A CHARACTERIZATION OF SUMS OF $2^n$TH POWERS OF GLOBAL MEROMORPHIC FUNCTIONS

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Abstract. Let $\mathcal{M}$ be a real analytic manifold. In this note we prove

Theorem. Let $X$ be a compact analytic set of $\mathcal{M}$ and $\Sigma$ its singular locus. Then, a meromorphic function $h$ on $X$ is a sum of $2n$th powers of meromorphic functions if and only if, for every analytic curve $\sigma : (-\epsilon, \epsilon) \rightarrow X$ not contained in $\Sigma$, it holds $h \circ \sigma = at^m + \cdots$, with $a > 0$ and $2n$ dividing $m$.

Introduction

Let $\mathcal{M}$ be a real analytic manifold. The goal of this note is to prove the following

Theorem. Let $X$ be a compact analytic set of $\mathcal{M}$ and $\Sigma$ its singular locus. Then, a meromorphic function $h$ on $X$ is a sum of $2n$th powers of meromorphic functions if and only if, for every analytic curve $\sigma : (-\epsilon, \epsilon) \rightarrow X$ not contained in $\Sigma$, it holds $h \circ \sigma = at^m + \cdots$, with $a > 0$ and $2n$ dividing $m$.

(We allow a slight abuse of notation when either $h \circ \sigma = 0$ or $h \circ \sigma$ is not defined.)

This includes the solution of Hilbert’s 17th Problem given in [Rz]. Indeed, for $n = 1$, the theorem above becomes

Corollary. A meromorphic function on $X$ is a sum of squares of meromorphic functions if and only if it is positive semidefinite off $\Sigma$.

On the other hand, this theorem implies that $h$ being a sum of $2n$th powers is a local question, which depends only on the germs $h_x$ for $x$ a limit of regular points.

Note that the singular locus has to be considered. Take, for instance, the “stereographic closure” of Whitney’s umbrella:

$$X = \{(t, x, y, z) \in \mathbb{R}^4 : t^2 + x^2 + y^2 + z^2 = 1, (1 - t)x^2 - yz^2 = 0\}.$$
This set is a compact analytic set, and \( h = z/(1-t) = (x/y)^2 \) is a sum of squares of meromorphic functions on \( X \) (actually a square), although it is negative at the singular point \((0, 0, 0, -1)\).

Our proof is based on Becker’s theory of 2n-th powers in formally real fields and uses two special features of the compact real analytic sets: their completeness with respect to real valuations, and the Artin–Lang specialization property. In addition, it also uses a real going-down for completions of excellent rings and Hironaka’s resolution of singularities.

Results similar to our theorem have been obtained for rational functions over real algebraic varieties \([Br-Sch, K-P, Sch]\), once the discussion of 2n-th powers was started by E. Becker \([B1, B2]\). In the analytic setting, the same theorem has been proved for \( X \) nonsingular of dimension 2 by W. Kucharz, \([K]\).

1. Preliminaries

Let \( M, X, \) and \( \Sigma \) be as in the introduction. We shall review here several facts from the theory of real analytic sets. All of them can be found in \([B-W], [C], [F], [T]\).

(1.1) Localizations. Let \( \mathcal{O} \) stand for the sheaf of germs of analytic functions of \( M \), and \( \mathcal{O}(M) \) for its ring of global sections; \( \mathcal{O}(M) \) is the ring of global analytic functions on \( M \). Then the ideal of \( X \),
\[
I = \{ f \in \mathcal{O}(M) : f|X = 0 \},
\]
generates a coherent sheaf of ideals \( \mathcal{I} \subset \mathcal{O} \), and \( X \) is the support of \( \mathcal{O}/\mathcal{I} \).
The ring of global sections of the latter sheaf is
\[
\Gamma(M, \mathcal{O}/\mathcal{I}) = \mathcal{O}(M)/I;
\]
this is the ring of global analytic functions on \( X \), denoted by \( \mathcal{O}(X) \). Thus, a global analytic function on \( X \) is the restriction to \( X \) of a global analytic function on \( M \).

