DEDEKIND DOMAINS AND GRADED RINGS

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Abstract. We prove that a Dedekind domain $R$, graded by a nontrivial torsionfree abelian group, is either a twisted group ring $k'[G]$ or a polynomial ring $k[X]$, where $k$ is a field and $G$ is an abelian torsionfree rank one group. It follows that $R$ is a Dedekind domain if and only if $R$ is a principal ideal domain. We also investigate the case when $R$ is graded by an arbitrary nontrivial torsionfree monoid.

We fix some notation and terminology. All rings $R$ are commutative with identity 1, and all semigroups $S$ are torsionfree. In case $S$ is a monoid, we denote by $e$ the identity of $S$ and by $\mathcal{U}(S)$ the group of invertible elements of $S$. For $s \in S$, denote by $(s)$ (respectively $(s)^1$) the subsemigroup (respectively submonoid) of $S$ generated by $s$. For more details on semigroups we refer to [3]. We say that $R$ is $S$-graded if $R = \bigoplus_{s \in S} R_s$, a direct sum of additive subgroups, such that $R_s R_t \subseteq R_{st}$, for all $s, t \in S$. The set $h(R) = \bigcup_{s \in S} R_s$ is the set of all homogeneous elements. If $T$ is a subset of $S$, then we put $R_{[T]} = \bigoplus_{t \in T} R_t$. Clearly, if $\text{Supp}(R) = \{s \in S | R_s \neq 0\}$, the support of $R$, then $R = R_{[\text{Supp}(R)]}$. Obviously $\text{Supp}(R)$ is a monoid if $R$ is a domain. If, moreover, $S$ is a group and $R_s R_t = R_{st}$ for all $s, t \in S$, then $R$ is called strongly $S$-graded. If $I$ is an ideal of $R$, we denote by $(I)_h$ the ideal generated by all homogeneous elements of $I$. If $I = (I)_h$, then $I$ is called a homogeneous ideal of $R$.

If $S$ is a monoid and $R$ is an $S$-graded integral domain, then $Q^G(R) = \{rc^{-1} | r \in R, c \in S, s \in S\}$, the graded quotient ring of $R$; if moreover, $S$ is cancellative, then $Q^G(R)$ is $G$-graded, where $G$ is the quotient group of $S$, and its component of degree $e$ is clearly a field. For further details on graded rings we refer to [6].

In recent years there has been a growing interest in divisibility properties of graded rings. For example, in [1, 7] graded rings which are factorial domains are investigated, while in [1, 2] graded rings that are Krull domains are studied. In this paper we investigate graded rings which are Dedekind domains.
We begin with two elementary lemmas.

**Lemma 1.** Let $G$ be a torsionfree abelian group, $R$ a $G$-graded ring and $S = \text{Supp}(R)$. If $R$ is a Dedekind domain, then $S$ is either a group or a torsionfree cancellative monoid with $\mathcal{Z}(S) = \{e\}$.

*Proof.* Since $R$ is a domain and $1 \in R$, $S$ is clearly a submonoid of $G$. Suppose $S \neq \mathcal{Z}(S)$. Since $T = S \setminus \mathcal{Z}(S)$ is an ideal of $S$, it follows that $R[T]$ is an ideal of $R$; and obviously $R/R[T] \cong R_{\mathcal{Z}(S)}$. Hence $R[T]$ is a nonzero prime ideal, and thus a maximal ideal of $R$. Therefore $R_{\mathcal{Z}(S)}$ is a field. Since $\mathcal{Z}(S)$ is a totally ordered group, it follows, using well-known techniques, that $\mathcal{Z}(S) = \{e\}$. This finishes the proof. □

**Lemma 2.** Assume $S$ is a nontrivial torsionfree cancellative monoid and $R$ is an $S$-graded Dedekind domain. If $S = \text{Supp}(\mathcal{A})$, then the quotient group $\langle S \rangle$ of $S$ has torsionfree rank 1.

