QUADRATIC FORMS WITH CUBE-FREE DISCRIMINANT

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ABSTRACT. Formulas are given for the number of genera of $\mathbb{Z}$-lattices with rank $n \geq 3$, signature $s$, and cube-free discriminant $\Delta$. The results are applied to study classification and orthogonal splittings in the indefinite case.

1. INTRODUCTION

Let $L$ be a $\mathbb{Z}$-lattice on a regular quadratic space $V$ of finite dimension $n \geq 3$ with signature $s$ and associated symmetric bilinear form $f: V \times V \to \mathbb{Q}$. For convenience, assume that $s \geq 0$ and $f(L, L) \subseteq \mathbb{Z}$. Let $x_1, \ldots, x_n$ be a $\mathbb{Z}$-basis for $L$ and put $dL = \det f(x_i, x_j)$, the discriminant of the lattice. The lattice is called even if $f(x, x) \in 2\mathbb{Z}$ for all $x \in L$, otherwise $L$ is odd. We study lattices $L$ with cube-free discriminant; thus $dL = \Delta$ where $\Delta = (-1)^{(n-s)/2}d^2D$, or $\Delta = (-1)^{(n-s)/2}2d^2D$, or $\Delta = (-1)^{(n-s)/2}4d^2D$ with $dD > 1$, squarefree and odd. When $\Delta$ is odd, even lattices $L$ exist if and only if $n$ is even and $D \equiv (-1)^{s/2} \mod 4$, and when $\Delta \equiv 2 \mod 4$, even $L$ exist if and only if $n$ is odd (by Chang [2, Satz 2]).

Denote by $G_e(n, s, \Delta)$ the number of genera of even lattices with rank $n$, (compatible) signature $s$, and discriminant $\Delta$, and by $G_0(n, s, \Delta)$ the corresponding number of genera of odd lattices. Let $e$ and $f$ be the number of primes dividing $d$ and $D$, respectively.

Theorem 1. Assume $\Delta = (-1)^{(n-s)/2}d^2D$. Then for even $n \geq 4$ and $D \equiv (-1)^{s/2} \mod 4$,

$$G_e(n, s, \Delta) = \begin{cases} 2^{e-1}(2^e + (-1)^{s/4}) & \text{if } D = 1, \\ 2^{2e+f-1} & \text{if } D > 1; \end{cases}$$

and for any $n \geq 3$,

$$G_0(n, s, \Delta) = 2^{2e+f}.$$
Theorem 2. Assume $\Delta = (-1)^{(n-s)/2} 2 d^2 D$. Then for odd $n \geq 3$,

$$G_e(n, s, \Delta) = \begin{cases} 
2^{e-1}(2^e + (-1)^{(s^2-1)/8}) & \text{if } D = 1, \\
2^{2e+f-1} & \text{if } D > 1; 
\end{cases}$$

and for any $n \geq 3$,

$$G_0(n, s, \Delta) = 2^{2e+f}.$$

Theorem 3. Assume $\Delta = (-1)^{(n-s)/2} 4 d^2 D$. Then for odd $n \geq 3$,

$$G_e(n, s, \Delta) = 2^{2e+f} ;
$$

for even $n \geq 4$,

$$G_e(n, s, \Delta) = \begin{cases} 
2^{e-1}(3 \cdot 2^e + (-1)^{s/4}) & \text{if } D = 1 \text{ and } s \equiv 0 \mod 4, \\
2^{2e+f} & \text{if } D \equiv -(-1)^{s/2} \mod 4, \\
3 \cdot 2^{2e+f-1} & \text{if } D > 1 \text{ and } D \equiv (-1)^{s/2} \mod 4; 
\end{cases}$$

and for any $n \geq 5$, and for $n = 4$ with $\Delta \equiv 12 \mod 16$,

$$G_0(n, s, \Delta) = 2^{2e+f+2}.$$

Also $G_0(3, s, \Delta) = 3 \cdot 2^{2e+f}$, provided $D > 1$.

In the indefinite case, where $|s| < n$, the class and genus coincide under our assumptions on $\Delta$, and hence these three theorems also determine the number of classes of lattices. Part of Theorem 1 was established in [6]; notation and terminology follow this earlier paper. A knowledge of the classification of lattices is implicit in these three theorems and will now be stated as a separate theorem. For $p$ dividing $\Delta$, let $L_p = L_0(p) \perp L_1(p)$ be a Jordan splitting (see [7, 91C]), where $L_0(p)$ is unimodular and $L_1(p)$ is either $p$-modular or $p^2$-modular.

