COMMUTATIVE BANACH ALGEBRAS WHICH SATISFY A BOCHNER-SCHOENBERG-EBERLEIN-TYPE THEOREM

SIN-EI TAKAHASI AND OSAMU HATORI

(Communicated by Paul S. Muhly)

Dedicated to Professor Junzo Wada

Abstract. A class of commutative Banach algebras which satisfy a Bochner-Schoenberg-Eberlein-type inequality is introduced. Commutative C*-algebras, the disk algebra and the Hardy algebra on the open disk are examples.

1. Introduction

In this paper we study commutative Banach algebras satisfying a Bochner-Schoenberg-Eberlein-type theorem. Let $G$ be a locally compact Abelian group. Let $G^\sim$ denote the dual group of $G$ and $M(G)$ denote the measure algebra of $G$ as usual. The algebra of all bounded continuous complex-valued functions on $G^\sim$ will be denoted by $C^b(G^\sim)$ and the Fourier-Stieltjes transform of $\mu$ in $M(G)$ by $\mu^\sim$. A theorem of Bochner-Schoenberg-Eberlein for the group algebra $L^1(G)$ states the following (cf. [10]):

Theorem (Bochner-Schoenberg-Eberlein). Let $\sigma$ be a function in $C^b(G^\sim)$ and $\beta$ be a positive real number. The following are equivalent:

(i) There exists $\mu$ in $M(G)$ which satisfies $\sigma = \mu^\sim$ and $\|\mu\| \leq \beta$.

(ii) For every finite number of complex numbers $c_1, \ldots, c_n$ and the same number of $\gamma_1, \ldots, \gamma_n$ in $G^\sim$ the inequality

$$\left\| \sum_{i=1}^{n} c_i \sigma(\gamma_i) \right\| \leq \beta \left\| \sum_{i=1}^{n} c_i \gamma_i \right\|_{L^\infty(G)}$$

holds.

Let $A$ be a (not necessary unital) commutative Banach algebra without order (i.e. which has no nonzero annihilators). $A^*$ denotes the dual space of $A$ and $\Phi_A$ the carrier space of $A$. We denote by $C(\Phi_A)$ the algebra consisting of all continuous complex-valued functions on $\Phi_A$ and $C_{BSE}(\Phi_A)$ by the set of all...
σ in \( C(\Phi_A) \) which satisfy the following: there exists a positive real number \( β \) such that for every finite number of complex numbers \( c_1, \ldots, c_n \) and the same number of \( ϕ_1, \ldots, ϕ_n \) in \( \Phi_A \) the inequality
\[
\left| \sum_{i=1}^{n} c_i \sigma(ϕ_i) \right| \leq β \left\| \sum_{i=1}^{n} c_i ϕ_i \right\| A^*
\]
holds. Also for each \( σ \) in \( C_{BSE}(\Phi_A) \) we denote by \( \|σ\|_{BSE} \) the infimum of such \( β \). Then we can easily see that for each \( x ∈ A \), its Gelfand transform \( \hat{x} \) belongs to \( C_{BSE}(\Phi_A) \) and \( \|x\|_\infty \leq \|\hat{x}\|_{BSE} \). Here \( \|\cdot\|_\infty \) denotes the supremum norm on \( \Phi_A \). Also we have the following:

**Lemma 1.**

(i) \( C_{BSE}(\Phi_A) \) is a subalgebra of \( C(\Phi_A) \).

(ii) \( \|\cdot\|_{BSE} \) is a complete algebra norm on \( C_{BSE}(\Phi_A) \).

(iii) \( C_{BSE}(\Phi_A) \) is semisimple.

