THE CORONA PROPERTY FOR BOUNDED ANALYTIC FUNCTIONS IN SOME BESOV SPACES

ARTUR NICOLAU

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Abstract. In this paper, the corona theorem for the algebra of bounded analytic functions in the unit disc which are in the Besov space \( B_p \), \( 1 < p < \infty \), is proved.

Let \( \Delta \) be the open unit disc in the complex plane and let \( H^\infty \) be the Banach space of all bounded analytic functions on \( \Delta \) with the norm

\[ ||f||_\infty = \sup\{|f(z)| : z \in \Delta\}. \]

For \( 1 < p < \infty \), let \( B_p \) be the class of all analytic functions \( f \) on \( \Delta \) such that

\[ \|f\|^p_{B_p(\Delta)} = \frac{1}{\pi} \int_{\Delta} |f'(z)|^p (1 - |z|)^{p-2} \, dm(z) < \infty. \]

It is easy to see that \( H^\infty \cap B_p \) is a Banach algebra with the norm \( ||f|| = ||f||_\infty + \|f\|_{B_p(\Delta)} \). In this note we consider the corona problem for this algebra.

Let \( \mathcal{M} \) be the maximal ideal space of \( H^\infty \cap B_p \) endowed with the Gelfand topology. It is clear that \( \Delta \) is naturally embedded in \( \mathcal{M} \). The corona problem consists of knowing if \( \Delta \) is dense in \( \mathcal{M} \). Here we answer this question in the affirmative. As is known (see [2, p. 191]), this turns out to be equivalent to the following result.

Theorem. Let \( 1 < p < \infty \). Given \( f_1, \ldots, f_n \in H^\infty \cap B_p \) such that

\[ \max_j |f_j(z)| \geq \delta > 0, \quad z \in \Delta \]

there exist \( g_1, \ldots, g_n \in H^\infty \cap B_p \) such that

\[ f_1 g_1 + \cdots + f_n g_n = 1. \]
Note that for $p = 2$, $H^\infty \cap B_2 A$ is the space of bounded analytic functions with finite Dirichlet integral. So our result contains the answer to a question of [3].

**Proof of the theorem.** By a normal families argument, we can assume that the corona data $f_1, \ldots, f_n$ are analytic on a neighborhood of the closed unit disk, and we have to find analytic functions $g_1, \ldots, g_n$ satisfying (2) and

$$
\|g_i\| \leq C \quad i = 1, \ldots, n,
$$

where $C$ is a constant depending on $\delta$, $\|f_1\|, \ldots, \|f_n\|$.

It is clear that

$$
\varphi_i(z) = \frac{f_j(z)}{\sum_{i=1}^{n} |f_i(z)|^2}
$$

is a nonanalytic solution of (2).

As in the case of $H^\infty$ (see [2, Chapter VIII]), our problem is equivalent to solving, with bounds, the following equations. For $1 \leq j, k \leq n$, find $b_{j,k}$ such that

$$
(3) \quad \bar{\partial} b_{j,k} = \varphi_j \bar{\partial} \varphi_k \quad \text{in } \Delta
$$

with

$$
(4) \quad \|b_{j,k}\|_{L^\infty(T)} + \int_\Delta |\nabla b_{j,k}(z)|^{p}(1 - |z|)^{p-2} dm(z) \leq C,
$$

where $C$ is a constant depending on $\delta$, $\|f_1\|, \ldots, \|f_n\|$.

It is sufficient to deal with an equation $\partial b = g$ where $g = \varphi_j \partial \varphi_k$. Applying (1), a calculation (see [2, p. 326]) gives

$$
(5) \quad |g(z)| \leq M \sum_{j=1}^{n} |f'_j(z)|,
$$

where $M$ is a constant depending on $\delta$.

In order to find a solution of (3) with bounded $L^\infty(T)$ norm it suffices to show that $|g(z)| dm(z)$ is a Carleson measure (see [2, p. 320]). Let us see that this is true.

Put $Q_h = \{z \in \Delta: |z| \geq 1 - h \text{ and } \theta - h \leq \text{Arg} z \leq \theta + h\}$. From (5) one has
\[
\int_{Q_h} |g(z)| \ dm(z) \leq M \sum_{j=1}^{n} \int_{Q_h} |f_j'(z)| \ dm(z) \\
= M \sum_{j=1}^{n} \int_{Q_h} |f_j'(z)|(1 - |z|)^{1-2/p} \ dm(z) \\
\leq M \sum_{j=1}^{n} \left[ \int_{Q_h} |f_j'(z)|^p (1 - |z|)^{p-2} \ dm(z) \right]^{1/p} \\
\times \left[ \int_{Q_h} (1 - |z|)^{-1-2/p/p} \ dm(z) \right]^{(p-1)/p} \\
\leq (p - 1) M \sum_{j=1}^{n} \left( \int_{\Delta} |f_j'(z)|^p (1 - |z|)^{p-2} \ dm(z) \right)^{1/p} \\
\leq (p - 1) M \sup_i \|f_i'\| \cdot h.
\]