Theorem 4. Let $L$ and $M$ be two lattices with the same cube-free discriminant $\Delta$ on the same quadratic space $V$. Then $L$ and $M$ are in the same genus (same class when $V$ is indefinite) if and only if

(i) $L_p$ and $M_p$ have the same type for $p \mid 2d$,

(ii) $dL_0(p) = dM_0(p)$ when $p \mid d$ and $L_1(p)$ is $p^2$-modular,

(iii) if $L$ is odd, with $L_1(2) = (4\eta)$ and $M_1(2) = (4\epsilon)$, then $\eta \equiv \epsilon \mod 4$.

The type of $L_p$ is defined in [7, 91C]. It is not necessary to include the local invariant $dL_0(p) = dM_0(p)$ for $p \mid dD$ when $L_1(p)$ is $p$-modular since this is ensured by the local invariant $S_p V$ (see [6]). Theorem 4 can also be derived from some of the more general results in [1] or [7].

2. Proof of Theorem 1

The formula for $G_e(n, s, \Delta)$ was derived in [6] and the method of proof presented is basic to determining $G_e$ and $G_0$ in other situations; only the local
structure of $L_2$ changes since the structure of $L_p$, $p$ odd, is the same as in [6]. Now assume $L$ is odd. When $n = 3$ there are exactly eight possibilities for $L_2$; namely, $(1, 1, \pm 1), (-1, -1, \pm 1), (1, 1, \pm 3), (-1, \pm 1, 3)$ (since $(1, 3) = (-1, -3), (-1, -1) = (3, 3)$, etc.). These eight lattices are distinguished by the eight possible values of the local invariants $S_2V$ and $dL_2$. For $n > 3$, by [5, Theorem 1], $L_2$ is of the form $(1, \ldots, 1) \perp N_2$ where $N_2$ has rank three and so is one of the eight possibilities above. As explained in detail in [6], for each of the $e$ primes dividing $d$ there are four possible local lattices $L_p$, and for each of the $f$ primes dividing $D$ there are two possible $L_p$. The choice for $V_\infty$ is fixed by the signature. Therefore, $\Delta$ and Hilbert reciprocity uniquely determine the choice of $L_2$. For each set of choices of local lattices, an analogue of Theorem 1 in [6] ensures the existence of a corresponding $\mathbb{Z}$-lattice. Hence $G_0(n, s, \Delta) = 2^{e+f}$. The above discussion includes a proof of Theorem 4 when $\Delta$ is odd; note that once $V$ and $\Delta$ are fixed, so are $S_2V$ and $dL_2$, and hence $L_2$ is uniquely determined by its type.

3. PROOF OF THEOREM 2

We consider first even lattices so that $n \geq 3$ is odd, and modify the proof of Theorem 3 in [6]. Let

$$r = \#\{p: p|d \text{ and } S_pV = -1\}$$

and

$$t = \#\{p: p|D \text{ and } S_pV = -1\}.$$ 

By Hilbert reciprocity, $r + t \equiv c \mod 2$ where $S_2V S_\infty V = (-1)^c$; we postpone calculating $c$ until it is needed. For odd primes $p$, the local structure of $L_p$ is essentially the same as in [6], since the extra 2 in the discriminant is a local unit and can be absorbed. However, there are now four possibilities for $L_2$; namely,

$$L_2 = H \perp \cdots \perp H \perp (2) \text{ when } D \equiv (-1)^{(s-1)/2} \mod 8,$$

$$L_2 = H \perp \cdots \perp H \perp (-2) \text{ when } D \equiv (-1)^{(s+1)/2} \mod 8,$$

$$L_2 = H \perp \cdots \perp H \perp B_2 \perp (2) \text{ when } D \equiv 3(-1)^{(s+1)/2} \mod 8,$$

$$L_2 = H \perp \cdots \perp H \perp B_2 \perp (-2) \text{ when } D \equiv 3(-1)^{(s-1)/2} \mod 8,$$

where $B_2$ is the binary anisotropic unimodular plane (since $B_2 \perp \langle 2 \cdot 3\eta \rangle = H \perp \langle -2\eta \rangle$ for $\eta = \pm 1, \pm 3$). These four cases can be distinguished by the local discriminant $dL_2$. Moreover, the value of $S_2V = S_2L_2$ is now determined by $D$, $n$, and $s$; hence $c$ is also a function of $D$, $n$, and $s$. The analogue of [6, Theorem 1] now holds when these four possibilities are incorporated. The analogue of [6, Theorem 2] then follows with the same proof (except the value of $S_2V$ is changed). The same calculation as in [6, Theorem 3] then establishes Theorem 2, except that when $D = 1$, we get $G_e(n, s, \Delta) = 2^{e-1}(2^e + (-1)^c)$. 