**Proof.** (i) We can easily see that \( C_{BSE}(\Phi_A) \) is a linear subspace of \( C(\Phi_A) \). Then we only show that the product of two functions in \( C_{BSE}(\Phi_A) \) is another one. To do this let \( σ_1, σ_2 ∈ C_{BSE}(\Phi_A), c_1, \ldots, c_n \) be complex numbers and \( ϕ_1, \ldots, ϕ_n ∈ \Phi_A \). For any \( ε > 0 \), choose \( x_1 ∈ A \) such that \( \|x_1\| ≤ 1 \) and
\[
\left\| \sum_{i=1}^{n} c_i σ_1(ϕ_i)x_i \right\| A^* \leq \sum_{i=1}^{n} c_i σ_1(ϕ_i)|x_i| + ε.
\]
Moreover choose \( x_2 ∈ A \) such that \( \|x_2\| ≤ 1 \) and
\[
\left\| \sum_{i=1}^{n} c_i ϕ_i(x_1)x_i \right\| A^* \leq \sum_{i=1}^{n} c_i ϕ_i(x_1)|x_i| + ε.
\]
Then we have
\[
\left\| \sum_{i=1}^{n} c_i (σ_1σ_2)(ϕ_i) \right\| \leq \|σ_2\|_{BSE} \left\| \sum_{i=1}^{n} c_i σ_1(ϕ_i)x_i \right\| A^* \leq \|σ_2\|_{BSE} \left( \sum_{i=1}^{n} c_i ϕ_i(x_1)x_i \left( σ_1(x) + ε \right) \right) \leq \|σ_2\|_{BSE} \left( \|σ_1\|_{BSE} \left\| \sum_{i=1}^{n} c_i ϕ_i(x_1)x_i \right\| A^* + ε \right) \leq \|σ_2\|_{BSE} \left( \|σ_1\|_{BSE} \left\| \sum_{i=1}^{n} c_i ϕ_i(x_1)x_i + ε \right\| A^* + ε \right) \leq \|σ_2\|_{BSE} \left( \|σ_1\|_{BSE} \left\| x_1x_2 + ε \right\| A^* + ε \right) .
\]
Since $\|x_1x_2\| \leq 1$ and $\varepsilon$ is arbitrary, it follows that $\sigma_1\sigma_2 \in C_{BSE}(\Phi_A)$ and $\|\sigma_1\sigma_2\|_{BSE} \leq \|\sigma_1\|_{BSE}\|\sigma_2\|_{BSE}$.

(ii) Observe that $\|\| \cdot \|_{BSE}$ is an algebra norm on $C_{BSE}(\Phi_A)$ by the same argument of the above proof. Then we only show that $\|\| \cdot \|_{BSE}$ is complete. Suppose that $\sigma_n \in C_{BSE}(\Phi_A)$ ($n = 1, 2, \ldots$) and $\lim_{n,m \to \infty} \|\sigma_n - \sigma_m\|_{BSE} = 0$. Then for each $\varphi \in \Phi_A$, $|\sigma_n(\varphi) - \sigma_m(\varphi)| \leq \|\sigma_n - \sigma_m\|_{BSE}$ ($n, m = 1, 2, \ldots$) and hence $\{\sigma_n(\varphi)\}$ converges to a complex number, say $\sigma(\varphi)$. In this case, we can see that $\{\sigma_n(\varphi)\}$ converges uniformly to $\sigma(\varphi)$ on $\Phi_A$. Therefore, $\sigma$ is a continuous function on $\Phi_A$. Now let $c_1, \ldots, c_k$ be complex numbers and $\varphi_1, \ldots, \varphi_k \in \Phi_A$. For any $\varepsilon > 0$, choose a number $N$ such that $\|\sigma_n - \sigma_m\|_{BSE} \leq \varepsilon$ for all $n, m \geq N$. Then we have

$$\left|\sum_{i=1}^{k} c_i(\sigma_n(\varphi_i) - \sigma_m(\varphi_i))\right| \leq \|\sigma_n - \sigma_m\|_{BSE} \left|\sum_{i=1}^{k} c_i\varphi_i\right|_{A^*}$$

$$< \varepsilon \left|\sum_{i=1}^{k} c_i\varphi_i\right|_{A^*}$$

for all $n, m \geq N$. Hence, by taking the limit with respect to $m$, we obtain that

$$\left|\sum_{i=1}^{k} c_i(\sigma_n(\varphi_i) - \sigma(\varphi_i))\right| \leq \varepsilon \left|\sum_{i=1}^{k} c_i\varphi_i\right|_{A^*}$$

for all $n \geq N$. This shows that $\sigma \in C_{BSE}(\Phi_A)$ and $\lim_{n \to \infty} \|\sigma_n - \sigma\|_{BSE} = 0$.

(iii) Observe that evaluations (at points of $\Phi_A$) are nonzero multiplicative linear functionals and the only point in $C_{BSE}(\Phi_A)$ annihilated by all of them is 0. Then $C_{BSE}(\Phi_A)$ is semisimple. Q.E.D.

**Remark.** By the above argument, $C_{BSE}(\Phi_A)$ is a semisimple commutative Banach algebra with respect to the BSE norm and contains $A^\sim = \{x^\sim : x \in A\}$.

