

ON 2-STEP SOLVABLE GROUPS OF FINITE MORLEY RANK

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ABSTRACT. We prove the following results: **Theorem 1.** *Let G be a connected, centerless, solvable group of class 2 and of finite Morley rank. Then we can interpret in G finitely many connected, solvable of class 2 and centerless algebraic groups $\tilde{G}_1, \dots, \tilde{G}_n$ over algebraically closed fields K_i in such a way that G interpretably imbeds in $\tilde{G} = \tilde{G}_1 \oplus \dots \oplus \tilde{G}_n$. Furthermore, $G' = (\tilde{G})'$. Let $F(G)$ denote the Fitting subgroup of G .* **Theorem 2.** *Let $G, \tilde{G}, \tilde{G}_i$ be as in Theorem 1. Then (i) $F(G) = F(\tilde{G}) \cap G$. (ii) $F(G)$ has a complement V in $G : G = F \rtimes V$. (iii) Elements of $F(G)$ are unipotent elements of G in \tilde{G} . (iv) If the characteristic of each base field K_i of \tilde{G}_i is different from 0, then V is definable and its elements are semi-simple in \tilde{G} .*

1. INTRODUCTION

This is a continuation of [Ne]; two of the questions asked in that article will be answered.

Theorem 1. *Let G be a connected, centerless, solvable group of class 2 and of finite Morley rank. Then we can interpret in G finitely many connected, solvable of class 2 and centerless algebraic groups $\tilde{G}_1, \dots, \tilde{G}_n$ over algebraically closed fields K_i in such a way that G definably imbeds in $\tilde{G} = \tilde{G}_1 \oplus \dots \oplus \tilde{G}_n$. Furthermore $G' = (\tilde{G})'$.*

The groups \tilde{G}_i were defined in [Ne] and in fact most of the Theorem 1 was proved in [Ne]. We only need to prove the algebraicity of each \tilde{G}_i .

Let \tilde{G}, \tilde{G}_i be as in the conclusion of Theorem 1. We say that an element $g \in \tilde{G}$ is unipotent (resp. semi-simple) if $g = g_1 \cdots g_n$ with $g_i \in \tilde{G}_i$ and g_i unipotent (resp. semi-simple) in the algebraic sense (see e.g. [Sp] for a definition of these terms).

The Fitting subgroup of G is the maximal, connected, normal, nilpotent, definable subgroup of G . A group of finite Morley rank G has a unique Fitting

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subgroup that we denote by $F(G)$, or by F if there is no possible confusion. With these notation and convention we prove the following result.

Theorem 2. *Let G , \tilde{G} , \tilde{G}_i be as in Theorem 1. Then*

- (i) $F(G) = F(\tilde{G}) \cap G$.
- (ii) $F(G)$ has a complement V in G : $G = F \rtimes V$.
- (iii) Elements of $F(G)$ are unipotent elements of G (in \tilde{G}).
- (iv) If the characteristic of each base field K_i of \tilde{G}_i is different from 0, then V is definable and its elements are semi-simple in \tilde{G} .

We do not know if the elements of the complement V are semi-simple (or if V can be chosen semi-simple) in case the characteristic of one of the fields K_i is zero.

Now a few words placing this article in the historical context: Gregory Cherlin [Ch] and Boris Zil'ber conjectured independently that infinite simple groups of finite Morley rank are algebraic groups over algebraically closed fields (known under the name of Cherlin's Conjecture). The conjecture is still open. Since maximal solvable connected subgroups (i.e., Borels) of a simple algebraic group are an important tool in their study and classification, one may legitimately think that solvable subgroups will also be important in the study of simple groups of finite Morley rank. This hypothetical hope is our main reason for studying solvable groups of finite Morley rank. A more down-to-earth reason is that centerless solvable groups of finite Morley rank seem to be easier to understand than the simple ones (thanks to Zil'ber's theorem [Zi] that states that in such a group one can interpret an algebraically closed field).

