$$I = \{f \in \mathcal{O}(M) : f|X \equiv 0\},$$

generates a coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}$, and X is the support of $\mathcal{O}/\mathcal{J}$. The ring of global sections of the latter sheaf is

$$\Gamma(M, \mathcal{O}/\mathcal{J}) = \mathcal{O}(M)/I;$$

this is the ring of global analytic functions on X, denoted by $\mathcal{O}(X)$. Thus, a global analytic function on X is the restriction to X of a global analytic function on M.

Now fix a point $x \in X$. We have the local ring

$$\mathcal{O}(X)_{\mathfrak{m}}, \qquad \mathfrak{m} = \{f \in \mathcal{O}(X) : f(x) = 0\}.$$

On the other hand, consider the stalk of $\mathcal{O}/\mathcal{J}$ at x,

$$(\mathcal{O}/\mathcal{J})_x = \mathcal{O}_x/I\mathcal{O}_x,$$

which we shall denote by $\mathcal{O}_x(X)$. The properties we need are

Lemma. *The rings $\mathcal{O}(X)_{\mathfrak{m}}$ and $\mathcal{O}_x(X)$ are excellent, and the canonical inclusion $\mathcal{O}(X)_{\mathfrak{m}} \to \mathcal{O}_x(X)$ induces an isomorphism of the completions.*

(More details can be found in [Rz, §2]. For excellent rings and related notions we refer to [M]).

(1.2) **Irreducible components and dimension.** Since X is compact, it has finitely many irreducible components, say $X_1, \dots, X_s$, whose ideals $I_1, \dots, I_s$ are the associated primes of I. Furthermore

$$\dim(X) = \max\{\dim(X_i) : 1 \le i \le s\}.$$

On the other hand, for $x \in X$,

$$\dim \mathcal{O}(X)_{\mathfrak{m}} = \max\{\dim(X_i) : x \in X_i\}.$$

(1.3) **Regular points.** This is quite a delicate notion in the real analytic setting. Here we shall adopt a definition of global nature:

Definition. A point $x \in X$ is a *regular point* if there are $f_1, \ldots, f_r \in I$ such that

(i) The jacobian of $f_1, \ldots, f_r$ has rank r at x.
(ii) The set $\{z \in X : f_1(z) = \cdots = f_r(z) = 0\}$ coincides with X in a neighborhood of x.

From 1.1 and the inverse mapping theorem, it follows easily that $x \in X$ is regular if and only if $\mathcal{O}(X)_{\mathfrak{m}}$ is a regular ring if and only if $\mathcal{O}_x(X)$ is a regular ring.

We denote

$$\operatorname{Reg}(X) = \{x \in X : x \text{ is a regular point}\},$$

and then

$$\operatorname{Reg}(X) = \bigcup_i \left(\operatorname{Reg}(X_i) \setminus \bigcup_{i \neq j} X_j \right). \tag{1.3.1}$$

The singular locus $\Sigma = X \setminus \operatorname{Reg}(X)$ is also an analytic set of M in the global sense: there is an analytic function $\Delta \in \mathcal{O}(X)$ such that

$$\Sigma = \{x \in X : \Delta(x) = 0\},$$

(and Δ does not vanish identically on any X_i).

(1.4) **Meromorphic functions.** Let $\mathcal{M}(X)$ stand for the total ring of fractions of the ring $\mathcal{O}(X)$; this $\mathcal{M}(X)$ is the ring of meromorphic functions on X. Thus, a meromorphic function is a quotient $h = f/g$ of two analytic functions, where the denominator g does not vanish identically on any X_i.

Finally we remark that the canonical map

$$\mathcal{M}(X) \to \mathcal{M}(X_1) \times \cdots \times \mathcal{M}(X_s) : f \to (f|X_1, \ldots, f|X_s) \tag{1.4.1}$$

is an isomorphism (Chinese remainder theorem).

Note also that each $\mathcal{M}(X_i)$ is a formally real field.

2. Proof of the necessary condition

It is clear from 1.3.1 that for the proof of this half of the theorem we may assume X irreducible, so that $\mathcal{O}(X)$ is a domain and $\mathcal{M}(X)$ a field.

Now, let $h \in \mathcal{M}(X)$ be a sum of $2n$th powers, and consider an analytic curve $\sigma : (-\varepsilon, \varepsilon) \to X$ not contained in Σ, with $h \circ \sigma = at^m + \cdots$, $a \neq 0$.