A multiplier $T$ on $A$ is the bounded linear operator on $A$ into itself which satisfies $xTy = (Tx)y$ for every $x$ and $y$ in $A$. $M(A)$ denotes the commutative Banach algebra consisting of all multipliers on $A$. It is well known that $T$ can be represented as a continuous complex-valued function $T^\sim$ on $\Phi_A$ (cf. [7]). We denote $M^\sim(A) = \{T^\sim : T \in M(A)\}$. Along these lines, the preceding theorem of Bochner–Schoenberg–Eberlein for a locally compact Abelian group is rewritten as $M^\sim(L^1(G)) = C_{BSE}(\Phi L^1(G))$, since we know that $L^\infty(G) \cong L^1(G)^*$, $G^\sim \cong \Phi L^1(G)$ and $M(G) \cong M(L^1(G))$. We will discuss commutative Banach algebras with such a condition. For this purpose we make the following

**Definition.** We say that a commutative Banach algebra $A$ without order is a BSE-algebra if $A$ satisfies the condition $M^\sim(A) = C_{BSE}(\Phi_A)$.

**Remark.** Precisely speaking the theorem of Bochner–Schoenberg–Eberlein in terms of BSE, we see that $M^\sim(A) = C_{BSE}(\Phi_A)$ and $\|T\| = \|T^\sim\|_{BSE}$ for all $T \in M(A)$ whenever $A = L^1(G)$ ($G$ : locally compact Abelian group). However we
see the following fact from Corollary 6 later on: if \( A \) is a BSE-algebra, then \( A \) is semisimple if and only if there exist positive real numbers \( \alpha \) and \( \beta \) such that 
\[
\alpha \| T \| \leq \| T^* \|_{\text{BSE}} \leq \beta \| T \|
\]
for all \( t \in M(A) \). Of course \( L^1(G) \) is semisimple. Therefore it seems reasonable to define BSE-algebras as in the above in view of topological isomorphism.

2. BSE-ALGEBRAS

Let \( \mathcal{C}^b(\Phi_A) \) be the algebra of all bounded continuous complex-valued functions on \( \Phi_A \). Then \( \mathcal{C}^\text{BSE}(\Phi_A) \) is contained in \( \mathcal{C}^b(\Phi_A) \) and hence an arbitrary BSE-algebra can have the following two types:

(I) \( M^\sim(A) = \mathcal{C}^\text{BSE}(\Phi_A) = \mathcal{C}^b(\Phi_A) \).

(II) \( M^\sim(A) = \mathcal{C}^\text{BSE}(\Phi_A) \subsetneq \mathcal{C}^b(\Phi_A) \).

For example, the group algebra of a finite group is type I-BSE and the group algebra of an infinite group is type II-BSE. In this section we show that a certain type I-BSE-algebra is isomorphic to a commutative \( C^* \)-algebra. We also show that both the disk algebra and the classical Hardy algebra are type II-BSE.

Now we begin with the following

**Lemma 2.** \( \mathcal{C}^\text{BSE}(\Phi_A) = \mathcal{C}^b(\Phi_A) \) if and only if there exists \( \beta < \infty \) such that for any finite number of \( c_1, \ldots, c_n \in \Delta \) and the same number of \( \varphi_1, \ldots, \varphi_n \in \Phi_A \), there exists \( x \) in \( A \) such that \( \| x \| \leq \beta \) and \( x^\sim(\varphi) = c_i \) (\( i = 1, \ldots, n \)). Here \( \Delta \) denotes the closed unit disk.

**Proof.** We first prove the "only if" part. Suppose that \( \mathcal{C}^\text{BSE}(\Phi_A) = \mathcal{C}^b(\Phi_A) \). Since the inequality \( \| \sigma \|_\infty \leq \| \sigma \|_{\text{BSE}} \) holds for every \( \sigma \in \mathcal{C}^\text{BSE}(\Phi_A) \), it follows from the open mapping theorem that there exists \( \beta < \infty \) such that \( \| \sigma \|_{\text{BSE}} \leq \beta \| \sigma \|_\infty \) for every \( \sigma \in \mathcal{C}^b(\Phi_A) \). Consider a finite number of \( c_1, \ldots, c_n \) in \( \Delta \) and the same number of \( \varphi_1, \ldots, \varphi_n \) in \( \Phi_A \). Choose a function \( \sigma \) in the unit ball of \( \mathcal{C}^b(\Phi_A) \) with \( \sigma(\varphi) = c_i \) for every \( i = 1, \ldots, n \). By Helly's theorem (cf. [8]) there exists \( x \in A \) such that \( \| x \| \leq \beta + 1 \) and \( x^\sim(\varphi) = c_i \) for \( i = 1, \ldots, n \), since for any complex numbers \( a_1, \ldots, a_n \), the inequalities