It will be interesting to generalize these theorems to arbitrary connected solvable but non-nilpotent groups (with or without center) of finite Morley rank. Another interesting problem is to classify simple B - N pairs of rank 1 and of finite Morley rank with B satisfying the hypothesis on G of Theorem 1. Should then B be an algebraic group? (See e.g. [Car] for a definition of a B - N pair). This problem is interesting for the following reason: what prevented us from proving that the group G of Theorem 1 is a product of algebraic groups is the hypothetical existence of infinite proper definable multiplicative subgroups of an algebraically closed field of finite Morley rank; it would be interesting to know if this is still a problem in simple groups; in other words, is the existence of bad fields a serious obstacle to prove Cherlin's Conjecture? Thus the above problem of B - N pairs of rank 1 would be a test-case.

The notation is as in [Ne]. The article is self-contained, however the reader who wants a better understanding of the involved machinery should be familiar (at least superficially) with §§5 and 6 of that article. In any event all the results of [Ne] used in this article are stated (without proofs) in §§2 and 3.

To prove Theorems 1 and 2 we need to understand the structure of \tilde{G}_i and its relevant properties explained in [Ne]. We give a survey of these results in §2. In §3 we explain the structure of local rings discovered by Cohen [Co]. In [E-N]

we gave an easy proof of Cohen's structure theorem which makes the structure of such a ring more transparent. The last sections are devoted to the proofs of the theorems.

2. THE GROUPS \tilde{G} AND \tilde{G}_i

Let G be a connected, centerless, solvable of class 2 group of finite Morley rank. Thus G' is abelian. By Theorem 1 of [Ne], $G = G' \rtimes T$ for some definable (so also connected and abelian) subgroup T . Since G is centerless T acts faithfully on the abelian group G' . Therefore we may view T as a subset of $\text{End } G'$. Let R be the ring generated by this action of T in the endomorphism ring of G' . If we write G' additively, it is easy to see that

$$R = \mathbf{Z}[T]/\text{ann } G'$$

where elements of the group ring $\mathbf{Z}[T]$ act on G' in a natural way: if $\sum n_i t_i \in \mathbf{Z}[T]$ with $n_i \in \mathbf{Z}$, $t_i \in T$ and if $x \in G'$ then

$$\left(\sum n_i t_i\right)(x) = \sum n_i t_i(x)$$

with the additive notation,

$$\left(\sum n_i t_i\right)(x) = \prod (x^{n_i})^{t_i}$$

with the multiplicative notation. In this article we will use the additive notation. Notice that G' is an R -module. Let \tilde{G} be $G' \rtimes R^*$ where R^* denotes the multiplicative group of invertible elements of R . By [Ne, Proposition 26],

$$R = \bigoplus_{1 \leq i \leq \ell} R_i$$

for some specific infinite ideals R_i of R . By [Ne, Lemma 29] each R_i is a local ring with nilpotent maximal ideal M_i . By [Ne, Lemma 32], each R_i is interpretable in G , thus they all have finite Morley rank. By Cherlin-Reineke [Ch-R], the maximal ideals M_i are all definable and by Macintyre [Mac 2], R_i/M_i is an algebraically closed field. Let $K_i = R_i/M_i$.

Let $G'_i = R_i G'$. Since R_i is interpretable and G' definable, G'_i is a definable subgroup of G . Clearly $G' = \bigoplus G'_i$ and R_j^* acts as identity on G'_i if $i \neq j$. So the group \tilde{G} defined as $G' \rtimes R^*$ is canonically isomorphic to $\bigoplus_{1 \leq i \leq \ell} (G'_i \rtimes R_i^*)$. We let \tilde{G}_i denote the group $G'_i \rtimes R_i^*$. Since $T < R^*$, we clearly have an interpretable imbedding of G in \tilde{G} : $G < \tilde{G}$.