First of all, we shall construct a valuation ν of the field $\mathscr{M}(X)$, using the curve σ. This curve defines the following commutative diagram

$$\begin{tikzcd} \mathscr{O}_x(X) \arrow[rr, "\varphi"] & & \mathbf{R}\{t\} \\ & \mathscr{O}(X)_{\mathfrak{m}} \arrow[ul] \arrow[ur, "\psi"'] & \end{tikzcd} \qquad \begin{array}{l} \varphi(f) = \text{expansion of } f\circ\sigma \text{ at } 0 \\ \psi = \varphi|\mathscr{O}(X)_{\mathfrak{m}} \end{array}$$

where $x = \sigma(0)$ and $\mathfrak{m}$ is the ideal corresponding to x. We have the prime ideals

$$\mathfrak{p} = \ker\varphi, \qquad \mathfrak{q} = \mathfrak{p}\cap\mathscr{O}(X)_{\mathfrak{m}} = \ker\psi$$

In this situation we claim:

(2.1) The localization $\mathscr{O}(X)_{\mathfrak{q}}$ is a regular ring.

Indeed, by the Lemma in 1.1, the homomorphism $\mathscr{O}(X)_{\mathfrak{q}} \to \mathscr{O}_x(X)_{\mathfrak{p}}$ is flat. From [M, 21.D, Theorem 51(i), p. 155], it follows that $\mathscr{O}(X)_{\mathfrak{q}}$ is regular if $\mathscr{O}_x(X)_{\mathfrak{p}}$ is also regular. Consequently, we deal with the latter ring.

Here we use the ideals R_k introduced in [T, Chapter II] to formulate several jacobian criteria. First of all, the singular locus of X is the set

$$\Sigma = \{z\in X : \mathscr{O}_z(X) \text{ is not regular}\}.$$

Then, as X is irreducible, by the Lemma in 1.1 and 1.2 we find

$$\dim\mathscr{O}_z(X) = \dim\mathscr{O}(X)_{\mathfrak{n}} = \dim(X),$$

where $\mathfrak{n}$ = maximal ideal of z. Hence

$$\Sigma = \{x\in X : \mathscr{O}_z(X) \text{ is not regular of dimension } \dim(X)\}.$$

Now, $\mathscr{O}_z(X) = \mathscr{O}_z/I\mathscr{O}_z$ and, from [T, Chapter II, Theorem 7.9], we get

$$\text{germ } \Sigma_x \text{ of } \Sigma \text{ at } x = V(R_k(I\mathscr{O}_x)),$$

with $k = \dim M - \dim X$. Finally, the germ σ_x of (the image of) σ at x is contained in $V(\mathfrak{p})$ but not in Σ_k, so that $V(\mathfrak{p})$ is not contained in $\Sigma_x = V(R_k(I\mathscr{O}_x))$. In conclusion, $\mathfrak{p}$ does not contain $R_k(I\mathscr{O}_x)$ and, from [T, Chapter II, Theorem 3.1], we deduce that $\mathscr{O}_x(X)_{\mathfrak{p}}$ is a regular ring.

As remarked before, this implies 2.1. □

On the other hand, the homomorphism ψ induces an embedding $\kappa(\mathfrak{q}) \subset qf(\mathbf{R}\{t\})$, where $\kappa(\mathfrak{q})$ stands for the residue field of $\mathfrak{q}$. Then we restrict the ordinary valuation of $qf(\mathbf{R}\{t\})$ to a valuation $\overline{\nu}$ of $\kappa(\mathfrak{q})$. Coming back to our meromorphic function h, since $h\circ\sigma = at^m + \cdots$, $a\neq 0$, is defined, $h\in\mathscr{O}(X)_{\mathfrak{q}}$ and its class $\overline{h}$ in $\kappa(\mathfrak{q})$ verifies $\overline{\nu}(\overline{h}) = m$.

Now we need 2.1: since $\mathscr{O}(X)_{\mathfrak{q}}$ is regular, $\overline{\nu}$ lifts to a valuation w of $qf(\mathscr{O}(X)_{\mathfrak{q}}) = \mathscr{M}(X)$ with residue field $\kappa(\mathfrak{q})$. Furthermore, the ring W of w dominates $\mathscr{O}(X)_{\mathfrak{q}}$ and so $h\in W$. Finally, consider the valuation ν, composite of $\overline{\nu}$ and w. The canonical epimorphism $W \to \kappa(\mathfrak{q})$ maps the ring V of ν onto the ring $\overline{V}$ of $\overline{\nu}$.