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We must now compute \( c \mod 2 \). In this case

\[
L_2 = H \perp \cdots \perp H \perp \langle (-1)^{(s-1)/2} \rangle
\]

so that, using Hilbert symbols as in [6],

\[
S_2 V = (-1)^{(n^2-1)/8} \langle (-1)^{(s-1)/2} \rangle \langle (-1)^{(n+1)/2} \rangle.
\]

It follows that \( c \equiv \frac{1}{8}(s^2 - 1) \mod 2 \), as required.

Now assume \( L \) is odd and \( n \geq 3 \) arbitrary. For fixed \( n \) it is easily seen that there are exactly eight possible lattices \( L_2 \) distinguished by the values of \( dL_2 \) and \( S_2 V \); namely, \( L_2 = \langle 1, \ldots, 1 \rangle \perp N_2 \) where \( N_2 = \langle \pm 1, 2 \rangle, \langle \pm 1, 2 \cdot 3 \rangle, \langle 3, \pm 2 \rangle, \) or \( \langle -3, \pm 2 \cdot 3 \rangle \). Thus \( G_0(n, s, \Delta) = 2^{2e+f} \), as in the proof of Theorem 1. A proof of Theorem 4 for this situation is implicitly included above.

4. Proof of Theorem 3

Let \( \Delta = (-1)^{(n-s)/2} 4d^2 D \) and assume first that \( L \) is even. The local structure for odd primes is the same as in [6]. When \( n \geq 3 \) is odd, there are two distinct possibilities for \( L_2 \) for each \( n \), \( s \), and \( D \), namely,

\[
L_2 = H \perp \cdots \perp H \perp \langle 4\eta \rangle \quad \text{with } \eta \equiv (-1)^{(s-1)/2} D \mod 8,
\]

and

\[
L_2 = H \perp \cdots \perp H \perp B_2 \perp \langle 4\eta \rangle \quad \text{with } \eta \equiv (-1)^{(s+1)/2} 3D \mod 8.
\]

Modifying the arguments in [6], it follows that each of these \( L_2 \) gives \( 2^{2e+f-1} \) distinct genera when \( D > 1 \), so that \( G_e(n, s, \Delta) = 2^{2e+f} \). For \( D = 1 \), each of these \( L_2 \) gives \( 2^{s-1}(2^e + (-1)^c) \) distinct genera, where \( (-1)^c = S_2 V S_\infty V \). These two \( L_2 \) lie on spaces whose Hasse symbols \( S_2 V \) have opposite signs (even when \( D > 1 \)) and hence the two terms \( 2^{s-1}(-1)^c \) in the expressions counting the genera cancel. Thus \( G_e(n, s, \Delta) = 2^{2e} \). This also establishes Theorem 4 in this case since, although \( \eta \mod 8 \) is an invariant of \( L_2 \), the Hasse symbol \( S_2 V \) and \( \Delta \) already distinguish the possibilities.

Next consider \( n \geq 4 \) even. There are now ten possibilities for \( L_2 = H \perp \cdots \perp H \perp N_2 \) with \( N_2 \) one of the following rank four lattices:

(i) if \( dN_2 = 4 \), then \( N_2 = H \perp 2H, B_2 \perp 2B_2 \), or \( H \perp \langle 2, -2 \rangle \),

(ii) if \( dN_2 = -4.3 \), then \( N_2 = H \perp 2B_2, B_2 \perp 2H \) or \( H \perp \langle 2, 2.3 \rangle \),

(iii) if \( dN_2 = -4 \), then \( N_2 = H \perp \langle 2, 2 \rangle \) or \( H \perp \langle -2, -2 \rangle \),

(iv) if \( dN_2 = 4.3 \), then \( N_2 = H \perp \langle 2, -2.3 \rangle \) or \( H \perp \langle -2, 2.3 \rangle \).