Now fix a point \( x \in X \). We have the local ring
\[
\mathcal{O}(X)_m, \quad m = \{ f \in \mathcal{O}(X) : f(x) = 0 \}.
\]
On the other hand, consider the stalk of \( \mathcal{O}/\mathcal{I} \) at \( x \),
\[
(\mathcal{O}/\mathcal{I})_x = \mathcal{O}_x/I\mathcal{O}_x,
\]
which we shall denote by \( \mathcal{O}_x(X) \). The properties we need are

**Lemma.** The rings \( \mathcal{O}(X)_m \) and \( \mathcal{O}_x(X) \) are excellent, and the canonical inclusion \( \mathcal{O}(X)_m \to \mathcal{O}_x(X) \) induces an isomorphism of the completions.

(More details can be found in \([Rz, \S2]\). For excellent rings and related notions we refer to \([M]\)).

(1.2) Irreducible components and dimension. Since \( X \) is compact, it has finitely many irreducible components, say \( X_1, \ldots, X_s \), whose ideals \( I_1, \ldots, I_s \) are the associated primes of \( I \). Furthermore
\[
\dim(X) = \max\{\dim(X_i) : 1 \leq i \leq s\}.
\]
On the other hand, for \( x \in X \),
\[
\dim \mathcal{O}(X)_m = \max \{ \dim(X_i) : x \in X_i \}.
\]

(1.3) **Regular points.** This is quite a delicate notion in the real analytic setting. Here we shall adopt a definition of global nature:

**Definition.** A point \( x \in X \) is a regular point if there are \( f_1, \ldots, f_r \in I \) such that

(i) The jacobian of \( f_1, \ldots, f_r \) has rank \( r \) at \( x \).

(ii) The set \( \{ z \in X : f_1(z) = \cdots = f_r(z) = 0 \} \) coincides with \( X \) in a neighborhood of \( x \).

From 1.1 and the inverse mapping theorem, it follows easily that \( x \in X \) is regular if and only if \( \mathcal{O}(X)_m \) is a regular ring if and only if \( \mathcal{O}_x(X) \) is a regular ring.

We denote
\[
\text{Reg}(X) = \{ x \in X : x \text{ is a regular point} \},
\]
and then
\[
\text{(1.3.1)} \quad \text{Reg}(X) = \bigcup_i \left( \text{Reg}(X_i) \setminus \bigcup_{i \neq j} X_j \right).
\]

The singular locus \( \Sigma = X \setminus \text{Reg}(X) \) is also an analytic set of \( M \) in the global sense: there is an analytic function \( \Delta \in \mathcal{O}(X) \) such that
\[
\Sigma = \{ x \in X : \Delta(x) = 0 \},
\]
(and \( \Delta \) does not vanish identically on any \( X_i \)).

(1.4) **Meromorphic functions.** Let \( \mathcal{M}(X) \) stand for the total ring of fractions of the ring \( \mathcal{O}(X) \); this \( \mathcal{M}(X) \) is the ring of meromorphic functions on \( X \). Thus, a meromorphic function is a quotient \( h = f/g \) of two analytic functions, where the denominator \( g \) does not vanish identically on any \( X_i \).

Finally we remark that the canonical map
\[
\text{(1.4.1)} \quad \mathcal{M}(X) \to \mathcal{M}(X_1) \times \cdots \times \mathcal{M}(X_s) : f \mapsto (f|X_1, \ldots, f|X_s)
\]
is an isomorphism (Chinese remainder theorem).

Note also that each \( \mathcal{M}(X_i) \) is a formally real field.

2. **Proof of the necessary condition**

It is clear from 1.3.1 that for the proof of this half of the theorem we may assume \( X \) irreducible, so that \( \mathcal{O}(X) \) is a domain and \( \mathcal{M}(X) \) a field.