After this preparation we use [B2, 1.9, p. 146]: as h is a sum of $2n$th powers in $\mathscr{M}(X)$, $2n$ divides $\nu(h)$. In other words, there is $g \in \mathscr{M}(X)$ such that g^{2n}/h is a unit of V. In particular, $g^{2n}/h \in V \subset W$ and $g^{2n} = h(g^{2n}/h) \in W$. Thus g and h belong to W, and, taking residue classes in $\kappa(\mathfrak{q})$, we have $\overline{g}$, $\overline{h}$ such that $\overline{g}^{2n}/\overline{h} = \overline{g^{2n}/h}$ is a unit in V. We conclude $2n$ divides $\overline{\nu}(\overline{h}) = m$.

To end the proof we have to check that $a > 0$. Suppose, by way of contradiction, that $a < 0$. Then there would be regular points $z = \sigma(t)$, $t > 0$, with $h(z) < 0$. Hence, it is enough to check that h is positive semidefinite off Σ. For that, take any representation of h as a sum of $2n$th powers:

$$h = \sum f_i^{2n}/g^{2n}, \qquad f_i, g \in \mathscr{O}(X), g \neq 0.$$

Clearly h is positive semidefinite in $\mathrm{Reg}(X) \setminus \{g = 0\}$, and this set is dense in $\mathrm{Reg}(X)$ because X is irreducible. We are done.

Remark. The long argument used above is needed to handle the curves σ on which h can be evaluated, i.e. $h = f/g$ with $g \circ \sigma \neq 0$, but this evaluation cannot be done on any expression $h = \sum f_i^{2n}/g_i^{2n}$ because $g_i \circ \sigma = 0$, for some i.

3. Proof of the sufficient condition

Let $h \in \mathscr{M}(X)$ not be a sum of $2n$th powers. We have to find an analytic curve $\sigma: (-\varepsilon, \varepsilon) \to X$, not contained in Σ such that $h \circ \sigma = at^m + \cdots$, $a \neq 0$, and either $a < 0$ or $2n$ does not divide m.

Since h is not a sum of $2n$th powers in $\mathscr{M}(X)$, applying the isomorphism 1.4.1, we find that $h|X_i$ is not a sum of $2n$th powers in $\mathscr{M}(X_i)$ for some i. Then choose an analytic function g which does not vanish identically on X_i and such that

$$X_i \setminus \{g = 0\} = \mathrm{Reg}(X_i) \setminus \bigcup_{j \neq i} X_i = \mathrm{Reg}(X) \setminus \bigcup_{j \neq i} X_i = X \setminus \{g = 0\}$$

and $g^{2n} h = f$ is an analytic function.

Clearly, it is enough to find $\sigma: (-\varepsilon, \varepsilon) \to X$ not contained in $\{g = 0\}$ such that $f \circ \sigma = at^m + \cdots$, $a \neq 0$, and either $a < 0$ or $2n$ does not divide m. Furthermore, formulated this way we can restrict ourselves to the case $X = X_i$, i.e., to the case X is irreducible. In conclusion, we shall prove:

(3.1) *Claim.* Assume X is irreducible. Let $f, g \in \mathscr{O}(X)$, $g \neq 0$, and f not a sum of $2n$th powers in $\mathscr{M}(X)$. Then, there is an analytic curve $\sigma: (-\varepsilon, \varepsilon) \to X$ not contained in $\{g = 0\}$, such that $f \circ \sigma = at^m + \cdots$, $a \neq 0$, and either $a < 0$ or $2n$ does not divide m.

First, we distinguish a special case:

Case when f is not a sum of squares in $\mathscr{M}(X)$. Here f must be negative in some ordering of the field $\mathscr{M}(X)$, and so will be $g^2 f$. We deduce from [Rz, Corollary 2.4], that there is some point $x \in X$ with $g(x)^2 f(x) < 0$. Finally, pick any analytic curve $\sigma\colon (-\varepsilon, \varepsilon) \to X$ with $\sigma(0) = x$. Clearly $g \circ \sigma \neq 0$ and

$$f \circ \sigma = a + bt + \cdots, \quad \text{with } a = f(\sigma(0)) = f(x) < 0. \quad \square$$

