AN INTERPOLATION THEOREM
IN SYMMETRIC FUNCTION F-SPACES

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Abstract. It is well known that every separable or perfect symmetric Banach
function space $X$ is an interpolation space between $L^1$ and $L^\infty$ (see [1]
and [4]). In this paper we prove that every symmetric function $F$-space is
interpolation between $L^0$ and $L^\infty$, where $L^0$ is the space of all measurable
functions whose support has finite measure. Moreover, for any function $f \in
L^0 + L^\infty$ the norm $\|f\|_{L^0 + L^\infty}$ is computed in the terms of the nonincreasing
rearrangement function $f^*$ of $f$ as well as in terms of its distribution function
d$f$.

1. Introduction

In the sequel $(I, m)$ will denote the Lebesgue measure space on $I = [0, 1]$ or on $I = [0, \infty)$. By $S = S(I, m)$ we mean the space of all equivalence
classes of measurable real-valued functions defined on $I$.

Assume that $E = E(m)$ is a subgroup of $S(I, m)$ endowed with a functional
$\|\|_E : E \to \mathbb{R}_+$ that is even vanishing only at zero and satisfying the triangle
inequality. Then $E$ is said to be a function $F^*$-group. A function $F^*$-group
$(E, \|\|_E)$ such that $E$ is a linear space and the operation of multiplication
by scalars is continuous is called a function $F^*$-space. Any function $F^*$-group
($F^*$-space) that is a complete metric space with respect to the metric $d(f, g) =
\|f - g\|_E$ is called a function $F$-group ($F$-space). For details concerning $F$-
spaces see [7].

By $L^0 = L^0(m)$ we shall denote the space of all measurable functions defined
on $I$ whose support has finite measure. This space is endowed with the group-
norm $\|f\|_0 = m(\text{supp} f)$, where $\text{supp} f = \{t \in I : f(t) \neq 0\}$.

By $L^\infty = L^\infty(m)$ we shall mean the Banach space of all $m$-essentially bounded
functions endowed with the norm $\|f\|_\infty = \text{ess} \sup_{t \in I} |f(t)|$.

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The intersection and the algebraic sum of the spaces $L^0$ and $L^\infty$ shall be written as $L^0 \cap L^\infty$ and $L^0 + L^\infty$, respectively. The norms in these spaces will be defined as follows

$$\|f\|_{L^0 \cap L^\infty} = \max\{\|f\|_0, \|f\|_\infty\},$$

$$\|f\|_{L^0 + L^\infty} = \inf\{\|g\|_0 + \|h\|_\infty : f = g + h, \ g \in L^0, \ h \in L^\infty\}.$$

The spaces $L^0$ and $L^0 \cap L^\infty$ are function $F$-groups, and $L^0 + L^\infty$ is a function $F$-space (all are symmetric, see below).

In $f \in S(I, m)$, then the distribution function $d_f$ is defined by

$$d_f(s) = m(\{t \in I : |f(t)| > s\}), \quad s > 0.$$

The nonincreasing rearrangement of $f$ is given by

$$f^*(t) = \inf\{s > 0 : d_f(s) < t\}, \quad \inf\phi = \infty.$$

Clearly, $d_f$ and $f^*$ are nonincreasing right-continuous functions on $[0, \infty)$. Notice that if $mI = \infty$, then these functions may also take value $\infty$.

A function $F$-group ($F$-space) $(E, \|\|_E)$ is called symmetric (or rearrangement invariant) if:

1. $g \in S, f \in E,$ and $|g| \leq |f|$, a.e., imply $g \in E$ and $\|g\|_E \leq \|f\|_E$.
2. For any $g \in S$ equimeasurable with $f \in E$, i.e., if $d_f = d_g$, then one has $g \in E$ and $\|g\|_E = \|f\|_E$.

Conditions 1 and 2 are equivalent to the following:

1. If $g \in S, f \in E, g^*(t) \leq f^*(t),$ and all $t \in I$, then $g \in E$ and $\|g\|_E \leq \|f\|_E$. Some properties of symmetric function $F$-spaces have been studied in [3].

2. IMBEDDING THEOREM AND EXAMPLES

Our first result is related to the problem given at the end of [2].