Now, let \( h \in \mathcal{M}(X) \) be a sum of \( 2n \)th powers, and consider an analytic curve \( \sigma : (-\varepsilon, \varepsilon) \to X \) not contained in \( \Sigma \), with \( h \circ \sigma = a t^m + \cdots, \ a \neq 0 \).
First of all, we shall construct a valuation $\nu$ of the field $\mathcal{M}(X)$, using the curve $\sigma$. This curve defines the following commutative diagram

$\mathcal{O}_x(X) \xrightarrow{\varphi} \mathcal{R}\{t\} \xrightarrow{\psi} \mathcal{O}(X)_m$

where $x = \sigma(0)$ and $m$ is the ideal corresponding to $x$. We have the prime ideals

$p = \ker \varphi, \quad q = p \cap \mathcal{O}(X)_m = \ker \psi$

In this situation we claim:

(2.1) The localization $\mathcal{O}(X)_q$ is a regular ring.

Indeed, by the Lemma in 1.1, the homomorphism $\mathcal{O}(X)_q \rightarrow \mathcal{O}_x(X)_p$ is flat. From [M, 21.D, Theorem 51(i), p. 155], it follows that $\mathcal{O}(X)_q$ is regular if $\mathcal{O}_x(X)_p$ is also regular. Consequently, we deal with the latter ring.

Here we use the ideals $R_k$ introduced in [T, Chapter II] to formulate several jacobian criteria. First of all, the singular locus of $X$ is the set

$$\Sigma = \{z \in X: \mathcal{O}_z(X) \text{ is not regular}\}.$$  

Then, as $X$ is irreducible, by the Lemma in 1.1 and 1.2 we find

$$\dim \mathcal{O}_z(X) = \dim \mathcal{O}(X)_n = \dim(X),$$

where $n = \text{maximal ideal of } z$. Hence

$$\Sigma = \{x \in X: \mathcal{O}_z(X) \text{ is not regular of dimension } \dim(X)\}.$$  

Now, $\mathcal{O}_z(X) = \mathcal{O}_z/I\mathcal{O}_z$ and, from [T, Chapter II, Theorem 7.9], we get

$$\text{germ } \Sigma_x \text{ of } \Sigma \text{ at } x = V(R_k(I\mathcal{O}_x)), $$

with $k = \dim M - \dim X$. Finally, the germ $\sigma_x$ of (the image of) $\sigma$ at $x$ is contained in $V(p)$ but not in $\Sigma_k$, so that $V(p)$ is not contained in $\Sigma_x = V(R_k(I\mathcal{O}_x))$. In conclusion, $p$ does not contain $R_k(I\mathcal{O}_x)$ and, from [T, Chapter II, Theorem 3.1], we deduce that $\mathcal{O}_x(X)_p$ is a regular ring.

As remarked before, this implies 2.1. $\square$

On the other hand, the homomorphism $\psi$ induces an embedding $\kappa(q) \subset qf(\mathcal{R}\{t\})$, where $\kappa(q)$ stands for the residue field of $q$. Then we restrict the ordinary valuation of $qf(\mathcal{R}\{t\})$ to a valuation $\overline{\nu}$ of $\kappa(q)$. Coming back to our meromorphic function $h$, since $h \circ \sigma = at^m + \cdots, a \neq 0$, is defined, $h \in \mathcal{O}(X)_q$ and its class $\overline{h}$ in $\kappa(q)$ verifies $\overline{\nu}(\overline{h}) = m$.

Now we need 2.1: since $\mathcal{O}(X)_q$ is regular, $\overline{\nu}$ lifts to a valuation $\nu$ of $qf(\mathcal{O}(X)_q) = \mathcal{M}(X)$ with residue field $\kappa(q)$. Furthermore, the ring $W$ of $\nu$ dominates $\mathcal{O}(X)_q$ and so $h \in W$. Finally, consider the valuation $\nu$, composite of $\overline{\nu}$ and $\nu$. The canonical epimorphism $W \rightarrow \kappa(q)$ maps the ring $V$ of $\nu$ onto the ring $\overline{V}$ of $\overline{\nu}$.  

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After this preparation we use [B2, 1.9, p. 146]: as \( h \) is a sum of \( 2n \)th powers in \( \mathcal{M}(X) \), \( 2n \) divides \( \nu(h) \). In other words, there is \( g \in \mathcal{M}(X) \) such that \( g^{2n}/h \) is a unit of \( V \). In particular, \( g^{2n}/h \in V \subseteq W \) and \( g^{2n} = h(g^{2n}/h) \in W \). Thus \( g \) and \( h \) belong to \( W \), and, taking residue classes in \( \kappa(q) \), we have \( g, h \) such that \( g^{2n}/h = g^{2n}/h \) is a unit in \( V \). We conclude \( 2n \) divides \( \nu(h) = m \).