Once the precedent case is solved we consider the

Case when f is a sum of squares in $\mathscr{M}(X)$. By [B2, 1.9, p. 146], there is a real valuation ν of the field $\mathscr{M}(X)$ such that $2n$ does not divide $\nu(f)$. Fix an ordering α of $\mathscr{M}(X)$ compatible with ν and denote by V the convex hull of $\mathbf{R}$ in $\mathscr{M}(X)$ with respect to α. This V is a valuation ring with residue field $\mathbf{R}$ which dominates the local ring $A = \mathscr{O}(X)_{\mathfrak{m}}$ of some point $x \in X$ [Rz, Lemma 2.3]. Now, since α is compatible with ν, the valuation ring of ν contains V and so it is $V_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ of V. Hence, the value group of $V_{\mathfrak{p}}$ is a quotient of the one of V, and, as $2n$ does not divide the value of f with respect to $V_{\mathfrak{p}}$, it does not divide the one with respect to V either. All this means that we can suppose V is the valuation ring of ν.

We have $A \to V \subset K = qf(A) = \mathscr{M}(X)$. The integral closure $\overline{A}$ of A in K is contained in V, and V dominates a local ring $B = \overline{A}_{\overline{\mathfrak{m}}}$, where $\overline{\mathfrak{m}}$ is a maximal ideal of $\overline{A}$, lying over the maximal ideal $\mathfrak{m}$ of A:

Now we consider the adic completions $A^{\wedge}$ and $B^{\wedge}$. Since A is excellent (1.1), there is a unique zero divisor $\mathfrak{q}^{\wedge}$ of $A^{\wedge}$ such that $B^{\wedge}$ is the integral closure of $A^{\wedge}/\mathfrak{q}^{\wedge}$ in $K^{\wedge} = qf(A^{\wedge}/\mathfrak{q}^{\wedge})$ [EGA, 7.6.2, p. 209]. We have the diagram

$$\begin{tikzcd}
A^{\wedge} \arrow[r] & A^{\wedge}/\mathfrak{q}^{\wedge} \arrow[r] & (A^{\wedge}/\mathfrak{q}^{\wedge})^{-} = B^{\wedge} & \subset K^{\wedge} \\
A \arrow[u] \arrow[ur] \arrow[rr] & & B \arrow[u] & \subset K \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

In this picture there is a room for the local ring $A^* = \mathscr{O}_x(X)$. First, by 1.1, $A^{\wedge}$ is the adic completion of A^*. Thus, the zero divisors of $A^{\wedge}$ are the extensions of the ones of A^* [T, Chapter III, Corollary 4.8] and so $\mathfrak{q}^{\wedge} = \mathfrak{q}^* A^{\wedge}$ for a zero-divisor $\mathfrak{q}^*$ of A^*. Finally, denote by B^* the integral closure of $A^*/\mathfrak{q}^*$ in $K^* = qf(A^*/\mathfrak{q}^*)$. We have come to the diagram

$$\begin{tikzcd}
A^{\wedge} \arrow[r] & A^{\wedge}/\mathfrak{q}^{\wedge} \arrow[r] & (A^{\wedge}/\mathfrak{q}^{\wedge})^{-} = B^{\wedge} & \subset K^{\wedge} \\
A^* \arrow[u] \arrow[r] & A^*/\mathfrak{q}^* \arrow[u] \arrow[r] & (A^*/\mathfrak{q}^*)^{-} = B^* \arrow[u] & \subset K^* \arrow[u, phantom, "\cup"] \\
A \arrow[u] \arrow[ur] \arrow[rr] & & B \arrow[u] & \subset K \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Indeed, B^* is local [T, Chapter II, Proposition 2.5] and, again by [EGA] loc.cit., $B^{\wedge}$ is the adic completion of B^*. Hence, $B^{\wedge}$ dominates B^* and B^* dominates B, and the maximal ideal of B generates the one of B^*.

Now consider the ordering α we had in K. Since V is α-convex and V dominates B, α is a central ordering in the local excellent domain B, in the sense of [Rz, §1]. Then, by [Rz, Theorem 1.1], α extends to a total ordering

$\alpha^\frown$ in $B^\frown$. Let us denote by α^* the restriction of $\alpha^\frown$ to B^*, and by V^* the convex hull of $\mathbf{R}$ in K^* with respect to α^*. The valuation ring V^* is an extension of V, has residue field $\mathbf{R}$ and dominates B^*. We summarize the situation in the next diagram

$$\begin{tikzcd}
A^* \arrow[r] & A^*/\mathfrak{q}^* \arrow[r] & (A^*/\mathfrak{q}^*)^- = B^* \arrow[r] & V^* \subset K^* \\
A \arrow[u] \arrow[ur] \arrow[rr] & & B \arrow[u] \arrow[r] & V \arrow[u] \subset K
\end{tikzcd} \quad \cup \ ; \text{ all residue fields are } \mathbf{R}.$$