To see that this list is complete, observe that \( H \perp \langle 2\eta \rangle = B_2 \perp \langle -2.3\eta \rangle \) for any \( \eta \equiv 1 \mod 2 \). Thus when \( D \equiv (-1)^{s/2} \mod 4 \) there are three possible \( L_2 \), while for \( D \equiv (-1)^{s/2} \mod 4 \) there are only two possible \( L_2 \). For each choice of \( L_2 \) a slightly modified form of Theorem 1 in [6] holds. When \( D > 1 \) it follows that \( G_e(n, s, \Delta) = 3.2^{2e+f-1} \) for \( D \equiv (-1)^{s/2} \mod 4 \), and \( G_e(n, s, \Delta) = 2^{2e+f} \) for \( D \equiv (-1)^{s/2} \mod 4 \). Next assume \( D = 1 \). Then \( \Delta \equiv \pm 4 \mod 32 \) and only
the cases (i) and (iii) can occur. For (iii), where \( s \equiv 2 \mod 4 \), the two spaces \( V_2 \) and \( V'_2 \) that occur have \( S_2 V_2 = -S_2 V'_2 \), so that the two terms \( 2^{e-1}(-1)^c \) cancel, and \( G_e(n, s, \Delta) = 2e \). Finally, for (i), where \( s \equiv 0 \mod 4 \), we must consider \( N_2 = H \perp 2H \), \( B_2 \perp 2B_2 \), or \( H \perp (2, -2) \). The first and third \( \mathbb{Z}_2 \)-lattices lie on the same hyperbolic space and hence have the same Hasse symbol \((-1)^{n(n+2)/8}\). Calculation shows that \( B_2 \perp 2B_2 \) lies on a space with the opposite Hasse symbol. It follows that \( G_e(n, s, \Delta) = 2^{e-1}(3.2e + (-1)^{s/4}) \).

Finally, we consider the local structure of \( L_2 \) when \( L \) is an odd lattice. For each \( n \geq 5 \) there exist 32 distinct \( L_2 \) divided into three basic types according to the nature of the nonunimodular component \( L_1(2) \) of \( L_2 \).

(i) There are eight distinct lattices of rank \( n = 5 \) when the nonunimodular component \( L_1(2) \) is \( 2H \) or \( 2B_2 \); namely, \( \langle 1, 1, \pm 1 \rangle \perp L_1(2) \) and \( \langle -1, -1, \pm 1 \rangle \perp L_1(2) \) (the equivalences \( \langle \pm 3 \rangle \perp 2B_2 = \langle \mp 1 \rangle \perp 2H \) can be used to remove any \( \langle \pm 3 \rangle \) components). Similarly, when \( n > 5 \), there are eight distinct lattices obtained from the above by adjoining the rank \( n - 5 \) lattice \( \langle 1, \ldots, 1 \rangle \); these lattices are distinguished by the local invariants \( S_2 V \) and \( dL_2 \). Note that when \( n = 3 \) only four lattices exist; one is excluded for each of the four possible values for \( dL_2 \). When \( n = 4 \) only six distinct lattices exist; the two excluded lattices have \( dL_2 \equiv 4 \mod 16 \). Note also, the lattices in this family do not diagonalize locally.

(ii) There are eight distinct lattices of rank \( n \geq 3 \) with a rank two nonunimodular component \( L_1(2) \) that diagonalizes (adjoin \( 2 \) to each of the lattices given at the end of the proof of Theorem 2). These eight lattices are distinguished by the invariants \( S_2 V \) and \( dL_2 \).

(iii) There are sixteen distinct rank four lattices with \( L_1(2) = \langle 4\eta \rangle \); namely, \( \langle 1, 1, \pm 1, 4\eta \rangle \) and \( \langle -1, -1, \pm 1, 4\eta \rangle \) where \( \eta = \pm 1 \). These lattices are distinguished by the invariants \( S_2 V \), \( dL_2 \), and \( \eta \mod 4 \). (If \( L_1(2) \) is \( \mathbb{Z}_2 x \) and \( \mathbb{Z}_2 y \) in two Jordan splittings of \( L_2 \), then \( x = ay + 4z \) for some \( z \in L_2 \) with \( f(y, z) = 0 \), and hence \( f(x, x) \equiv f(y, y) \mod 16 \), making \( \eta \mod 4 \) an invariant.) Higher rank lattices can be obtained by adjoining \( \langle 1, \ldots, 1 \rangle \). However, when \( n = 3 \), four of the sixteen possible lattices do not exist, one for each of the four values of \( dL_2 \).