**Theorem 1.** (a) If $E$ is a nontrivial symmetric function $F$-group then

$$L^0 \cap L^\infty \subset E.$$

(b) If $E$ also satisfies one of the following conditions:

(i) $\|1_A\|_E < \infty$ implies $mA < \infty$, or

(ii) $E$ is a symmetric function $F$-space,

then

$$E \subset L^0 + L^\infty.$$

**Proof.** (a) Let $0 \neq f \in E$. Define $A_n = \{t \in I : |f(t)| \geq \frac{1}{n}\}, \ n = 1, 2, \ldots$. We have $A_n \uparrow$ and $\bigcup_{n=1}^\infty A_n = \text{supp} f$. Thus, $mA_k > 0$ for some $k \in \mathbb{N}$. Since $k^{-1}1_{A_k} \leq |f|$, we have $k^{-1}1_{A_k} \in E$. By the assumption on $E$ we have $1_{A_k} \in E$.

Now, let $g \in L^0 \cap L^\infty$. There exist pairwise disjoint sets $B_1, B_2, \ldots, B_\ell$ such that $m(B_i) = m(A_k)$ for $i = 1, 2, \ldots, \ell$ and supp$g \subset \bigcup_{i=1}^\ell B_i$. All functions
$1_{B_i}(i = 1, 2, \ldots, \ell)$ and $1_{A_k}$ are pairwise equimeasurable. Since, in view of our assumption, $E$ is symmetric, we have

$$\|1_{B_1}\|_E = \|1_{B_2}\|_E = \cdots = \|1_{B_\ell}\|_E = \|1_{A_k}\|_E.$$ 

Hence

$$\|1_{\text{suppg}}\|_E \leq \left\| \sum_{i=1}^{\ell} 1_{B_i} \right\|_E \leq \sum_{i=1}^{\ell} \|1_{B_i}\|_E = \ell \|1_{A_k}\|_E < \infty$$

and so $\|g\|_E \leq \|g\|_\infty \|1_{\text{suppg}}\|_E \leq (\|g\|_\infty + 1) \|1_{\text{suppg}}\|_E < \infty$, where $\|g\|_\infty$ denotes the integer part of the number $\|g\|_\infty$. Thus, $g \in E$ and the inclusion (1) is proved.

(b) Assume first that $E$ satisfies condition (i). Let $f \in E$ and $A = \{t \in I : |f(t)| \leq 1\}$. Then $f1_A \in L^\infty$ and

$$\|1_{A\setminus A}\|_E \leq \|f1_{A\setminus A}\|_E \leq \|f\|_E < \infty.$$ 

Thus, in view of (i), $m(I \setminus A) < \infty$, i.e., $f1_{I \setminus A} \in L^0$.

Now assume that $E$ satisfies condition (ii) and $f \in E$. Define $A_n = \{t \in I : |f(t)| \geq n\}, n = 1, 2, \ldots$. We shall prove that $mA_k < \infty$ for some $k \in \mathbb{N}$. In fact, if not, then we have $mA_n = \infty$ for any $n \in \mathbb{N}$. Choose $B_n \subset A_n$ with $mB_n = 1$ for any natural $n$. All functions $1_{B_n}$ are equimeasurable, and so

$$0 < \|1_{B_n}\|_E = \|1_{B_n}\|_E \leq \left\| \frac{1}{n} f1_{B_n} \right\|_E \leq \left\| \frac{1}{n} f \right\|_E \rightarrow 0 \text{ as } n \rightarrow \infty.$$ 

Thus, $\|1_{B_n}\|_E = 0$, a contradiction. Since $f1_{A_k} \in L^0$ and $f1_{I \setminus A_k} \in L^\infty$, the proof is finished.

Remarks. (1). O'Neil proved in [5] that the symmetric Orlicz group $L^\psi$ generated by generalized Young function $\psi$ satisfies $L^0 \cap L^\infty \subset L^\psi \subset L^0 + L^\infty$. Moreover, $L^\psi$ is an $F$-space, i.e., the scalar multiplication is continuous in $L^\psi$ if and only if $\lim_{u \to 0} \psi(u) = 0$.

(2). Let us note that in (b) it can be obtained that if $f \in E$ then for every $\varepsilon > 0$ there are functions $f_1 \in L^\infty$ and $f_2 \in L^0$, with $f_1 + f_2 = f$ such that $\|f_1\|_\infty \leq \varepsilon$ in the first case and $\|f_2\|_0 \leq \varepsilon$ in the second one.