Now we apply Hironaka's resolution of singularities I and II ([H1], cf. also [H2, pp. 5.8-5.9] to the local excellent ring B and the ideal $J = fB$: there is a proper birational morphism $Z \to \operatorname{Spec} B$, where Z is a regular scheme and $J\mathscr{O}_z$ is simple everywhere, i.e., for each $z \in Z$ there is a regular system of parameters $x_1, \ldots, x_k$ of $\mathscr{O}_{Z,z}$ such that $f\mathscr{O}_{Z,z} = (x_1^{p_1} \cdots x_k^{p_k})$.

In our situation, we choose the point z as follows. Since $Z \to \operatorname{Spec} B$ is proper and birational the valuation V dominates a unique local ring of Z: this is our $\mathscr{O}_{Z,z}$. But $\mathscr{O}_{Z,z} = B[h_1, \ldots, h_s]_\mathfrak{n}$ for some elements $h_1, \ldots, h_s \in K$ and an ideal $\mathfrak{n} \subset B[h_1, \ldots, h_s]$. Finally notice that the residue field of $\mathfrak{n}$ is $\mathbf{R}$, because this is the residue field of V and V dominates $\mathscr{O}_{Z,z}$. As $\mathbf{R} \subset B[h_1, \ldots, h_s]$, we can replace $h_1, \ldots, h_s$ by $h_1 - c_1, \ldots, h_s - c_s$ for suitable $c_i \in \mathbf{R}$ to have $h_1, \ldots, h_s \in \mathfrak{n}$. Hence

Conclusion.

(3.1.3) The ring $B_1 = B[h_1, \ldots, h_s]_\mathfrak{n}$ is regular, $h_1, \ldots, h_s \in \mathfrak{n}$, and $\mathfrak{n}$ has a regular system of parameters $x_1, \ldots, x_k$ such that the element $f \in A$ factorizes in B in the form $f = ux_1^{p_1} \cdots x_k^{p_k}$, where u is a unit and $p_i \geq 0$.

Once we have this, we come back to our valuation ν. It holds:

$$\nu(f) = p_1\nu(x_1) + \cdots + p_k\nu(x_k),$$

and as $2n$ does not divide $\nu(f)$, it does not divide p_i for some i, say $i = 1$. To finish the proof of 3.1 we need a local homomorphism $\varphi: B \to \mathbf{R}\{t\}$ such that

(3.1.4) $\qquad \varphi(x_1) = t; \qquad \varphi(x_j) = a_j t^{2n}, \qquad a_j \neq 0, \text{ for } j > 1,$

(3.1.5) $\qquad \varphi(g) \neq 0.$

Indeed, from $A = \mathscr{O}(X)_\mathfrak{m} \to B_1 \xrightarrow{\varphi} \mathbf{R}\{t\}$ we obtain an analytic curve $\sigma: (-\varepsilon, \varepsilon) \to X$ with $f \circ \sigma = \varphi(f)$, $g \circ \sigma = \varphi(g)$. Then, in view of 3.1.4 we have $f = at^m + \cdots$, where:

$a = a_1a_2 \cdots a_k \neq 0$, $a_1 = \varphi(u)(0) \neq 0$ because $\varphi(u)$ is a unit of $\mathbf{R}\{t\}$,

and $2n$ does not divide $m = p_1 + 2n\sum_{j>1} p_j$, since it does not divide p_1.

Finally, by 3.1.5, $g \circ \sigma \neq 0$, so that σ is not contained in $\{g = 0\}$. □

Consequently, let us find φ verifying 3.1.4 and 3.1.5.

First, consider $B^*[h_1, \dots, h_s]$. Clearly V^* contains this ring, and so it must dominate a localization $B_1^* = B^*[h_1, \dots, h_s]_{\mathfrak{n}^*}$. This gives a local inclusion $B_1 \to B_1^*$, which is faithfully flat, because