*Proof.* Let $Q = Q^S(R)$, the graded quotient ring of $R$. Note that $Q$ is also a Dedekind domain. Since $Q$ is $\langle S \rangle$-graded and because $Q_e$ is a field, it is well known that $Q$ is a twisted group ring $Q_e[G]$ of a group $G$ over $Q_e$. Moreover, $S = \text{Supp}(R)$ yields $G = \langle S \rangle$. Let $F$ be a maximal free subgroup of $G$, then $G/F$ is a torsion group. Therefore, $Q$ is integral over $Q_e[F]$ and since $\dim(Q) = 1$ (note that $S$ is nontrivial) we obtain that $\dim(Q_e[F]) = 1$. Now $Q_e[F]$ is isomorphic with a Laurent polynomial ring over the field $Q_e$ in rank($F$) variables. It follows that rank($F$) = 1. The result follows. □

We consider the two cases mentioned in Lemma 1 separately. First we assume $\mathcal{Z}(S) = \{e\}$.

**Proposition 3.** Let $R$ be an $S$-graded ring, where $S = \text{Supp}(R)$ is a nontrivial torsionfree cancellative monoid with $\mathcal{Z}(S) = \{e\}$. Then $R$ is a Dedekind domain if and only if $R \cong k[X]$, a polynomial ring over a field $k$.

*Proof.* Let $T = S \setminus \{e\}$. Then $T$ is a nontrivial ideal of $S$. As in Lemma 1, it follows that $R[T]$ is a maximal ideal of $R$. Hence $R_e \cong R/R[T]$ is a field.

We first consider the case that $S = \mathbb{N}$, the nonnegative integers. Put $M = R_{[N_0]} = \bigoplus_{n > 0} R_n$, the unique maximal homogeneous ideal of $R$. Therefore, the ideal generated by $R_1$ is equal to $M^n$ for some $n \geq 1$. But then $n = 1$, and hence $R_m = R_1 R_{m-1} = R_1^n$, for every $m \geq 1$. Consequently $R = R_0 \oplus \sum_{n \geq 1} R_1^n$. Let $0 \neq r_1 \in R_1$; then $Rr_1$ is a nonzero homogeneous ideal and thus $Rr_1 = M$. Hence $R_1 = R_0 r_1$ and for each $n \geq 1$, $R_n = R_0 r_1^n$. So $R = \bigoplus_{n \in \mathbb{N}} R_0 r_1^n$, a polynomial ring in $r_1$ over $R_0$.

We now consider the general case. Let $s \in S$, $s \neq e$, then $\langle s \rangle^1 \cong \mathbb{N}$. Because of Lemma 2, $G = \langle S \rangle$ has torsionfree rank one. Let $\text{grp}(s)$ be the cyclic subgroup of $G$ generated by $s$, and let $\overline{G} = G/\text{grp}(s)$. It follows that $R$ is also a $\overline{G}$-graded ring with identity component $R_{\langle s \rangle^1}$. The latter follows from the fact that $\mathcal{Z}(S) = \{e\}$. It then follows from [1] that $R_{\langle s \rangle^1}$ is a Krull domain.
Now since $G$ is a torsion group and because $\mathbb{Z}(S) = \{ e \}$, we obtain that for every $e \neq t \in S$, there exist $n, m \in \mathbb{N}_0$ such that $t^n = s^m$. Consequently, $R$ is integral over $R_{(s)^1}$ and, therefore, the latter ring is of dimension 1. Hence $R_{(s)^1}$ is a Dedekind domain. It follows from the first case that $R_s = R_0 r_s$, $0 \neq r_s \in R_s$. Therefore, $R = R_0 [S]$, a twisted monoid ring. We now prove that $S \cong \mathbb{N}$; this will finish the proof. Since $S$ is torsionfree cancellative and has no nontrivial invertible elements, there exists a linear order $<$ on $S$ such that all elements of $S$ are positive. We assert that $S \setminus \{ e \}$ has a minimum element for this order. For if not, then $S$ has an infinite descending chain

$$s_1 > s_2 > s_3 > \cdots > s_n > \cdots > e.$$ 

But then one obtains the following infinite strictly ascending chain of ideals in $R$:

$$\sum_{s \geq s_1} R_s \subset \sum_{s \geq s_2} R_s \subset \cdots \subset \sum_{s \geq s_n} R_s \subset \cdots;$$