To end the proof we have to check that \( a > 0 \). Suppose, by way of contradiction, that \( a < 0 \). Then there would be regular points \( z = \sigma(t), t > 0 \), with \( h(z) < 0 \). Hence, it is enough to check that \( h \) is positive semidefinite off \( \Sigma \). For that, take any representation of \( h \) as a sum of \( 2n \)th powers:

\[
h = \sum_i f_i^{2n}/g^{2n}, \quad f_i, g \in \mathcal{O}(X), g \neq 0.
\]

Clearly \( h \) is positive semidefinite in \( \text{Reg}(X) \setminus \{g = 0\} \), and this set is dense in \( \text{Reg}(X) \) because \( X \) is irreducible. We are done.

**Remark.** The long argument used above is needed to handle the curves \( \sigma \) on which \( h \) can be evaluated, i.e. \( h = f/g \) with \( g \circ \sigma \neq 0 \), but this evaluation cannot be done on any expression \( h = \sum_i f_i^{2n}/g_i^{2n} \) because \( g_i \circ \sigma = 0 \), for some \( i \).

### 3. Proof of the sufficient condition

Let \( h \in \mathcal{M}(X) \) not be a sum of \( 2n \)th powers. We have to find an analytic curve \( \sigma: (-\varepsilon, \varepsilon) \to X \), not contained in \( \Sigma \) such that \( h \circ \sigma = at^m + \cdots, a \neq 0 \), and either \( a < 0 \) or \( 2n \) does not divide \( m \).

Since \( h \) is not a sum of \( 2n \)th powers in \( \mathcal{M}(X) \), applying the isomorphism 1.4.1, we find that \( h|X_i \) is not a sum of \( 2n \)th powers in \( \mathcal{M}(X_i) \) for some \( i \). Then choose an analytic function \( g \) which does not vanish identically on \( X_i \) and such that

\[
X_i \setminus \{g = 0\} = \text{Reg}(X_i) \setminus \bigcup_{j \neq i} X_j = \text{Reg}(X) \setminus \bigcup_{j \neq i} X_j = X \setminus \{g = 0\}
\]

and \( g^{2n}h = f \) is an analytic function.

Clearly, it is enough to find \( \sigma: (-\varepsilon, \varepsilon) \to X \) not contained in \( \{g = 0\} \) such that \( f \circ \sigma = at^m + \cdots, a \neq 0 \), and either \( a < 0 \) or \( 2n \) does not divide \( m \). Furthermore, formulated this way we can restrict ourselves to the case \( X = X_i \), i.e., to the case \( X \) is irreducible. In conclusion, we shall prove:

(3.1) **Claim.** Assume \( X \) is irreducible. Let \( f, g \in \mathcal{O}(X), g \neq 0 \), and \( f \) not a sum of \( 2n \)th powers in \( \mathcal{M}(X) \). Then, there is an analytic curve \( \sigma: (-\varepsilon, \varepsilon) \to X \) not contained in \( \{g = 0\} \), such that \( f \circ \sigma = at^m + \cdots, a \neq 0 \), and either \( a < 0 \) or \( 2n \) does not divide \( m \).

First, we distinguish a special case:

**Case when \( f \) is not a sum of squares in \( \mathcal{M}(X) \).** Here \( f \) must be negative in some ordering of the field \( \mathcal{M}(X) \), and so will be \( g^2f \). We deduce from [Rz,
Corollary 2.4], that there is some point $x \in X$ with $g(x)^2 f(x) < 0$. Finally, pick any analytic curve $\sigma: (-\varepsilon, \varepsilon) \to X$ with $\sigma(0) = x$. Clearly $g \circ \sigma \neq 0$ and

$$f \circ \sigma = a + bt + \cdots,$$

with $a = f(\sigma(0)) = f(x) < 0$. 

