ON THE FUNDAMENTAL GROUPS OF MANIFOLDS
WITH ALMOST-NONNEGATIVE RICCI CURVATURE

GUOFANG WEI

(Communicated by Jonathan M. Rosenberg)

Abstract. We give an upper bound on the growth of \( \pi_1(M) \) for a class of manifolds \( M \) with Ricci curvature \( \text{Ric}_M \geq -\varepsilon \), diameter \( d(M) = 1 \), and volume \( \text{vol}(M) \geq v \).

In [4], Milnor proved that every finitely generated subgroup of the fundamental group of a manifold \( M^n \) with nonnegative Ricci curvature is of polynomial growth with degree \( \leq n \). It is conjectured by Gromov [2] that the fundamental group of a near-elliptic manifold (in the sense of Gromov) is of polynomial growth. The purpose of this note is to present the following theorem.

**Theorem 1.** For any constant \( v > 0 \), there exists \( \varepsilon = \varepsilon(n, v) > 0 \) such that if a complete manifold \( M^n \) admits a metric satisfying the conditions \( \text{Ric}_M \geq -\varepsilon \), \( d(M) = 1 \), and \( \text{vol}(M) \geq v \), then the fundamental group of \( M \) is of polynomial growth with degree \( \leq n \).

Our proof depends essentially on a recent result of M. Anderson [1].

**Theorem 2** (M. Anderson). In the class of compact \( n \)-dimensional Riemannian manifolds \( M \) such that \( \text{Ric}_M \geq (n - 1)\mu \), \( \text{vol}(M) \geq v \), and \( d(M) \leq D \), there are only finitely many isomorphism classes of \( \pi_1(M) \).

**Proof of Theorem 1.** Choose a base point \( \hat{x}_0 \) in the universal covering \( \tilde{M} \rightarrow M \), and let \( x_0 = p(\hat{x}_0) \) and \( g_1, \ldots, g_r \) be a set of generators of the fundamental group \( \pi_1(M) \) viewed as deck transformations in the isometry group of \( \tilde{M} \). Denote \( \Gamma(s) = \{ \text{distinct words in } \pi_1(M) \text{ of length } \leq s \} \), \( \gamma(s) = \#\Gamma(s) \), and \( l = \max_{1 \leq i \leq r}(d(\hat{x}_0, g_i(\hat{x}_0))) \).

Choose a fundamental domain \( F \) of \( \pi_1(M) \) which contains \( \hat{x}_0 \); then

\[
\bigcup_{g \in \Gamma(s)} g(F) \subset B_{s l + d}(\hat{x}_0),
\]

where \( d = d(M) = 1 \). Therefore,

\[
\gamma(s) \cdot \text{vol}(M) \leq \text{vol}(B_{s l + 1}(\hat{x}_0)).
\]
Now suppose, on the contrary, that for any $\epsilon > 0$, there is a manifold $M^n$ with a metric satisfying $\text{Ric}_M \geq -\epsilon$, $d(M) = 1$, $\text{vol}(M) \geq v$, and $\pi_1(M)$ is not of polynomial growth with degree $\leq n$. By the proof of Theorem 2, $\pi_1(M)$ has a presentation which obeys the following:

1. The number of generators $g_1, \ldots, g_N$ is uniformly bounded with $N \leq N(v/D^n, HD^2)$,
2. $d(g_j(x_0), x_0) \leq 3D$,
3. every relation is of the form $g_i g_j = g_k$.

The statements (2) and (3) have already been proved by Gromov [3]. By our assumption, $\pi_1(M)$ is not of polynomial growth with degree $\leq n$. In particular, when taking the above generators, we can find real numbers $s_i$ for all $i$ such that

$$\gamma(s_i) > s_i^n.$$ 

It is crucial that this relation is independent of $\epsilon$, as follows from (1) and (3).

On the other hand, by (1) we have

$$\gamma(s) \leq \frac{1}{v} \int_0^{3s+1} \left( \frac{\sinh \sqrt{\epsilon} t}{\sqrt{\epsilon}} \right)^{n-1} dt.$$ 

For any fixed, sufficiently large $s_0$, there is $\epsilon_0 = \epsilon(s_0)$ such that for all $s \leq s_0$, $\epsilon \leq \epsilon_0$,

$$\gamma(s) \leq \frac{6^n}{nv} s^n.$$ 

Now take $i_0 > 6^n/nv$. Then for $\epsilon < \epsilon(s_{i_0})$, using (2) and (3), we get a contradiction.

We would like to mention that Peter Peterson, working from a different orientation and with different technique, has obtained a slightly weaker result. Instead of a lower volume bound, he imposes a lower bound on the contractibility radius and arrives at the same conclusion.

ACKNOWLEDGMENTS

The author would like to thank Detlef Gromoll and Michael Anderson for very helpful conversations.

REFERENCES


Department of Mathematics, State University of New York at Stony Brook, Stony Brook, New York 11794–3651

Current address: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139