\[
\left| \sum_{i=1}^n a_i c_i \right| = \left| \sum_{i=1}^n a_i \sigma(\varphi_i) \right| \leq \| \sigma \|_{\text{BSE}} \left\| \sum_{i=1}^n a_i \varphi_i \right\|_{A^*} \\
= \beta \| \sigma \|_\infty \left\| \sum_{i=1}^n a_i \varphi_i \right\|_{A^*} \\
= \beta \left\| \sum_{i=1}^n a_i \varphi_i \right\|_{A^*}
\]

holds. We can prove the "if" part in a similar calculation. Q.E.D.

**Theorem 3.** A semisimple type I-BSE-algebra with a bounded approximate identity is isomorphic to a commutative \( C^* \)-algebra and conversely.
Proof. If $A$ is isomorphic to a commutative $C^*$-algebra, then $A \cong C_0(\Phi_A)$ and hence $M(A)^\sim = C^b(\Phi_A)$. So by the preceding lemma, $A$ is type I-BSE. Conversely let $A$ be a semisimple type I-BSE-algebra with a bounded (say, by $\beta < \infty$) approximate identity $\{e_j\}$. Define $\tau_x$ by the relation $\tau_x(y) = xy$ ($y \in A$), then the set $I = \{\tau_x : x \in A\}$ is an ideal of $M(A)$ which is isomorphic to $A$. Note that $\|\tau_x\| \geq \|\tau_x(e_j/\|e_j\|)\| = (1/\|e_j\|)\|xe_j\| \geq (1/\beta)\|xe_j\| \rightarrow (1/\beta)\|x\|$, so the operator norm on $I$ is a complete norm. Therefore $I$ is closed in $M(A)$. Also by the semisimplicity of $A$, the algebra homomorphism: $T \rightarrow T^\sim$ of $M(A)$ onto $M^\sim(A)$ is injective. Note also that $\|T^\sim\|_{\infty} \leq \beta\|T\|$ for all $T \in M(A)$. Since $A$ is type I-BSE, $M^\sim(A) = C^b(\Phi_A)$, so $M(A) \cong C^b(\Phi_A)$ by the open mapping theorem. Consequently, $I^\sim = \{T^\sim : T \in I\}$ becomes a commutative $C^*$-algebra and $I^\sim \cong A$. Q.E.D.

Problem 1. Does Theorem 2 hold without the assumption of a bounded approximate identity?

Theorem 4. (i) $C_{\text{BSE}}(\Phi_A)$ equals the set of all $\sigma \in C^b(\Phi_A)$ for which there exists a bounded net $\{x_\lambda\}$ in $A$ with $\lim x_\lambda(\varphi) = \sigma(\varphi)$ for all $\varphi \in \Phi_A$.

(ii) $C_{\text{BSE}}(\Phi_A) = C^b(\Phi_A) \cap (A^{**}\Phi_A)$, where $A^{**}$ denotes the second dual of $A$.

Proof. (i) Let a function $\sigma$ in $C^b(\Phi_A)$ be such that there exist $\beta < \infty$ and a net $\{x_\lambda\}$ in $A$ with $\|x_\lambda\| \leq \beta$ for all $\lambda$ and $\lim x_\lambda(\varphi) = \sigma(\varphi)$ for all $\varphi \in \Phi_A$. To show $\sigma \in C_{\text{BSE}}(\Phi_A)$, let $c_1, \ldots, c_n$ be complex numbers in $\Delta$ and $\varphi_1, \ldots, \varphi_n$ be points in $\Phi_A$. Then we have

$$\left| \sum_{i=1}^n c_i \sigma(\varphi_i) \right| \leq \sum_{i=1}^n c_i x_\lambda(\varphi_i) + \sum_{i=1}^n c_i \{x_\lambda(\varphi_i) - \sigma(\varphi_i)\} \leq \beta \left\| \sum_{i=1}^n c_i \varphi_i \right\|_{A^{\sim}} + \sum_{i=1}^n c_i \{x_\lambda(\varphi_i) - \sigma(\varphi_i)\}.$$