$$B^*[h_1, \dots, h_s] = B^* \otimes_B B[h_1, \dots, h_s]$$

and $B \to B^*$ is flat. It follows from [M, 13.B, Theorem 19, p. 79] that

$$(*) \qquad \dim(B_1^*) = \dim(B_1) + \dim(B_1^*/\mathfrak{n}B_1^*).$$

We claim

$$(**) \qquad \mathfrak{n}B_1^* = \mathfrak{n}^*.$$

For $h_1, \dots, h_s \in \mathfrak{n}$ implies $B_1^*/\mathfrak{n}B_1^* = B^*/B^* \cap \mathfrak{n}B_1^*$, and, since the maximal ideal $\overline{\mathfrak{m}}$ of B generates the one $\overline{\mathfrak{m}}^*$ of B^*, we deduce $B^* \cap \mathfrak{n}B_1^* \supset \mathfrak{m}B^* = \overline{\mathfrak{m}}^*$, which gives $B_1^*/\mathfrak{n}B_1^* = B^*/\overline{\mathfrak{m}}^* = \mathbf{R}$. Thus, $\mathfrak{n}B_1^*$ is a maximal ideal and it must coincide with $\mathfrak{n}^*$.

From $(*)$, $(**)$ and [M, 21.D, Theorem 51, p. 155], we conclude that B_1^* is regular and

(3.1.6) $\qquad x_1, \dots, x_k$ are a regular system of parameters of B_1^*.

Our next step is

(3.1.7) $\qquad$ There is a local embedding $B_1^* \to \mathbf{R}\{x_1, \dots, x_k\}$.

Indeed, the ring B^* is an analytic ring, i.e., there is a local epimorphism $\mathbf{R}\{y\} \to B^*$, $y = (y_1, \dots, y_r)$ (by [T, Chapter II, Proposition 2.3] and the fact that the residue field of B^* is $\mathbf{R}$). We extend it to another one,

$$\mathbf{R}\{y\}[z] \to B^*[h_1, \dots, h_s], \qquad z = (z_1, \dots, z_s),$$

by $z_i \to h_i$. Since $h_1, \dots, h_s \in \mathfrak{n}^*$, the inverse image of $\mathfrak{n}^*$ is the maximal ideal (y, z), and we get a local epimorphism

$$\mathbf{R}\{y\}[z]_{(y,z)} \to B^*[h_1, \dots, h_s]_{\mathfrak{n}^*} = B_1^*.$$

Finally, if I is the kernel, we obtain an isomorphism

$$\mathbf{R}\{y\}[z]_{(y,z)}/I \to B_1^*.$$

Thus, $\mathbf{R}\{y\}[z]_{(y,z)}/I$ is a local regular ring of dimension $r + s - k$, and the inverse images $x_1', \dots, x_k'$ of $x_1, \dots, x_k$ generate the maximal ideal $(y, z)/I$.

Now, $\mathbf{R}\{y\}[z]_{(y,z)}$ being regular, there are $x_{k+1}', \dots, x_{r+s}'$ which generate I. Then $x_1', \dots, x_{r+s}'$ generate (y, z) and also generate $(y, z)\mathbf{R}[[y, z]]$, because $\mathbf{R}[[y, z]]$ is the completion of $\mathbf{R}\{y\}[z]_{(y,z)}$. Hence, by the inverse mapping theorem,

$$\frac{D(x_1', \dots, x_{r+s}')}{D(y_1, \dots, y_r, z_1, \dots, z_s)}(0) \neq 0,$$

and, since $x'_1, \dots, x'_{r+s}$ actually dwell in $\mathbf{R}\{y, z\}$, they generate $(y, z)\mathbf{R}\{y, z\}$. All this gives

$$\mathbf{R}\{y\}[z]_{(y,z)}/I \to \mathbf{R}\{y, z\}/I\mathbf{R}\{y, z\} \to \mathbf{R}\{x'_1, \dots, x'_k\},$$

which, composed with the inverse of $\mathbf{R}/\{y\}[z]_{(y,z)}/I \to B_1^*$, is the embedding required in 3.1.7. $\square$

After 3.1.7, we are ready to produce $\varphi\colon B_1 \to \mathbf{R}\{t\}$, verifying 3.1.4 and 3.1.5. We put

$$\varphi\colon B_1 \to B_1^* \to \mathbf{R}\{x_1, \dots, x_k\} \xrightarrow{\tau} \mathbf{R}\{t\},$$

$$\tau(x_1) = t, \quad \text{and} \quad \tau(x_j) = a_j t^{2n}, \qquad a_j \neq 0, \ \text{for } j \neq 1.$$

Thus, φ fulfills 3.1.4. In addition, one can always choose the a_j's to have

$$\tau(g) = g(t, a_2 t^{2n}, \dots, a_k t^{2n}) \neq 0,$$

since $g \neq 0$ in $\mathbf{R}\{x_1, \dots, x_k\}$. This is the condition 3.1.5.

As explained before, this definition of φ completes the proof. $\square$

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