And so, (3) can be solved by means of bounded functions and one has

(6) \[\inf\{\|H\|_{L^\infty(T)} : H \text{ solves } (3)\} \leq C.\]

In order to obtain a solution of (3) bounded with respect to the norm

\[\|b\|_{B_p(A)}^p = \int_{\Delta} |\nabla b(z)|^p (1 - |z|)^{p-2} \ dm(z)\]

let us take

\[H_0(z) = \frac{1}{\pi} \int_{\Delta} \frac{g(\xi)}{\xi - z} \ dm(\xi).\]

One has \(\partial H_0 = g\) in \(\Delta\). So, applying (5),

\[\int_{\Delta} |\partial H_0(z)|^p (1 - |z|)^{p-2} \ dm(z) \leq C.\]

Furthermore, \(\partial H_0\) is the Beurling transform of \(g\). Since \((1 - |z|)^{p-2}\) is an \(A_p\) weight for \(1 < p < \infty\) (see [1, p. 411]), one has

\[\int_{\Delta} |\partial H_0(z)|^p (1 - |z|)^{p-2} \ dm(z) \leq K(p) \int_{\Delta} |g(z)|^p (1 - |z|)^{p-2} \ dm(z) \leq K(p) C\]

because of (5). So

(7) \[\|H_0\|_{B_p(\Delta)} \leq C.\]

Nevertheless, the problem is to solve the \(\partial\) equation (3) by means of a function \(b\) satisfying simultaneously the two bounds

\[\|b\|_{L^\infty(T)} \leq C \quad \text{and} \quad \|b\|_{B_p(\Delta)} \leq C.\]
To do this, for $1 < p < \infty$, let us consider the Besov class $B_p(T)$ formed by those functions in $L^p(T)$ such that

$$\|f\|_{B_p(T)}^p = \int_{-\pi}^{\pi} \frac{1}{h^2} \int_{-\pi}^{\pi} |f(e^{it+h}) - f(e^{it})|^p \, dt \, dh < \infty.$$ 

If $f \in L^p(T)$ and $u$ denotes its Poisson integral, it is well known (see [5, p. 152]) that there exist an absolute constant $M$ such that

$$M^{-1} \int_{\Delta} |\nabla u(z)|^p (1 - |z|)^{p-2} \, dm(z) \leq M \int_{\Delta} |\nabla u(z)|^p (1 - |z|)^{p-2} \, dm(z).$$

(8)

Claim. $\|H_0\|_{B_p(T)} \leq C$.

Of course, since $H_0$ is not harmonic, the claim cannot be deduced automatically from (7) and (8). Assume the claim is true and let us finish the proof of the theorem.

Since $\partial H_0 = g$ and (6), one has

$$\inf\{\|H_0 - F\|_{\infty} : F \in \text{BMOA}\} = \inf\{\|H\|_{\infty} : H \text{ solves } (3)\} \leq C,$$

where BMOA is the space of analytic functions on $\Delta$ with boundary values of bounded mean oscillation.

Peller and Hruscev proved that $B_p(T)$ has the best approximation property, for $1 < p < \infty$ (see [4, p. 103]). So, there exists a unique $F_0 \in \text{BMOA}$ satisfying

$$\|H_0 - F_0\|_{\infty} = \inf\{\|H_0 - F\|_{\infty} : F \in \text{BMOA}\} \leq C$$

and furthermore

(9)

$$\|F_0\|_{B_p(T)} \leq K \|H_0\|_{B_p(T)}.$$

Therefore $H_0 - F_0$ is a solution of the $\partial\bar{\partial}$ equation (3), satisfying $\|H_0 - F_0\|_{\infty} \leq C$. Now, apply (7), (8), (9), and the claim to get

$$\left(\int_{\Delta} |\nabla (H_0 - F_0)(z)|^p (1 - |z|)^{p-2} \, dm(z)\right)^{1/p} \leq \left(\int_{\Delta} |\nabla H_0(z)|^p (1 - |z|)^{p-2} \, dm(z)\right)^{1/p} + 2 \left(\int_{\Delta} |F_0'(z)|^p (1 - |z|)^{p-2} \, dm(z)\right)^{1/p} \leq C.$$

So $H_0 - F_0$ satisfies (3) and (4).

Proof of the claim. First of all, we remark that

$$\|H_0\|_{\text{BMO}(T)} \leq C.$$
with the constant $C$ depending only on the data of the corona problem. Because of [6, Theorem 1.1.2.], one only has to check that $|\nabla H_0(z)|\, dm(z)$ is a Carleson measure with norm only depending on $\delta$, $\|f_1\|$, \ldots, $\|f_n\|$, and in fact, this has been done in the proof of (6).