Once the precedent case is solved we consider the

Case when $f$ is a sum of squares in $\mathcal{M}(X)$. By [B2, 1.9, p. 146], there is a real valuation $\nu$ of the field $\mathcal{M}(X)$ such that $2n$ does not divide $\nu(f)$. Fix an ordering $\alpha$ of $\mathcal{M}(X)$ compatible with $\nu$ and denote by $V$ the convex hull of $\mathbf{R}$ in $\mathcal{M}(X)$ with respect to $\alpha$. This $V$ is a valuation ring with residue field $\mathbf{R}$ which dominates the local ring $A = \mathcal{O}(X)_m$ of some point $x \in X$ [Rz, Lemma 2.3]. Now, since $\alpha$ is compatible with $\nu$, the valuation ring of $\nu$ contains $V$ and so it is $V_p$ for some prime ideal $p$ of $V$. Hence, the value group of $V_p$ is a quotient of the one of $V$, and, as $2n$ does not divide the value of $f$ with respect to $V_p$, it does not divide the one with respect to $V$ either. All this means that we can suppose $V$ is the valuation ring of $\nu$.

We have $A \to V \subset K = qf(A) = \mathcal{M}(X)$. The integral closure $\overline{A}$ of $A$ in $K$ is contained in $V$, and $V$ dominates a local ring $B = \overline{A}_m$, where $m$ is a maximal ideal of $\overline{A}$, lying over the maximal ideal $m$ of $A$.

Now we consider the adic completions $A^\wedge$ and $B^\wedge$. Since $A$ is excellent (1.1), there is a unique zero divisor $q^\wedge$ of $A^\wedge$ such that $B^\wedge$ is the integral closure of $A^\wedge/q^\wedge$ in $K^\wedge = qf(A^\wedge/q^\wedge)$ [EGA, 7.6.2, p. 209]. We have the diagram

$$A^\wedge \to A^\wedge/q^\wedge \to (A^\wedge/q^\wedge)^\wedge = B^\wedge \subset K^\wedge$$

In this picture there is a room for the local ring $A^* = \mathcal{O}_x(X)$. First, by 1.1, $A^\wedge$ is the adic completion of $A^*$. Thus, the zero divisors of $A^\wedge$ are the extensions of the ones of $A^*$ [T, Chapter III, Corollary 4.8] and so $q^\wedge = q^* A^\wedge$ for a zero-divisor $q^*$ of $A^*$. Finally, denote by $B^*$ the integral closure of $A^*/q^*$ in $K^* = qf(A^*/q^*)$. We have come to the diagram

$$A^\wedge \to A^\wedge/q^\wedge \to (A^\wedge/q^\wedge)^\wedge = B^\wedge \subset K^\wedge$$

Indeed, $B^*$ is local [T, Chapter II, Proposition 2.5] and, again by [EGA] loc.cit., $B^\wedge$ is the adic completion of $B^*$. Hence, $B^\wedge$ dominates $B^*$ and $B^\wedge$ dominates $B$, and the maximal ideal of $B$ generates the one of $B^*$. Now consider the ordering $\alpha$ we had in $K$. Since $V$ is $\alpha$-convex and $V$ dominates $B$, $\alpha$ is a central ordering in the local excellent domain $B$, in the sense of [Rz, §1]. Then, by [Rz, Theorem 1.1], $\alpha$ extends to a total ordering
Let us denote by $\alpha^*$ the restriction of $\alpha$ to $B^*$, and by $V^*$ the convex hull of $R$ in $K^*$ with respect to $\alpha^*$. The valuation ring $V^*$ is an extension of $V^*$ has residue field $R$ and dominates $B^*$. We summarize the situation in the next diagram

$$
\begin{array}{c}
A^* \\ \downarrow \\
A \\
\end{array} \rightarrow \begin{array}{c}
A^*/q^* \\ \downarrow \\
A^*/q^* \\
\end{array} \rightarrow \begin{array}{c}
B^*/q^* \\ \downarrow \\
B^*/q^* \\
\end{array} \rightarrow \begin{array}{c}
K^* \\
\cup \\
K^* \\
\end{array} \\
\text{all residue fields are } R.