Example 1. Let $E = S$ on $I = [0, \infty)$ with the $F$-group norm

$$\|f\|_E = \frac{m(\text{suppf})}{1 + m(\text{suppf})} \quad \text{(taking by definition } \frac{\infty}{1 + \infty} = 1).$$

Then $E$ does not satisfy any condition (i) and (ii) from Theorem 1, and $E \not\subset L^0 + L^\infty$.

Thus, it is interesting to pose the following problem:

Is the alternative of conditions (i) and (ii) necessary for (2)? Now, we give examples of symmetric function $F$-spaces and $F$-groups connected with Orlicz and Marcinkiewicz spaces.
Example 2. Let \( \psi \) be an Orlicz function, i.e., a continuous increasing function on \([0, \infty)\) such that \( \psi(0) = 0 \). The Orlicz space \( L^\psi \) is the space of all \( f \in S \) such that \( I_\psi(\lambda f) = \int f \psi(\lambda |f(t)|) \, dm < \infty \) for some \( \lambda > 0 \) depending on \( f \).

The functional \( \| \cdot \|_\psi \) defined on \( L^\psi \) by \( \|f\|_\psi = \inf\{\lambda > 0 : I_\psi(f/\lambda) \leq \lambda\} \) is an \( F \)-norm. Define a new Orlicz function \( \psi \) by

\[
\psi(u) = \frac{\phi(u)}{1 + \phi(u)}.
\]

Then \( L^\psi = L^0 + L^\psi \) and \( \|f\|_\psi \leq \|f\|_+ \leq 4\|f\|_\psi \), where

\[
\|f\|_+ = \inf\{\|f_0\|_0 + \|f_1\|_\psi : f = f_0 + f_1, \ f_0 \in L^0, \ f_1 \in L^\psi\}.
\]

In fact, from the inequality \( \psi(u) \leq \min(1, \phi(u)) \) we have \( L^0 \subset L^\psi \) and \( L^\psi \subset L^\psi \), which imply \( L^0 + L^\psi \subset L^\psi \) and the first inequality on the \( F \)-norms. On the other hand, if \( f \in L^\psi \) and \( \|f\|_\psi < \lambda \) then \( I_\psi(f/\lambda) \leq \lambda \), and if \( A = \{t \in I : |f(t)| > \lambda \phi^{-1}(1)\} \), then

\[
\frac{mA}{2} \leq I_\psi(f1_A/\lambda) \leq I_\psi(f/\lambda) \leq \lambda.
\]

Hence \( f1_A \in L^0 \) and

\[
I_\psi(f1_{I \setminus A}/\lambda) \leq 2I_\psi(f1_{I \setminus A}/\lambda) \leq 2I_\psi(f/\lambda) \leq 2\lambda,
\]

i.e., \( f1_{I \setminus A} \in L^\psi \). Thus \( f \in L^0 + L^\psi \) and \( \|f\|_+ \leq \|f1_A\|_0 + \|f1_{I \setminus A}\|_\psi \leq mA + 2\lambda \leq 4\lambda \).

Example 3. Let \( \psi \) be an Orlicz function. Define the Marcinkiewicz space \( M^\psi \) generated by functional

\[
\|f\|_{M^\psi} = \inf\{\lambda > 0 : \sup_{t > 0} \frac{f^*(t)}{\psi^{-1}(\lambda/t)} \leq \lambda\}.
\]

Then \( M^\psi \) is the symmetric function \( F \)-group. It is sufficient to prove the triangle inequality for \( \| \cdot \|_{M^\psi} \). If \( \|f\|_{M^\psi} < \lambda_1 \) and \( \|g\|_{M^\psi} < \lambda_2 \), then \( f^*(t) \leq \lambda_1 \psi^{-1}(\lambda_1/t) \) and \( g^*(t) \leq \lambda_2 \psi^{-1}(\lambda_2/t) \) for any \( t > 0 \). Hence, by the property of rearrangement and the above,

\[
(f + g)^*(t) \leq f^* \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} t \right) + g^* \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} t \right)
\]

and so \( \|f + g\|_{M^\psi} \leq \lambda_1 + \lambda_2 \), which gives the triangle inequality. Let us note that

\[
\|f\|_{M^\psi} = \inf\{\lambda > 0 : \sup_{s > 0} \psi(s/\lambda)d_f(s) \leq \lambda\}
\]

and

\[
\|f\|_{M^\psi} \leq \|f\|_\psi \quad \text{for} \quad f \in L^\psi.
\]
$M^\psi$ is a symmetric function $F$-space, i.e., the scalar multiplication is continuous in $M^\psi$ if and only if $\psi^{-1}$ satisfies the $\Delta_2$-condition (see [5, Theorem 9.9]). Moreover, $M^\psi(0, 1)$ is separable if and only if $\lim_{u \to \infty} \psi(u) = c < \infty$ and $M^\psi(0, \infty)$ is not separable (see [5, Theorems 9.14 and 9.15]).