To prove the claim, we have to show that

$$
\int_{-\pi}^{\pi} \frac{1}{h^2} \int_{-\pi}^{\pi} |H_0(e^{i(s+h)}) - H_0(e^{is})|^p \, ds \, dh \leq C.
$$

Since $\|H_0\|_{BMO(\mathbb{T})} \leq C$, one has $\int_{-\pi}^{\pi} |H_0(e^{i\theta})|^p \, d\theta \leq AC$ where $A$ is an absolute constant. Therefore, by symmetry on $h$, in order to prove (10), it suffices to verify

$$
\int_0^{1/2} \frac{1}{h^2} \int_{-\pi}^{\pi} |H_0(e^{i(s+h)}) - H_0(e^{is})|^p \, ds \, dh \leq C.
$$

Let us just reproduce a proof of the second inequality in (8) and let us see that the harmonicity is not used.

Take $r = 1 - h$ and let $\partial H_0/\partial n$, $\partial H_0/\partial \theta$ be the derivatives of $H_0$ with respect to the radius and the argument. We have

$$
|H_0(e^{i(s+h)}) - H_0(e^{is})| \\
\leq |H_0(e^{i(s+h)}) - H_0(re^{i(s+h)})| + |H_0(re^{i(s+h)}) - H_0(re^{is})| \\
+ |H_0(re^{is}) - H_0(e^{is})| \\
\leq \int_r^1 \left| \frac{\partial H_0}{\partial n}(\xi e^{i(s+h)}) \right| \, d\xi + \int_0^h \left| \frac{\partial H_0}{\partial \theta}(re^{i(s+\phi)}) \right| \, d\phi + \int_r^1 \left| \frac{\partial H_0}{\partial n}(\xi e^{is}) \right| \, d\xi.
$$

Apply Minkowski integral inequality [5, p. 271], to get

$$
\left( \int_{-\pi}^{\pi} |H_0(e^{i(s+h)}) - H_0(e^{is})|^p \, ds \right)^{1/p} \\
\leq \int_r^1 \left( \int_{-\pi}^{\pi} \left| \frac{\partial H_0}{\partial n}(\xi e^{i(s+h)}) \right|^p \, ds \right)^{1/p} \, d\xi \\
+ \int_0^h \left( \int_{-\pi}^{\pi} \left| \frac{\partial H_0}{\partial \theta}(re^{i(s+\phi)}) \right|^p \, ds \right)^{1/p} \, d\phi \\
+ \int_r^1 \left( \int_{-\pi}^{\pi} \left| \frac{\partial H_0}{\partial n}(\xi e^{is}) \right|^p \, ds \right)^{1/p} \, d\xi \\
= 2 \int_r^1 \left( \int_{-\pi}^{\pi} \left| \frac{\partial H_0}{\partial n}(\xi e^{is}) \right|^p \, ds \right)^{1/p} \, d\xi + h \left( \int_{-\pi}^{\pi} \left| \frac{\partial H_0}{\partial \theta}(re^{is}) \right|^p \, ds \right)^{1/p} \\
= (I) + (II).
$$
Changing to planar coordinates and applying (7), one gets
\[
\int_0^{1/2} \frac{1}{h^2} (II)^p \, dh = \int_0^{1/2} h^{p-2} \int_{-\pi}^\pi \left| \frac{\partial H^0_0(re^{i\theta})}{\partial \theta} \right|^p \, ds \, dh \\
= \int_0^{1/2} h^{p-2} \int_{-\pi}^\pi \left| \frac{\partial H^0_0((1-h)e^{i\theta})}{\partial \theta} \right|^p \, ds \, dh \\
\leq 2 \int_\Delta |\nabla H^0_0(z)|^p (1-|z|)^{p-2} \, dm(z) \leq C.
\]

For the term \((I)\), put \(x = 1 - \xi\) and apply Hardy’s inequality ([5, p. 272]) to obtain
\[
\int_0^{1/2} \frac{1}{h^2} (I)^p \, dh = 2^p \int_0^{1/2} \frac{1}{h^2} \left[ \int_1^h \left( \int_{-\pi}^\pi \left| \frac{\partial H^0_0(\xi e^{i\theta})}{\partial n} \right|^p \, ds \right)^{1/p} \, d\xi \right]^p \, dh \\
= 2^p \int_0^{1/2} \frac{1}{h^2} \left[ \int_0^h \left( \int_{-\pi}^\pi \left| \frac{\partial H^0_0((1-x)e^{i\theta})}{\partial n} \right|^p \, ds \right)^{1/p} \, dx \right]^p \, dh \\
\leq 2^p K(p) \int_0^{1/2} h^{-2+p} \int_{-\pi}^\pi \left| \frac{\partial H^0_0((1-h)e^{i\theta})}{\partial n} \right|^p \, ds \, dh \\
\leq 2^{p+1} K(p) \int_\Delta |\nabla H^0_0(z)|^p (1-|z|)^{p-2} \, dm(z) \leq C
\]

because of (7).

This gives (10) and therefore we have proved the claim. This completes the proof of the theorem. □

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REFERENCES


DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA, BARCELONA, SPAIN