Now we apply Hironaka's resolution of singularities I and II ([H1], cf. also [H2, pp. 5.8-5.9]) to the local excellent ring $B$ and the ideal $J = fB$: there is a proper birational morphism $Z \rightarrow \text{Spec } B$, where $Z$ is a regular scheme and $J\mathfrak{O}_z$ is simple everywhere, i.e., for each $z \in Z$ there is a regular system of parameters $x_1, \ldots, x_k$ of $\mathfrak{O}_{Z,z}$ such that $J\mathfrak{O}_{Z,z} = (x_1^{p_1} \cdots x_k^{p_k})$.

In our situation, we choose the point $z$ as follows. Since $Z \rightarrow \text{Spec } B$ is proper and birational the valuation $V$ dominates a unique local ring of $Z$: this is our $\mathfrak{O}_{Z,z}$. But $\mathfrak{O}_{Z,z} = B[h_1, \ldots, h_s]_n$ for some elements $h_1, \ldots, h_s \in K$ and an ideal $n \subset B[h_1, \ldots, h_s]$. Finally notice that the residue field of $n$ is $R$, because this is the residue field of $V$ and $V$ dominates $\mathfrak{O}_{Z,z}$. As $R \subset B[h_1, \ldots, h_s]$, we can replace $h_1, \ldots, h_s$ by $h_1 - c_1, \ldots, h_s - c_s$ for suitable $c_i \in R$ to have $h_1, \ldots, h_s \in n$. Hence

**Conclusion.**

(3.1.3) The ring $B_1 = B[h_1, \ldots, h_s]_n$ is regular, $h_1, \ldots, h_s \in n$, and $n$ has a regular system of parameters $x_1, \ldots, x_k$ such that the element $f \in A$ factorizes in $B$ in the form $f = u x_1^{p_1} \cdots x_k^{p_k}$, where $u$ is a unit and $p_i \geq 0$.

Once we have this, we come back to our valuation $\nu$. It holds:

$$\nu(f) = p_1 \nu(x_1) + \cdots + p_k \nu(x_k),$$

and as $2n$ does not divide $\nu(f)$, it does not divide $p_i$ for some $i$, say $i = 1$.

To finish the proof of 3.1 we need a local homomorphism $\varphi: B \rightarrow R\{t\}$ such that

(3.1.4) $\varphi(x_1) = 1$; \hspace{1cm} $\varphi(x_j) = a_j t^{2n}$, \hspace{1cm} $a_j \neq 0$, for $j > 1$;

(3.1.5) $\varphi(g) \neq 0$.

Indeed, from $A = \mathfrak{O}(X)_m \rightarrow B_1 \xrightarrow{\varphi} R\{t\}$ we obtain an analytic curve $\sigma: (-\varepsilon, \varepsilon) \rightarrow X$ with $f \circ \sigma = \varphi(f)$, $g \circ \sigma = \varphi(g)$. Then, in view of 3.1.4 we have $f = at^m + \cdots$, where:

$a = a_1 a_2 \cdots a_k \neq 0$, $a_1 = \varphi(u)(0) \neq 0$ because $\varphi(u)$ is a unit of $R\{t\}$,
and $2n$ does not divide $m = p_1 + 2n \sum_{j > 1} p_j$, since it does not divide $p_1$.

Finally, by 3.1.5, $g \circ \sigma \neq 0$, so that $\sigma$ is not contained in $\{g = 0\}$. $\square$

Consequently, let us find $\varphi$ verifying 3.1.4 and 3.1.5.
First, consider $B^*[h_1, \ldots, h_s]$. Clearly $V^*$ contains this ring, and so it must dominate a localization $B^*_1 = B^*[h_1, \ldots, h_s]$. This gives a local inclusion $B_1 \to B_1^*$, which is faithfully flat, because

$$B^*[h_1, \ldots, h_s] = B^* \otimes_B B[h_1, \ldots, h_s]$$

and $B \to B^*$ is flat. It follows from [M, 13.B, Theorem 19, p. 79] that

$$\dim(B_1^*) = \dim(B_1) + \dim(B_1^*/nB_1^*).$$

We claim

$$nB_1^* = n^*.$$  

For $h_1, \ldots, h_s \in n$ implies $B_1^*/nB_1^* = B^*/B^* \cap nB_1^*$, and, since the maximal ideal $\overline{m}$ of $B$ generates the one $\overline{m}^*$ of $B^*$, we deduce $B^* \cap nB_1^* \supset mB^* = \overline{m}^*$, which gives $B_1^*/nB_1^* = B^*/\overline{m}^* = R$. Thus, $nB_1^*$ is a maximal ideal and it must coincide with $n^*$.