Taking the limit with respect to $\lambda$, we obtain $\sigma \in C_{\text{BSE}}(\Phi_A)$. Conversely let $\sigma \in C_{\text{BSE}}(\Phi_A)$. Let $\Lambda$ be the net consisting of all finite subsets of $\Phi_A$. Then by Helly’s theorem, for each $\epsilon > 0$ and $\lambda \in \Lambda$, there exists $x = x(\lambda, \epsilon) \in A$ such that $\|x\| \leq \|\sigma\|_{\text{BSE}} + \epsilon$ and $x(\varphi) = \sigma(\varphi)$ for all $\varphi \in \lambda$. This shows that $\lim x(\lambda, \epsilon) = \sigma$ pointwise on $\Phi_A$.

(ii) $C^b(\Phi_A) \cap (A^{**}\Phi_A) \subset C_{\text{BSE}}(\Phi_A)$ is straightforward. To show the reverse inclusion, let $\sigma \in C_{\text{BSE}}(\Phi_A)$. Then by (i), there exist $\beta < \infty$ and a net $\{x_\lambda\}$ in $A$ with $\|x_\lambda\| \leq \beta$ for all $\lambda$ and $\lim x_\lambda(\varphi) = \sigma(\varphi)$ for all $\varphi \in \Phi_A$. Let $\pi$ be the natural embedding of $A$ into $A^{**}$. Then there exists a subnet $\{x_\lambda'\}$ of $\{x_\lambda\}$ and $F \in A^{**}$ such that $w^*\lim \pi(x_\lambda') = F$. Therefore for each $\varphi \in \Phi_A$,

$$\sigma(\varphi) = \lim x_\lambda'(\varphi) = \lim \pi(x_\lambda')(\varphi) = F(\varphi),$$

so that $\sigma = F|\Phi_A$. Q.E.D.
Remarks. By the above proof, for each $\sigma \in C_{BSE}(\Phi_A)$,
\[
\|\sigma\|_{BSE} = \inf\{\beta > 0 : \exists \text{ a net } \{x_\lambda\} \subset A \text{ with } \|x_\lambda\| \leq \beta \quad (\forall \lambda),
\lim \tilde{x}_\lambda(\varphi) = \sigma(\varphi) \quad (\forall \varphi \in \Phi_A)\}.
\]
Also the algebra $C_{BSE}(\Phi_A)$ is closed in the following sense: If $\sigma$ is a function in $C^b(\Phi_A)$ such that there exist $\beta < \infty$ and a net $\{\sigma_\lambda\}$ in $C_{BSE}(\Phi_A)$ with $\|\sigma_\lambda\| \leq \beta$ for every $\lambda$ and $\lim \sigma_\lambda(\varphi) = \sigma(\varphi)$ for every $\varphi \in \Phi_A$, then $\sigma$ must be in $C_{BSE}(\Phi_A)$.

Definition. If $\{e_\lambda\}$ is a bounded net in $A$ satisfying the conditions $\lim \varphi(xe_\lambda) = \varphi(x)$ for every $x \in A$ and $\varphi \in \Phi_A$, then we call it a bounded weak approximate identity of $A$ in the sense of Jones–Lahr (cf. [2] or [6]).

Corollary 5. $M^\sim(A) \subset C_{BSE}(\Phi_A)$ if and only if $A$ has a bounded weak approximate identity in the sense of Jones–Lahr.

Proof. Since $M^\sim(A)$ contains the identity of $C^b(\Phi_A)$, the “only if” part follows immediately from Theorem 4(i). To show the “if” part, assume that $A$ has a bounded (say, by $\beta < \infty$) weak approximate identity $\{e_\lambda\}$ in the sense of Jones–Lahr. Let $T$ be a multiplier on $A$. Then for any complex numbers $c_1, \ldots, c_n$ and the same number of $\varphi_1, \ldots, \varphi_n$ in $\Phi_A$,
\[
\left| \sum_{i=1}^n c_i T^\sim(\varphi_i) \right| \leq \left| \sum_{i=1}^n c_i T^\sim(\varphi_i)e_\lambda(\varphi_i) \right| + \left| \sum_{i=1}^n c_i T^\sim(\varphi_i)\{1 - e_\lambda(\varphi_i)\} \right|
\]
\[
\leq \|Te_\lambda\| \left| \sum_{i=1}^n c_i \varphi_i \right|_{A^*} + \left| \sum_{i=1}^n c_i T^\sim(\varphi_i)\{1 - e_\lambda(\varphi_i)\} \right|
\]
\[
\leq \beta \|T\| \left| \sum_{i=1}^n c_i \varphi_i \right|_{A^*} + \left| \sum_{i=1}^n c_i T^\sim(\varphi_i)\{1 - e_\lambda(\varphi_i)\} \right|
\]
Hence, by taking the limit with respect to $\lambda$, we obtain $\|T^\sim\|_{BSE} \leq \beta \|T\|$ and hence $T^\sim \in C_{BSE}(\Phi_A)$. Q.E.D.