These 32 lattices account for the extra factor of 4 in \( G_0(n, s, \Delta) \), \( n \geq 5 \). Only 24 of these lattices exist for \( n = 3 \), three for each value of \( dL_2 \) (which is determined by \( \Delta \)). An adjustment is also needed when \( n = 4 \). This completes the proofs of Theorems 3 and 4.

5. Orthogonal splittings

For certain discriminants \( \Delta \) an indefinite lattice \( L \) with \( dL = \Delta \) diagonalizes if and only if \( L_2 \) has an orthogonal basis. This has been studied in [4]. The above theorems lead to further results. As in [4], let \( \mathcal{D} \) (i) denote the set of discriminants of lattices \( L \) on spaces \( V \), with Witt index at least \( i = i(V) \), that diagonalize over \( \mathbb{Z} \) whenever the localization \( L_2 \) diagonalizes. Also, let
\( \mathcal{D}(\infty) \) denote the stable version consisting of discriminants where \( dL \in \mathcal{D}(\infty) \) means the lattice \( L \perp H^m \) diagonalizes for \( m \) sufficiently large, assuming \( L \) diagonalizes.

**Theorem 5.** Let \( \Delta \) be cube-free and not divisible by any prime \( p \equiv 1 \mod 4 \). Then \( \Delta \in \mathcal{D}(\infty) \).

More precisely, for \( e + f \geq 1 \), it follows from Theorems 1, 2, 3, and 4 that \( \pm d^2D, \pm 2d^2D \in \mathcal{D}(e+f) \) and \( \pm 4d^2D \in \mathcal{D}(e+f+1) \), provided each prime \( p \) dividing \( dD \) has \( p \equiv 3 \mod 4 \). The situation is far more complicated when primes \( p \equiv 1 \mod 4 \) divide \( \Delta \) (see [4]). We sketch the argument for \( \Delta = \pm 4d^2D \in \mathcal{D}(e+f+1) \). It suffices to show that there exist \( 3.2^{2e+f} \) distinct classes of diagonalized lattices with rank \( n \), signature \( s \), and discriminant \( \Delta \), since one quarter of the lattices counted in \( G_0(n, s, \Delta) \) do not diagonalize locally for \( p = 2 \). These lattices can be constructed as orthogonal sums by using as building blocks \( \langle \pm 4 \rangle \) and \( \langle 2, -2 \rangle \) for \( p = 2 \), \( \langle \pm p \rangle \) for each \( p \mid D \), \( \langle \pm p^2 \rangle \), \( \langle p, -p \rangle \) and either \( \langle p, p \rangle \) or \( \langle -p, -p \rangle \) for each \( p \mid d \), and \( \langle \pm 1 \rangle \) to fill out the required rank and signature (in some situations more care is needed). There are three choices when \( p = 2 \), two choices for each \( p \) dividing \( D \), and four choices for each \( p \) dividing \( d \). Since \( p \equiv 3 \mod 4 \) for each \( p \mid dD \), so that \(-1\) is not a local square, these \( 3.2^{2e+f} \) choices produce distinct sets of localizations of \( L \), and hence cover all possible classes.

The following result follows from the fact that \( G_0(n, s, \Delta) \) is independent of \( n \) and \( s \). A related theorem for square-free discriminants was established in [5], but by a different method.

**Theorem 6.** Let \( L \) be an indefinite odd lattice with rank \( n \geq 3 \) and cube-free discriminant \( \Delta \neq 0 \mod 4 \). Then \( L \) has an orthogonal splitting

\[
L = \langle \pm 1, \ldots, \pm 1 \rangle \perp T,
\]

where \( T \) is an indefinite odd ternary lattice.

Modified results hold when \( \Delta \equiv 0 \mod 4 \).

**Theorem 7.** Let \( L \) be an indefinite even lattice with rank \( n \geq 3 \) and cube-free discriminant \( \Delta \). Then \( L \) has an orthogonal splitting

\[
L = H \perp \cdots \perp H \perp E_8 \perp \cdots \perp E_8 \perp T,
\]