3. AN INTERPOLATION THEOREM

The first result is closely connected with the approximation spaces introduced by Peetre and Sparr in [6].

**Proposition 1.** Let $f \in S$ and $0 \leq s$, $t < ml$. Then

\[
(3) \quad f^*(t) = \inf \{ \| f - g \|_\infty : \| g \|_0 \leq t \} = \inf \{ \| f - f 1_A \|_\infty : mA \leq t \}
\]

and

\[
(4) \quad d_f(s) = \inf \{ \| f - h \|_0 : \| h \|_\infty \leq s \}.
\]

**Proof.** Let $f^*(t) < \infty$ and $B = \{ x \in I : |f(x)| > f^*(t) \}$. Then $mB = d_f(f^*(t)) \leq t$ and

\[
E(t, f) = \inf \{ \| f - g \|_\infty : \| g \|_0 \leq t \} \leq \inf \{ \| f - f 1_A \|_\infty : mA \leq t \} \leq \| f - f 1_B \|_\infty = \| f 1_{I \setminus B} \|_\infty \leq f^*(t).
\]

If $f^*(t) = \infty$ then $ml = \infty$ and $f \notin L^\infty$. Hence for any $g \in L^0$ with $\| g \|_0 \leq t$, we have $f - g \notin L^\infty$ and $E(t, f) = \infty$. Conversely, if $E(t, f) < \infty$, then for any $\varepsilon > 0$ there exists $g \in L^0$ with $\| g \|_0 \leq t$ such that $\| f - g \|_\infty < E(t, f) + \varepsilon$. Assuming $\| f - g \|_0 = u$ we get $|f| - |g| \leq |f - g| \leq u$, a.e. in $I$, and so

\[
\{ x \in I : |f(x)| > u \} \subset \{ x \in I : |g(x)| > 0 \} \cup A,
\]

where $A \subset I$ is a set of measure zero. Thus,

\[
d_f(u) \leq \| g \|_0 \leq t \quad \text{and} \quad f^*(t) \leq u = \| f - g \|_\infty < E(t, f) + \varepsilon.
\]

Since $\varepsilon > 0$ is arbitrary, the proof of (3) is complete. If $h \in L^\infty$ is such that $\| h \|_\infty \leq s$, then

\[
|f(x) - h(x)| \geq |f(x)| - |h(x)| \geq |f(x)| - s \quad \text{a.e. in } I
\]

and so

\[
A_s = \{ x \in I : |f(x)| > s \} \subset \{ x \in I : |f(x) - h(x)| > 0 \} \cup A,
\]

where $A \subset I$ is a set of measure zero. Thus $d_f(s) = mA_s \leq \| f - h \|_0$. Moreover, taking $h_0 = f 1_{I \setminus A_s}$ we have $\| h_0 \|_\infty \leq s$ and $\| f - h_0 \|_0 = \| f 1_{A_s} \|_0 = d_f(s)$. Thus

\[
\inf \{ \| f - h \|_0 : \| h \|_\infty \leq s \} = d_f(s).
\]

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Proposition 2. Let \( E \) be a symmetric function \( F \)-space.

(a) If \( A \) is a subset of \( I \) of finite measure and \( f, f_n \in E \) are such that \( f_n \to f \) uniformly in \( A \setminus B \), with \( mB = 0 \), then \( \|(f - f_n)1_A\|_E \to 0 \) as \( n \to \infty \).

(b) The set of countable valued real functions is dense in \( E \).

(c) An operator \( \sigma_a(a > 0) \) defined on \( E \) by \( \sigma_a f(t) = f(t/a) \) is continuous in \( E \) and

\[
\|\sigma_a f\|_E \leq K_a \|f\|_E
\]

for any \( f \in E \) and for a constant \( K_a > 0 \) independent of \( f \).