a contradiction. Let $s_1$ be the minimum element in $S \setminus \{ e \}$. So $M = \sum_{s \geq s_1} R_s$ is the unique maximal homogeneous ideal of $R$ and $RR_{s_1} = M$. Consequently, for every $s \in S \setminus \{ e \}$, $R_s \subset RR_{s_1}$ and thus $ss_1^{-1} \in S$. If $s_1^n < s < s_1^{n+1}$, $n \geq 1$, then $1 < ss_1^{-n} < s_1$, a contradiction since $ss_1^{-n} \in S$. Therefore,

$$e < s_1 < s_1^2 < \cdots < s_1^n < \cdots$$

is a strictly ascending chain of elements of $S$ which cannot be refined in $S$. Suppose there exists $t \in S$ such that $t > s_1^n$ for all $n \in \mathbb{N}$. Then by an argument as above, such a minimal element $t$ exists. But then $ts_1^{-1} < t$ and, for every $n \in \mathbb{N}$, $ts_1^{-1} > s_1^n$; a contradiction. Hence $S = \langle s_1 \rangle \cong \mathbb{N}$. \hfill $\square$

**Proposition 4.** Let $R$ be a $G$-graded ring, where $G = \text{Supp}(R)$ is a nontrivial torsionfree abelian group. If $R$ is a Dedekind domain, then $R = k'[G]$, a twisted group ring over a field $k$, and $G$ has torsionfree rank one.

**Proof.** It follows from Lemma 2 that $G$ is of rank one. Hence to prove the result, it is sufficient to show that $R_e$ is a field, or equivalently that $R$ has no nonzero homogeneous prime ideals. So we assume $R_e$ is not a field and derive a contradiction.

Let $P$ be a nonzero homogeneous prime ideal. Then $R/P$ is a field and a $G$-graded ring. Therefore $R/P$ is trivially graded, and thus $P = p + \sum_{g \in G \setminus \{ e \}} R_g$ where $p$ is a maximal ideal of $R_e$. Write $P = P(p)$. Conversely, let $p$ be a nonzero prime ideal of $R_e$. Then $Rp$ is an ideal of $R$ such that $Rp \cap R_e = p$. Let $M$ be a homogeneous ideal of $R$ maximal with respect to $M \cap R_e = p$. One easily verifies that $M$ is a prime ideal of $R$, and thus by the previous $M = P(p)$. Now fix a nonzero prime ideal $p$ in $R_e$. Let $T$ be the ring $R$ localized to the multiplicative set $R_e \setminus p$. Then $T$ is also a $G$-graded Dedekind domain, and by the above $T$ has only one nonzero homogeneous prime ideal, namely $P(p)$ localized to $R_e \setminus p$. We may assume $T = R$. 


So let \( R \) be a \( G \)-graded Dedekind domain with unique nonzero homogeneous prime ideal \( P = P(p) = p + \sum_{g \in G \setminus \{e\}} R_g \). It follows that for every \( 0 \neq x \in P \cap h(R) \) there exists \( n(x) \geq 0 \) such that \( Rx = P^{n(x)} \). Since \( P = \sum_{x \in P \cap h(R)} Rx \), we obtain that \( P = Rx_g \) for some \( x_g \in R_g, \ g \in G \). Assume \( g \neq e \). Then \( R_g = R_{x_g} \) and thus \( RR_g = Rx_g = P \). Therefore, for every \( h \in G \setminus \{e\} \), \( R_h = R_g R_{g^{-1}h} \) and \( p = R_g R_{g^{-1}} \). Consequently, \( P^k \supseteq p + \sum_{n>0} R_{g^{-n}} \), for all \( k \geq 1 \); a contradiction as \( p \neq 0 \). Therefore, \( g = e \). In this case it follows that \( p = R_e x_e \) and \( R_g = R_g x_e \) for \( g \neq e \). We obtain \( RR_g = RR_g Rx_e \). This yields \( R = Rx_e \) and thus \( p = R_e \), a contradiction. This finishes the proof. \( \square \)

**Corollary 5.** Let \( R \) be a \( G \)-graded ring, where \( G \) is a torsionfree abelian group with the ascending chain condition on cyclic subgroups; and assume \( |\text{Supp}(R)| > 1 \). Then, \( R \) is a Dedekind domain if and only if \( R \cong k[X] \) or \( R \cong k[X, X^{-1}] \) for some field \( k \).