From (*), (**) and [M, 21.D, Theorem 51, p. 155], we conclude that $B_1^*$ is regular and

$$(3.1.6) \quad x_1, \ldots, x_k$$

are a regular system of parameters of $B_1^*$.

Our next step is

$$(3.1.7) \quad \text{There is a local embedding } B_1^* \to R\{x_1, \ldots, x_k\}.$$  

Indeed, the ring $B^*$ is an analytic ring, i.e., there is a local epimorphism $R\{y\} \to B^*$, $y' = (y_1, \ldots, y_r)$ (by [T, Chapter II, Proposition 2.3] and the fact that the residue field of $B^*$ is $R$). We extend it to another one,

$$R\{y\}[z] \to B^*[h_1, \ldots, h_s], \quad z = (z_1, \ldots, z_s),$$

by $z_i \to h_i$. Since $h_1, \ldots, h_s \in n^*$, the inverse image of $n^*$ is the maximal ideal $(y, z)$, and we get a local epimorphism

$$R\{y\}[z]_{(y, z)} \to B^*[h_1, \ldots, h_s]_{n^*} = B_1^*.$$  

Finally, if $I$ is the kernel, we obtain an isomorphism

$$R\{y\}[z]_{(y, z)}/I \to B_1^*.$$  

Thus, $R\{y\}[z]_{(y, z)}/I$ is a local regular ring of dimension $r + s - k$, and the inverse images $x'_{k+1}, \ldots, x'_{r+s}$ of $x_1, \ldots, x_k$ generate the maximal ideal $(y, z)/I$.

Now, $R\{y\}[z]_{(y, z)}$ being regular, there are $x_{k+1}^{'}, \ldots, x_{r+s}^{'}, z_1, \ldots, z_s$ which generate $I$. Then $x_1', \ldots, x_{r+s}'$ generate $(y, z)$ and also generate $(y, z)R[[y, z]]$, because $R[[y, z]]$ is the completion of $R\{y\}[z]_{(y, z)}$. Hence, by the inverse mapping theorem,

$$\frac{D(x_1', \ldots, x_{r+s}')} {D(y_1, \ldots, y_r, z_1, \ldots, z_s)(0)} \neq 0,$$

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and, since \( x'_1, \ldots, x'_r+s \) actually dwell in \( \mathbb{R}\{y, z\} \), they generate \((y, z)\mathbb{R}\{y, z\}\).

All this gives
\[
\mathbb{R}\{y\}[z]/(y, z)/I \rightarrow \mathbb{R}\{y, z\}/I/R\{y, z\} \rightarrow \mathbb{R}\{x'_1, \ldots, x'_k\},
\]
which, composed with the inverse of \( R/\{y\}[z]/I \rightarrow B_1^* \), is the embedding required in 3.1.7. □

After 3.1.7, we are ready to produce \( \varphi: B_1 \rightarrow R\{t\} \), verifying 3.1.4 and 3.1.5.

We put
\[
\varphi: B_1 \rightarrow B_1^* \rightarrow R\{x_1, \ldots, x_k\} \rightarrow R\{t\},
\]
\[
\tau(x_i) = t, \quad \text{and} \quad \tau(x_j) = a_j t^{2n}, \quad a_j \neq 0, \quad \text{for } j \neq 1.
\]

Thus, \( \varphi \) fulfills 3.1.4. In addition, one can always choose the \( a_j \)'s to have
\[
\tau(g) = g(t, a_2 t^{2n}, \ldots, a_k t^{2n}) \neq 0,
\]
since \( g \neq 0 \) in \( \mathbb{R}\{x_1, \ldots, x_k\} \). This is the condition 3.1.5.

As explained before, this definition of \( \varphi \) completes the proof. □

References


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