Let $S_{i1} < S_{i2} < \dots < S_{ik} = G'_i$ be the socle series of G'_i . Thus by definition S_{i1} is the subgroup generated by the minimal, G -normal, definable subgroups of \tilde{G}_i . S_{i1} is called the socle of \tilde{G}_i . It is normal, connected and definable. $S_{i,j+1}$ is the unique subgroup containing S_{ij} and having the following property: $S_{i,j+1}/S_{ij}$ is the socle of \tilde{G}_i/S_{ij} . S_{ij} is called the j th socle of \tilde{G}_i . Each quotient $S_{i,j+1}/S_{ij}$ is a direct sum of finitely many minimal definable

minimal normal subgroups of \tilde{G}_i/S_{ij} and the action of \tilde{G}_i/S_{ij} on these minimal subgroups is definably isomorphic to the action of the multiplicative group $(R_i/M_i)^*$ on the additive group $(R_i/M_i, +)$ by multiplication. These last sentences follow from [Ne, Corollary 9, Lemma 4] and the content of Zil'ber's Theorem explained at length in [Ne, §3]. By [Ne, Lemma 23], $M_i S_{i,j+1} \subseteq S_{ij}$. Since the elements of R_i^* are invertible, if $x \in S_{i,j+1} \setminus S_{ij}$ then $R_i^* x \cap S_{ij} = \emptyset$.

We will come back to the structure of \tilde{G}_i later in §3, after we get more information about local rings R_i .

3. THE RING R_i AND MORE ABOUT \tilde{G}_i

Let us drop the subscript i for the next two sections. Thus \tilde{G} , G , R , M , K , S_j , G' , R^* denote \tilde{G}_i , G_i , R_i , M_i , K_i , S_{ij} , G'_i , R_i^* respectively.

Each factor M^k/M^{k+1} is a K -vector space; by finiteness of Morley rank they have finite dimension. Since M is nilpotent, this shows that M is a finitely generated ideal of R . Hence M is definable. In fact by [Ne 1] every ideal of R is definable, connected and finitely generated. In particular R is Artinian and Noetherian.

We have to distinguish two cases: $\text{char } R = 0$ and $\text{char } R = p^{m+1} \neq 0$.

If $\text{char } R = 0$ then since M is nilpotent, $\text{char } K = 0$ also. In this case (in fact if $\text{char } K = \text{char } R$) we can find a copy of K in R (not necessarily definable); then R is isomorphic to $K[X_1, \dots, X_s]/I$ where I is an ideal containing a power of each monomial in X_1, \dots, X_s . M is also easy to recognize: $M \simeq \langle X_1, \dots, X_s \rangle / I$. This is the easy case.

Suppose $\text{char } R = p^{m+1} \neq 0$. We will explain the structure of R (see [Co] and also [E-N]). $1 + M$ is a subgroup of R^* . It is also a pure subgroup of bounded exponent. Therefore it has a complement that we denote by U :

$$R^* = (1 + M) \times U = U + M$$

where \times denotes the direct product sign in a multiplicative situation. It is easy to show that

$$U \simeq R^*/1 + M \simeq K^*.$$

Thus U is a divisible group. Also it is 0-definable (which is not the case if the characteristic is 0). In fact

$$U = (R^*)^{p^m}.$$

U is closed under product but $U \cup \{0\}$ is not closed under addition (unless $m = 0$). Let us close it under addition, i.e. let us consider the subring W of R generated by U . Then

$$W = U' + pU' + \dots + p^m U'$$

where $U' = U \cup \{0\}$. Furthermore the above "decomposition" has the following uniqueness property: if u_0, \dots, u_m and v_0, \dots, v_m are in U' and if $u_0 + pu_1 + \dots + p^m u_m = v_0 + \dots + p^m v_m$ then $u_j = v_j$ for all $j = 1, \dots, m$.

Notice also that $R = U' + M$ and $R^* = U + M$ and we again have uniqueness. Since U' is in bijection with K , it follows that W is in bijection with K^{m+1} .

W is a complete local ring with maximal ideal pW . Also $\text{char } W = \text{char } R = p^{m+1}$ and $W/pW \simeq K$. Given these data there is a unique such ring (see e.g. [Se, §5, Corollary of Proposition 10]). They are called Witt rings (not to be confused with the Witt rings of quadratic forms) and their structure is made explicit by Witt [W] and can also be found in [Jac 1] or [Jac 2]. However, we will not be concerned with their definition; for us it will be enough to know that if W is viewed as the variety K^{m+1} via the above decomposition then the addition and the multiplication of W are given by polynomial maps (see e.g. [Jac 1, Theorem 8.25]). We will refer to this fact by saying that W is an “algebraic ring”.