**Proof.** (a) From Theorem 1 and the fact that \( mA \) is finite it follows that \( 1_A \in E \). Now, the continuity of the multiplication operation yields

\[
\|(f - f_n)1_A\|_E \leq \|f - f_n\|_\infty \cdot 1_A \to 0 \quad \text{as} \quad n \to \infty.
\]

(b) Let \( f \in E \) and let \( \{I_n\}_{n=1}^\infty \) be a sequence of pairwise disjoint sets of positive finite measure such that \( \bigcup_{n=1}^\infty I_n = I \). For a given \( \varepsilon > 0 \) and \( n \in \mathbb{N} \) there is a sequence \( \{g_{n,m}\}_{m=1}^\infty \) of countable valued functions with supports in \( I_n \) and such that \( g_{n,m} \to f \) uniformly in \( I_n \setminus A_n \) where \( mA_n = 0 \), as \( m \to \infty \). Thus, in view of (a), we have

\[
\|f1_{I_n} - g_{n,m}\|_E \leq 2^{-n}\varepsilon
\]

for an \( m \in \mathbb{N} \), \( n = 1, 2, \ldots \). Denote, for brevity, \( g_{n,m} \), by \( g_n \) and define \( g = \sum_{n=1}^\infty g_n \). Then

\[
\|f - g\|_E = \left\| \sum_{n=1}^\infty (f1_{I_n} - g_n) \right\|_E \leq \sum_{n=1}^\infty \|f1_{I_n} - g_n\|_E \leq \sum_{n=1}^\infty 2^{-n}\varepsilon = \varepsilon.
\]

(c) By virtue of (b), the proof proceeds in the same way as the proof of Theorem 4.4 in [4].

**Theorem 2.** Let \( T: L^0 + L^\infty \) be a homomorphism, i.e.,

\[
T(f + g) = Tf + Tg \quad \text{and} \quad T(-f) = -Tf
\]

for any \( f, g \in L^0 + L^\infty \). Assume that \( T \) maps \( L^0 \) into \( L^0 \) and \( L^\infty \) into \( L^\infty \), and

\[
m(\text{supp} Tf) \leq M_0 m(\text{supp} f) \quad \forall f \in L^0,
\]

\[
\|Tf\|_\infty \leq M_1 \|f\|_\infty \quad \forall f \in L^\infty.
\]

Then \( T \) maps any symmetric function \( F \)-space \( E \) into itself and there exists a constant \( M > 0 \) such that

\[
\|Tf\|_E \leq M \|f\|_E
\]

for any \( f \in E \).

**Proof.** First we shall prove that

\[
(Tf)^*(M_0 t) \leq M_1 f^*(t)
\]

for any \( f \in L^0 + L^\infty \) and \( t > 0 \).
For this purpose note that for any \( g \in L^0 \) with \( \|g\|_0 \leq t \) we have \( \|Tg\|_0 \leq M_0\|g\|_0 \leq M_0t \) and, in view of (3)

\[
(Tf)^*(M_0t) \leq \|Tf - Tg\|_\infty = \|T(f - g)\|_\infty \leq M_1\|f - g\|_\infty,
\]

i.e.,

\[
(Tf)^*(M_0t) \leq M_1E(t, f) = M_1f^*(t).
\]

Let \( K > 0 \) be a constant such that \( \|\sigma_{M_0}\|_E \leq K\|f\|_E \) for any \( f \in E \) (see Proposition 2(c)). Since each \( f \in E \) belongs to \( L^0 + L^\infty \) (see Theorem 1) we can apply inequality (8) and we get \( \|Tf\|_E = \|(Tf)^*\|_E \leq K\|\sigma_{1/M_0}(Tf)^*\|_E \leq K\|M_1f^*\|_E \leq K([M_1] + 1)\|f^*\|_E = K([M_1] + 1)\|f\|_E \), where \([M_1]\) denotes the integer part of the number \( M_1 \). The proof is finished.

It is interesting, and useful as well, to describe the space \( L^0 + L^\infty \) and the \( F \)-norms \( K(u, f) = \inf\{\|g\|_0 + u\|h\|_\infty : f = g + h, \ g \in L^0, \ h \in L^\infty \} \) \((u > 0)\) of any function \( f \in L^0 + L^\infty \) in terms of its distribution function \( d_f \) as well as in terms of its nonincreasing rearrangement \( f^* \).