where \( T \) is an even lattice of rank \( t \) with

1. \( t = 3 \) if \( s \equiv 1 \mod 8 \),
2. \( t = 5 \) if \( s \equiv 5 \mod 8 \) (with \( t = 3 \) if \( i(L) \geq 3 \)),
3. \( t = 7 \) if \( s \equiv 7 \mod 8 \) (with \( t = 3 \) if \( i(L) \geq 2 \)),
4. \( t = 4 \) if \( s \equiv 2 \mod 8 \),
5. \( t = 6 \) if \( s \equiv 6 \mod 8 \) (with \( t = 4 \) if \( i(L) \geq 3 \)),
6. \( t = 8 \) if \( s \equiv 0 \mod 8 \) (with \( t = 4 \) if \( i(L) \geq 2 \)).
The key point in proving (i), for example, is that \( G_e(n, s, \Delta) = G_e(3, 1, \Delta) \) when \( s \equiv 1 \mod 8 \), and hence there exist exactly enough indefinite ternary lattices \( T \) to cover all the possibilities for \( L \); the number of hyperbolic planes \( H \) and even definite unimodular lattices \( E_8 \) is determined by the rank and signature of \( L \). The proof is similar when \( s \equiv 3 \mod 8 \), except now \( T \) is definite. Similar proofs hold in the other cases.

Remarks. The values \( t = 6, 7, \) and \( 8 \) in (v), (iii), and (vi) cannot, in general, be improved (see [5] for 6 and 7, for \( t = 8 \) adjoin (2.3.7) to the example for \( t = 7 \)). The other values for \( t \) are also best possible.

The splitting of indefinite lattices has also been considered by Gerstein [3] and Watson [8].

6. Square-free discriminants

We construct complete sets of representatives for the classes of even indefinite lattices with rank three, or four, and square-free discriminant. Let \( D = p_1 \cdots p_f \) where the \( p_i \) are \( f \) distinct odd primes. Choose \( \varepsilon_i = \pm 1, 1 \leq i \leq f \), with \( \prod_i \varepsilon_i = 1 \); there exist \( 2^{f-1} \) distinct choices for the \( \varepsilon_i \). First consider \( D \equiv 3 \mod 4 \). By Dirichlet's theorem, there exists a prime \( q \equiv 1 \mod 4 \) with \( (q/p_i) = \varepsilon_i, 1 \leq i \leq f \). It follows from quadratic reciprocity that \( (-D/q) = 1 \) and hence there exists an odd integer \( a > 0 \) with \( Da^2 \equiv -1 \mod 4q \). Put \( b = (Da^2 + 1)/4q \) and

\[
B = B(e_1, \ldots, e_n) = \begin{pmatrix} 2b & aD \\ aD & 2qD \end{pmatrix}.
\]

Then \( B \) corresponds to an even definite lattice with \( dB = D \). Also, locally, \( B_p = (2qD, 2q) \) for \( p|D \), and hence the \( 2^{f-1} \) lattices \( B(e_1, \ldots, e_n) \) are locally distinct (for some \( p \)). Since \( G_e(4, 2, -D) = 2^{f-1} \), it follows that the \( 2^{f-1} \) lattices \( H \perp B(e_1, \ldots, e_n) \) form a complete set of representatives for the classes of even lattices with \( n = 4, s = 2, \) and \( \Delta = -D \). Similarly, \( G_e(3, 1, -2D) = 2^{f-1} \), and hence \( (-2) \perp B(e_1, \ldots, e_n) \) gives a complete set of representatives for the classes of even lattices with \( n = 3, s = 1, \) and \( \Delta = -2D \).

Now let \( D \equiv 1 \mod 4 \). Choose \( \varepsilon_i \) as above, and a prime \( q \equiv 1 \mod 4 \) with \( (q/p_i) = \varepsilon_i, 1 \leq i \leq f \). Then \( (D/q) = 1 \) and there exists an odd integer \( a > 0 \) with \( Da^2 \equiv 1 \mod 4q \). Put \( c = (Da^2 - 1)/4q \) and

\[
C = C(e_1, \ldots, e_n) = \begin{pmatrix} 2c & aD \\ aD & 2qD \end{pmatrix}.
\]

Then \( C \) is even and indefinite with \( dC = -D \). The two sets of lattices \( H \perp C(e_1, \ldots, e_n) \) and \( (2) \perp C(e_1, \ldots, e_n) \) form complete sets of representatives for the classes of even lattices with square-free discriminant for \( n = 4, s = 0 \) and for \( n = 3, s = 1 \), respectively.

The above constructions can be modified to get complete sets of representatives for indefinite odd lattices with \( n = 3 \) or 4 and square-free discriminant.
However, now all $2^f$ choices for the $e_i$ are needed, and the diagonal 2's in the matrix for $B$, or $C$, should be omitted.

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