DARBOUX BAIRE-.5 FUNCTIONS

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Abstract. Let $I = [0, 1]$, and let $D$ denote the points of continuity of a function $f: I \to \mathbb{R}$. A Darboux function maps each connected set to a connected set. A function is Baire-1 (Baire-.5) if preimages of open sets are $F_\sigma$-sets ($G_\delta$-sets). We show that if $f$ is a Darboux Baire-.5 function, then the graph of the restriction of $f$ to $D$ is a dense subset of the whole graph of $f$. It is already known that there is a Darboux Baire-1 function which does not satisfy this conclusion.

A classical result from real analysis states that the set $D$ of points of continuity of an arbitrary function $f: I \to \mathbb{R}$ is a $G_\delta$-set. In 1966, Jones and Thomas showed that for any function $f: I \to I$ with a connected $G_\delta$-graph, $D$ is also a dense subset of $I$ [2]. Their argument still works for a Darboux function with a $G_\delta$-graph. It is not always the case that a Darboux Baire-1 or even a bounded approximately continuous function $f$ satisfies the stronger property that each point on the graph of $f$ has a point of $f(D)$ plotted nearby [1, Chapter II, Theorems 1, 2.4, and 6.5]. However, we show this property is satisfied by the Darboux Baire-.5 functions, which form a subcollection of the Darboux Baire-1 functions.

For a subset $A$ of $B$ in the plane, we say that $A$ is bilaterally $c$-dense in $B$ if in each open neighborhood of any point $(x, y) \in B$ lie $c$-many points of $A$ to the left and to the right of $(x, y)$.

Theorem. Suppose $f: I \to \mathbb{R}$ is a Darboux Baire-.5 function, and let $D$ denote the set of points at which $f$ is continuous. Then the graph of $f|D$ is bilaterally $c$-dense in the graph of $f$.

Proof. Since the graph of $f$ is a $G_\delta$-set, it is the intersection of a nested sequence of open subsets $G_1, G_2, \ldots$ of $I \times \mathbb{R}$. By [2], $D$ is a dense $G_\delta$-subset of $I$. We first show that the graph of $f|D$ is dense in the whole graph of $f$. Assume it is not. It follows that there is an open neighborhood $S = (a_1, b_1) \times (c_1, d_1)$ of a point $(x_0, f(x_0))$ in $I \times \mathbb{R}$ such that...
According to the hypothesis, $f^{-1}(c_1, d_1)$ is a $G_\delta$-set. Since $A = (a_1, b_1) \cap f^{-1}(c_1, d_1)$ is a $G_\delta$-set, it is topologically complete [3]. Let $l_x = \{x\} \times R$. For all rational numbers $r$ in $(c_1, d_1)$, define $h(r, c_1) = \{x \in A : f \cap l_x$ meets both $I \times \{r\}$ and $I \times \{c_1\}\}$ and $h(r, d_1) = \{x \in A : f \cap l_x$ meets both $I \times \{r\}$ and $I \times \{d_1\}\}$. Each of the sets $h(r, c_1)$ and $h(r, d_1)$ is closed in $A$, and each $x$ in $A$ belongs, for some value of $r$, to either $h(r, c_1)$ or $h(r, d_1)$.

According to the Baire Category Theorem, there is a rational number $r_0$ for which either $h(r_0, c_1)$ or $h(r_0, d_1)$—say $h(r_0, d_1)$—is somewhere dense in $A$. Then there is a subinterval $(a_2, b_2)$ of $(a_1, b_1)$ such that $h(r_0, d_1)$ contains the nonempty set $B = A \cap (a_2, b_2)$.

For all rational numbers $r < s$ in $[r_0, d_1]$, define $H(n, r, s) = \{x \in B :$ some component of $l_x \cap G_n$ meets both $I \times \{r\}$ and $I \times \{s\}\}$. As in [2], it can be shown that $H(n, r, s)$ is closed in $B$, and each point $x$ of $B$ belongs to some $H(n, r, s)$. Then some $H(n_1, r_1, s_1)$ is somewhere dense in $B$ and therefore contains a nonempty set $C = B \cap (a_3, b_3)$, where $(a_3, b_3)$ is a subinterval of $(a_2, b_2)$. Consequently, $f$ misses the set $C \times (r_1, s_1)$. However, for each $x \in C$, $f \cap l_x$ meets $I \times \{r_0\}$ and $I \times \{d_1\}$, and so $f \cap l_x$ meets $I \times \{r_1\}$ and $I \times \{s_1\}$. This implies every point of $\{x\} \times (r_1, s_1)$ is a limit point of $f|C$ whenever $x \in C$. Then $f$ meets $C \times (r_1, s_1)$, a contradiction. Therefore $f|D$ is dense in $f$ after all.

Since $f$ is a Darboux function, the graph of $f$ is bilaterally dense in itself. It was shown above that the graph of $f|D$ is dense in the graph of $f$. Moreover, the graph of $f|D$ is c-dense in itself because $D$ is c-dense in itself. It now follows that the graph of $f|D$ is bilaterally c-dense in the graph of $f$.

References


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