**Proof.** Since a torsionfree rank one abelian group with the ascending chain condition on cyclic subgroups is free \([4]\), the result follows from Lemmas 1 and 2 and Propositions 3 and 4. \( \square \)

**Remark.** Obviously, the condition in Proposition 4 is not sufficient, as, for example, a group algebra over an infinitely generated rank one group is not Noetherian.

Also, \( G \) does not need to have the ascending chain condition on cyclic subgroups. The following example is taken from \([2]\). Let \( G \) be an arbitrary torsionfree rank one abelian group and \( R = k[X_g | g \in G] \), a polynomial ring over the field \( k \). Obviously \( R \) is a unique factorization domain and is \( G \)-graded with \( \deg(X_{g_1} \cdots X_{g_r}) = g_1^{n_1} \cdots g_r^{n_r} \). Clearly \( Q = Q^G(R) = Q^G_e[G] \) is also a unique factorization domain. Moreover, \( Q \cong Q_e[Z][G/Z] \) is integral over \( Q_e[Z] \); so all nonzero prime ideals of \( Q \) are maximal. Therefore, \( Q \) is a Dedekind domain.

We obtain the following generalization of a result of Gilmer \([5, \text{Theorem 13.8}]\).

**Corollary 6.** Let \( S \) be a torsionfree cancellative monoid. Assume \( R \) is an \( S \)-graded ring with \( |\text{Supp}(R)| > 1 \). The following conditions are equivalent.

1. \( R \) is a Dedekind domain.
2. \( R \) is principal ideal domain.

In these cases, \( R \cong k^i[G] \), where \( G \) is a torsionfree rank one group, or \( R \cong k[X] \) with \( k \) a field.

**Proof.** We only need to observe that, when \( R \cong k^i[G] \cong [Z]^{G/Z} \) (\( G \) of rank 1) and \( R \) is a Dedekind domain, then \( R \) is a principal ideal domain. For this, let \( P \) be a prime ideal of \( R \). Then \( P = R r_1 + \cdots + R r_n \) for some \( r_i \in R \). Clearly there exists a finitely generated subgroup \( H \) of \( G \) such that \( r_i \in R_{\{H\}} \) for all \( 1 \leq i \leq n \). Since \( G \) is torsionfree of rank one, it follows that \( H \cong \mathbb{Z} \), \( R_{\{H\}} \cong k[X, X^{-1}] \) and therefore \( P \cap R_{\{H\}} = R_{\{H\}} r \) for some \( r \in R_{\{H\}} \). Hence \( P = Rr \). The result follows. \( \square \)
We now consider rings graded by noncancellative monoids.

**Proposition 7.** Let $R$ be an $S$-graded ring, where $S = \text{Supp}(R)$ is a nontrivial torsionfree noncancellative monoid. If $R$ is a Dedekind domain, then $|\text{Supp}(R)| = 2$ and $R = k + M$, where $k$ is a field and $M$ is a maximal ideal of $R$.

**Proof.** Since $S$ is torsionfree, $S = \bigcup_{\alpha \in \Gamma} S_\alpha$, the disjoint union of its cancellative Archimedean subsemigroups, with $\Gamma$ a semilattice. By $\leq$ we denote the partial order relation on $\Gamma$, that is $\alpha \leq \beta$ if $\alpha \beta = \alpha$. As $S$ is noncancellative, $|\Gamma| > 1$ and since $S$ is a monoid, $\Gamma$ has a maximum element, say $\delta$, with $e \in S_\delta$. Moreover, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$, a semilattice graded ring, where for every $\alpha \in \Gamma$, $R_\alpha = \bigoplus_{\gamma \in S_\alpha} R_\gamma$. Put $R' = \bigoplus_{\alpha \neq \delta} R_\alpha$, then $R'$ is a nonzero prime, and thus a maximal ideal of $R$. Hence $R_\delta$ is a field.