**Proposition 3.** The space \( L^0 + L^\infty \) consists of all functions \( f \) in \( S \) such that \( d_f(s) < \infty \) for some \( s > 0 \). Moreover, we have

\[
K(u, f) = \inf_{s>0}[su + d_f(s)] = \inf_{t>0}[t + uf^*(t)].
\]

**Proof.** Assume that \( d_f(s) < \infty \) for some \( s > 0 \), i.e., \( mA_s < \infty \) where \( A_s = \{x \in I : |f(x)| > s\} \). We have \( f1_{A_s} \in L^0, \ f1_{\sim A_s} \in L^\infty \), \( f = f1_{A_s} + f1_{\sim A_s} \) and so \( f \in L^0 + L^\infty \). Now, assume that \( f \in L^0 + L^\infty \), i.e., \( f = g + h \) with \( g \in L^0 \) and \( h \in L^\infty \). Let \( s > 2\|h\|_\infty \). Then

\[
\{x \in I : |f(x)| > s\} \subset \{x \in I : |g(x)| + |h(x)| > s\}
\]

\[
\subset \{x \in I : |g(x)| > s/2\} \cup \{x \in I : |h(x)| > s/2\}
\]

\[
= \{x \in I : |g(x)| > s/2\} \cup A.
\]

where \( mA = 0 \). Hence, \( d_f(s) \leq d_g(s/2) \leq m(\text{supp}g) < \infty \). If \( f \in L^0 + L^\infty \) then for any \( \varepsilon > 0 \) there exists a decomposition \( f = g + h \) such that \( \|g\|_0 + u\|h\|_\infty < K(u, f) + \varepsilon \). Let \( \|g\|_0 = a \) and \( \|h\|_\infty = b \). Then by Proposition 1,

\[
\inf_{t>0}[t + uf^*(t)] \leq a + uf^*(a)
\]

\[
= a + uE(a, f) \leq a + u\tilde{E}(a, f) \leq a + u\|h\|_\infty
\]

\[
= \|g\|_0 + u\|h\|_\infty < K(u, f) + \varepsilon,
\]

where \( \tilde{E}(a, f) = \inf\{\|f - g\|_\infty : \|g\|_0 = a\} \), and

\[
\inf_{s>0}[su + d_f(s)] \leq bu + d_f(b) \leq bu + \|g\|_0 = \|g\|_0 + u\|h\|_\infty < K(u, f) + \varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, we have that \( \inf_{t>0}[t + uf^*(t)] \) and \( \inf_{s>0}[su + d_f(s)] \) do not exceed \( K(u, f) \). On the other hand, if \( d_f(s_0) < \infty \), then for \( s > s_0 \) the
measure of the sets $A_s = \{x \in I: |f(x)| > s\}$ is finite and their intersection has measure zero. Therefore $\lim_{s \to \infty} d_f(s) = 0$ and so $f^*(t) < \infty$ for any $t > 0$. Setting $A = \{x \in I: |f(x)| > f^*(t)\}$ we have $mA = d_f(f^*(t)) \leq t$ and

$$K(u, f) \leq \|f 1_A\|_0 + u\|f 1_{I\setminus A}\|_\infty \leq mA + uf^*(t) \leq t + uf^*(t)$$

for any $t > 0$. Thus

$$K(u, f) \leq \inf_{t > 0} [t + uf^*(t)].$$

For any $\epsilon > 0$ there exists an $s_0 > 0$ such that $s_0u + d_f(s_0) < \inf_{s > 0} [su + d_f(s)] + \epsilon$. Then

$$K(u, f) \leq \|f 1_{A_{s_0}}\|_0 + u\|f 1_{I\setminus A_{s_0}}\|_\infty \leq d_f(s_0) + us_0 < \inf_{s > 0} [su + d_f(s)] + \epsilon$$

and (9) is proved.

Remarks. (3) Putting together the results in [6, Proposition 4.4] and Proposition 1, we also get the proof of (9).

(4) All our results are also true for arbitrary measure space $(\Omega, \Sigma, \mu)$.

(5) Inequality $E(t, f) \leq \tilde{E}(t, f)$ always holds and if the measure is non-atomic, then it is possible to prove even the equality $E(t, f) = \tilde{E}(t, f)$.

References