Corollary 6. Let $A$ be a BSE-algebra. Then $A$ is semisimple if and only if there exist positive real numbers $\alpha$ and $\beta$ such that $\alpha \|T\| \leq \|T^\sim\|_{BSE} \leq \beta \|T\|$ for all $T \in M(A)$.

Proof. Suppose that $A$ is semisimple. Then the map: $T \rightarrow T^\sim$ is an algebra isomorphism of $M(A)$ onto $C_{BSE}(\Phi_A)$ (= $M^\sim(A)$). Also there exists $\beta < \infty$ such that $\|T^\sim\|_{BSE} \leq \beta \|T\|$ for all $T \in M(A)$ as observed in the proof of the preceding corollary. Therefore we can find $\alpha < \infty$ such that $\alpha \|T\| \leq \|T^\sim\|_{BSE}$ for all $T \in M(A)$ by the open mapping theorem. Suppose conversely that there exist positive real numbers $\alpha$ and $\beta$ such that $\alpha \|T\| \leq \|T^\sim\|_{BSE} \leq \beta \|T\|$ for all $T \in M(A)$. Let $x \in A$ be such that $x^\sim(\varphi) = 0$ for all $\varphi \in \Phi_A$. Note that $(\tau_x)^\sim = x^\sim = 0$, so $\tau_x = 0$ by the assumption. Then $x = 0$, since $A$ has no nonzero annihilators. In other words, $A$ is semisimple. Q.E.D.

Remark. The multiplier algebra of a semisimple BSE-algebra $A$ is topologically isomorphic to $C_{BSE}(\Phi_A)$ by the above corollary.
Theorem 7. Both the disk algebra and the classical Hardy algebra are type II-BSE.

Proof. Let \( A \) be either the disk algebra or the classical Hardy algebra and \( D \) the open unit disk. We show that \( A = C_{\text{BSE}}(\Phi_A) \). To do this let \( f \in C_{\text{BSE}}(\Phi_A) \). By Theorem 4(i), there exist \( \beta < \infty \) and a net \( \{f_\lambda\} \) in \( A \) with the index set \( \Lambda \) such that \( \|f_\lambda\| \leq \beta \) for every \( \lambda \) and \( \lim f_\lambda(z) = f(z) \) for every \( z \in D \). If we can show that

\[
\lim \|f_\lambda - f\|_K = 0
\]

for all compact subset \( K \) of \( D \), then we conclude by Morera's theorem that \( f \in A \). Here \( \|f\|_K = \sup \{|f(z)|: z \in K\} \). Thus it only remains to show the equality (1). It is trivial that in any compact Hausdorff space \( S \) a net which has at most one cluster point must in fact converge. Now consider the closure \( \overline{S} \) of the set \( \{f_\lambda: \lambda \in \Lambda\} \) in the compact-convergence topology (denoted by \( \text{cc} \)) of \( H(D) \). Since \( S \) is bounded, it is \( \text{cc} \)-compact by Montel's theorem. Since \( \text{cc} \)-convergence implies pointwise convergence, every \( \text{cc} \)-cluster point of the net \( \{f_\lambda\} \) \( \text{cc} \)-converges, necessarily to \( f \). Q.E.D.

3. Ideals and quotient algebras of BSE-algebras

In this section we will consider the following problem: Are closed ideals and quotient algebras of BSE-algebras BSE? We will deal only with Banach algebras with discrete carrier space.

When a closed ideal \( I \) of a commutative Banach algebra \( A \) is essential as Banach \( A \)-module, that is, \( I \) equals the closed linear span of \( \{ax: a \in A, \ x \in I\} \), we will call \( I \) an essential ideal.

Theorem 8. Let \( A \) be a BSE-algebra with discrete carrier space and \( I \) an essential closed ideal of \( A \). Then

(i) \( M^\sim(A/I) = C_{\text{BSE}}(\Phi_{A/I}) \).