Let $e' \neq \delta$ and $P_\alpha = \bigoplus_{\beta \geq \alpha} R_\beta$, where the sum runs over all $\beta$ that are either incomparable with $\alpha$ or $\beta < \alpha$. Then $P_\alpha$ is an ideal of $R$ and $R/P_\alpha = \bigoplus_{\beta \geq \alpha} R_\beta \neq 0$ is a domain. So $P_\alpha$ is a maximal ideal in $R$ or $P_\alpha = \{0\}$. But in the first case $\bigoplus_{\beta \geq \alpha} R_\beta$ would be a field, which is impossible as each element of $R_\alpha \subseteq R'$ is not invertible. Hence for every $\alpha \neq \delta$, $P_\alpha = \{0\}$, that is $\{\beta \in \Gamma : \beta \leq \alpha \text{ or } \beta \text{ incomparable with } \alpha\} = \emptyset$. Therefore $\Gamma = \{\delta, \alpha\}$, $\alpha \neq \delta$ and $\alpha \delta = \alpha$; $R = R_\delta \oplus R_\alpha$ and $S = S_\delta \cup S_\alpha$. Now if $S_\delta \neq \{e\}$, then $S_\delta$ is a nontrivial torsionfree abelian group and $R_\delta$ is a field graded by $S_\delta$, which is impossible. So $S = \{e\} \cup S_\alpha$. Now since $S$ is not cancellative there exist $s \in S_\alpha$ and $t, t' \in S$ such that $st = st'$ and $t \neq t'$. But as $S_\alpha$ itself is cancellative we obtain that, say, $t = e$ and thus $t' \in S_\alpha$. This yields that $t'$ is an idempotent, and consequently, $S_\alpha = G$ is a group.

Let $1_G$ be the identity of $G$. We claim that $G = \{1_G\}$. For if not, then $R = (R_e + R_{1_G}) \oplus \sum_{g \in G \setminus \{e\}} R_g$ is a Dedekind domain graded by the nontrivial torsionfree group $G$ (the identity component being $R_e + R_{1_G}$). Lemma 4 implies that $R_e + R_{1_G}$ is a field. A contradiction as $R_{1_G}$ is a nonzero ideal of the latter ring. This proves the claim; and therefore $R = R_e + R_{1_G}$. This finishes the proof. \[\square\]

Note that there are plenty of Dedekind domains of the type $R = k + M$. For example polynomial rings $k[X]$ or power series $k[[X]]$ where $k$ is a field, or $\mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$. In case $R$ is a principal ideal domain we can prove that $R$ is embedded in a formal power series ring and contains a polynomial ring.

**Corollary 8.** With the notations and assumptions as in Proposition 7. If, moreover, $R$ is a principal ideal domain and $M = RX$, $X \in R$, then $k[X] \subseteq R = k + M \subseteq k[[X]].$

**Proof.** Let $R_e = k$, a field, then $R = k + RX$. It follows that $R = k + kX + kX^2 + \cdots + kX^n + RX^n$. Thus for every $r \in R$, and $n \geq 0$, there exist
r_0, r_1, \ldots, r_n \in k \text{ and } b_n \in RX \text{ such that } r = r_0 + r_1X + \cdots + r_nX^n + b_nX^n.

One easily verifies that the r_i's are uniquely determined by r. Hence we obtain a well-defined map

\[ \varphi : R \to k[[X]] : r \mapsto r_0 + r_1X + \cdots + r_nX^n + \cdots. \]

It follows that \( \varphi \) is a ring homomorphism. Moreover, since \( \bigcap_{n \in \mathbb{N}} RX^n = \{0\} \), \( \varphi \) is a monomorphism. This proves the result. \( \square \)

Of course not every principal ideal domain \( R \) is of the form \( k + M \). Let \( R = R[\frac{1}{X}] = R[X, X+X], \) that is the localization of \( R[X] \) with respect to the prime ideal generated by \( X^2 + X + 1 \). Clearly \( R \) is a principal ideal domain, and it is easily verified that \( R \) is not of the form \( k + M \) for some field \( k \) and nonzero ideal \( M \).

References


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