(ii) \( C_{\text{BSE}}(\Phi_I) \subset M^\sim(I) \).

In particular, if the quotient \( I : A \) of \( I \) (cf. [9] for definition) is contained in \( I \), then \( A/I \) is BSE. Also if \( I \) has a bounded weak approximate identity in the sense of Jones-Lahr and has no nonzero annihilators, then \( I \) is BSE.

Proof. (i) Since \( A \) is BSE, it has a bounded weak approximate identity in the sense of Jones-Lahr by Corollary 5. Then \( A/I \) also has an approximate identity in the sense of Jones-Lahr. Hence, by Corollary 5, we have that \( M^\sim(A/I) \subset C_{\text{BSE}}(\Phi_{A/I}) \). To show the reverse inclusion, let \( \sigma' \in C_{\text{BSE}}(\Phi_{A/I}) \). Since both \( \Phi_A \) and \( \Phi_{A/I} \) are discrete, it follows from Theorem 4(ii) that

\[
C_{\text{BSE}}(\Phi_A) = A^{**} | \Phi_A
\]

and

\[
C_{\text{BSE}}(\Phi_{A/I}) = (A/I)^{**} | \Phi_{A/I}.
\]
By (3) we can take $F' \in (A/I)^{**}$ with $F'|\Phi_{A/I} = \sigma'$. Set $I^\perp = \{ f \in A^* : f|I = 0 \}$. For each $f \in I^\perp$, define $f^*$ by the relation $f^*(x') = f(x)$ ($x' = x + I \in A/I$). The map: $f \to f^*$ is an isometric isomorphism of $I^\perp$ onto $(A/I)^*$. We define $\tilde{F}'$ by the relation $\tilde{F}'(f) = F'(f^*)$ ($f \in I^\perp$). Then $\tilde{F}' \in (I^\perp)^*$. and hence there is $F \in A^{**}$ such that $\|F\| = \|\tilde{F}'\|$ and $F|I^\perp = \tilde{F}'$ from the Hahn–Banach extension theorem. By (2) and the BSE property of $A$, there exists $T \in M(A)$ with $T^\sim = F|\Phi_A$. Since $I$ is essential and $T(xy) = xTy$ for all $x$, $y \in A$ (for $A$ has no nonzero annihilators), we see $T(I) \subset I$ and hence $T'$ defined by $T'(x') = (Tx)'$ ($x \in A$) belongs to $M(A/I)$. In this case $T'^\sim = \sigma'$. Actually for any $x \in A$ and $\varphi \in \Phi_A \cap I^\perp$, 

$$x^\sim(\varphi)T'^\sim(\varphi') = x^\sim(\varphi')T'^\sim(\varphi) = (T'x')^\sim(\varphi') = (Tx)'^\sim(\varphi')$$

so that $T'^\sim = T^\sim(\varphi)$ and hence $\sigma'(\varphi') = F'(\varphi') = \tilde{F}'(\varphi) = F(\varphi) = T^\sim(\varphi) = T'^\sim(\varphi')$. Therefore $\sigma' = T'^\sim$, since $\Phi_{A/I} = \{ \varphi' : \varphi \in \Phi_A \cap I^\perp \}$. We thus conclude that $\sigma' \in M^\sim(A/I)$. In other words, $M^\sim(A/I) \supset C_{BSE}(\Phi_{A/I})$.

(ii) Let $\omega \in C_{BSE}(\Phi_I)$. Note that $\Phi_I = \{ \varphi|I : \varphi \in \Phi_A \cap I^\perp \}$, which is also discrete (see [9, Theorem 3.1.18]). So a complex-valued function $\sigma$ on $\Phi_A$ defined by

$$\sigma(\varphi) = \begin{cases} \omega(\varphi|I) & (\varphi \in \Phi_A \cap I^\perp) \\ 0 & (\varphi \in \Phi_A \backslash I^\perp) \end{cases}$$

belongs to $C_{BSE}(\Phi_A)$. In fact for every finite number of complex numbers $c_1, \ldots, c_n$ and the same number of $\varphi_1, \ldots, \varphi_n$ in $\Phi_A$, using $\sum'$ to denote the summation over $\varphi_i \in \Phi_A \backslash I^\perp$,

$$\left| \sum_{i=1}^n c_i \sigma(\varphi_i) \right| = \left| \sum' c_i \sigma(\varphi_i) \right| = \left| \sum c_i \omega(\varphi_i|I) \right|$$

$$\leq \|\omega\|_{BSE} \left\| \sum' c_i \varphi_i |I \right\|_{I^*}$$

$$= \|\omega\|_{BSE} \left\| \sum_{i=1}^n c_i \varphi_i |I \right\|_{I^*}$$

$$\leq \|\omega\|_{BSE} \left\| \sum_{i=1}^n c_i \varphi_i \right\|_{A^*},$$

so that $\sigma \in C_{BSE}(\Phi_A)$. As $A$ is BSE, take $T \in M(A)$ such that $T^\sim = \sigma$ and put $S = T|I$. Then $S \in M(I)$. Also for any $x \in I$ and $\varphi \in \Phi_A \backslash I^\perp$, we have

$$S^\sim(\varphi|I)x^\sim(\varphi|I) = (Sx)^\sim(\varphi|I) = (Tx)^\sim(\varphi|I) = T^\sim(\varphi)x^\sim(\varphi)$$

$$= \sigma(\varphi)x^\sim(\varphi|I) = \omega(\varphi|I)x^\sim(\varphi|I).$$
Then $\omega = S_1^* \in M_1^\sim(I)$. In other words, $C_{BSE}(\Phi_I) \subseteq M_1^\sim(I)$.

In particular, if $I: A \subseteq I$ then $A/I$ has no nonzero annihilators and hence it is BSE by (i). Also if $I$ has a bounded weak approximate identity in the sense of Jones–Lahr, then $C_{BSE}(\Phi_I) = M_1^\sim(\Phi_I)$ by (ii) and Corollary 5. Therefore if further $I$ has no nonzero annihilators, then $I$ is BSE. Q.E.D.

**Remark.** Let $A$ be a commutative Banach algebra and $I$ a closed ideal of $A$. Then $I: A \subseteq I$ either if $A$ has an approximate identity or if $I$ is modular. If also $A$ has a bounded weak approximate identity in the sense of Jones–Lahr and $I$ is kernel then $I: A \subseteq I$.

**Corollary 9.** Every quotient algebra of the group algebra on a compact Abelian group is BSE.

**Remark.** We don’t know if a closed ideal of the group algebra of an arbitrary compact Abelian group has a bounded weak approximate identity in the sense of Jones–Lahr.

### 4. Counterexamples

Let $l^1$ be the commutative Banach algebra of all absolutely convergent sequences of complex numbers with pointwise multiplication. Then

$$l^1 = C_{BSE}(\Phi_{l^1}) \subseteq M^\sim(l^1) = C^b(\Phi_{l^1}).$$

Therefore $l^1$ is not BSE and does not have a bounded weak approximate identity in the sense of Jones–Lahr. Let $N$ be the semigroup of all natural numbers and $L(N)$ be the semigroup algebra of $N$. We can show that $M^\sim(L(N))$ is not contained in $C_{BSE}(\Phi_{L(N)})$. So $L(N)$ is also not BSE and does not have a bounded weak approximate identity in the sense of Jones–Lahr. However the semigroup algebra on the semigroup of all positive real numbers possesses a bounded weak approximate identity in the sense of Jones–Lahr (cf. [6]). In spite of this fact we don’t know whether this algebra is BSE or not.

We conclude this section with the following remark pointed out by K. Izuchi: The measure algebra $M(G)$ of a nondiscrete locally compact Abelian group $G$ is not BSE. In fact, let $G^\sim$ be the dual group of $G$ and choose a measure $\mu$ in $M(G)$ such that its Fourier–Stieltjes transform $\hat{\mu}$ vanishes at infinity and its Gelfand transform $\mu^\vee$ is not identically zero on $\Phi_{M(G)} \setminus G^\sim$ (cf. [1, 3 or 4]). Next define $f$ by $f = \mu^\sim$ on $G^\sim$ and $f = 0$ on $\Phi_{M(G)} \setminus G^\sim$. We see easily that $f$ belongs to $C_{BSE}(\Phi_{M(G)})$ but there does not exist a measure $\nu$ in $M(G)$ with $\nu^\vee = f$.


### Acknowledgments

The authors would like to express their hearty thanks to the referee for the present simple proof of Theorem 7 and several useful comments and to Professors Hisashi Choda and Keiji Izuchi for their useful suggestions and comments.
References


Department of Basic Technology, Yamagata University, Yonezawa 992, Japan
Department of Mathematics, Tokyo Medical College, 6-1-1 Shinjuku, Shinjuku-ku, Tokyo 160, Japan