Let us go back to G' . We would like to have a better understanding of the action of U on G' (if $\text{char } R = 0$, then take U to be a copy of K^* found in R). Let S_{j+1}/S_j be one of the quotients considered in §2. We have already noticed that S_{j+1}/S_j is a direct sum of finitely many G/S_j -minimal subgroups. Choose a non-zero representative from each one of those minimal subgroups, say x_1, \dots, x_n . Thus

$$S_{j+1}/S_j = \overline{Rx_1} \oplus \dots \oplus \overline{Rx_n}$$

where $\overline{}$ means “modulo S_j ”. But $R = U' + M$ and $Mx_j \subseteq S_j$, thus

$$S_{j+1}/S_j = \overline{U'x_1} \oplus \dots \oplus \overline{U'x_n}.$$

The reader should be aware that if $\text{char } R \neq 0, p$ then $U'x_k$ is never a subgroup but only a unique set of representatives of Rx_k modulo S_j . Doing this for all j , we can find elements x_1, \dots, x_n of G' such that

$$G' = U'x_1 + \dots + U'x_n$$

with the property that if $u_1, \dots, u_n \in U', v_1, \dots, v_n \in U'$ and if $u_1x_1 + \dots + u_nx_n = v_1x_1 + \dots + v_nx_n$, then $u_j = v_j$ for all $j = 1, \dots, n$. We will say that x_1, \dots, x_n is a U' -base of G' . Now it is clear how U' acts on G' ; it acts componentwise:

$$u(u_1x_1 + \dots + u_nx_n) = uu_1x_1 + \dots + uu_nx_n.$$

Since the product uu_j depends polynomially on u and u_j , this remark will be valuable to prove Theorem 1. In fact at this point the proof Theorem 1 should be clear to the expert.

4. \tilde{G}_i IS AN ALGEBRAIC GROUP

We continue to omit the subscript i as in the previous section.

By definition $\tilde{G} = G' \rtimes R^*$. Thus to prove Theorem 1 we need to show that G' and R^* are algebraic groups and the action of R^* on G' is an algebraic

action. We fix a U' -base x_1, \dots, x_n of G' constructed in the previous section. Let p be the characteristic of K . p may be 0. For x_i ($i = 1, \dots, n$) and j an integer we can write

$$p^j x_i = \sum_k \alpha_{ijk} x_k$$

for some unique $\alpha_{ijk} \in U'$. We fix α_{ijk} throughout this section. Notice that if $x_i \in S_l$, the l th socle, then $p^j x_i \in S_{l-j}$ (because $MS_l \subseteq S_{l-1}$ and $p \in M$) and hence if $x_i \in S_j$ then $\alpha_{ijk} = 0$ for all k .

Lemma 1. G' is an algebraic group.

Proof. We identify G' with the variety K^n via the decomposition of G' with respect to the U' -base x_1, \dots, x_n :

$$G' = U'x_1 + \dots + U'x_n,$$

remembering that U' is in (interpretable) bijection with K . Let

$$x = \alpha_1 x_1 + \dots + \alpha_n x_n, \quad y = \beta_1 x_1 + \dots + \beta_n x_n$$

be two elements of G' . Then

$$x + y = \tau_1 x_1 + \dots + \tau_n x_n$$

for some unique $\tau_1, \dots, \tau_n \in U'$. We need to show that each τ_i depends on $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ polynomially, i.e. that there are n polynomials f_1, \dots, f_n , independent of $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$, with coefficients in $U' (\simeq K)$ and in $2n$ variables such that

$$f_i(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) = \tau_i$$

for each $i = 1, \dots, n$. We will prove this by induction on S_j (the j th socle of G').

Assume first that $x, y \in S_1$. We may delete $\alpha_i x_i$'s, $\beta_i x_i$'s for which $x_i \notin S_1$ to assume that each x_i figuring in the above expression of x or y is in S_1 . Then, with this convention, $p x_i = 0$. Now compute

$$x + y = \left(\sum_i \alpha_i x_i \right) + \left(\sum_i \beta_i x_i \right) = \sum_i (\alpha_i + \beta_i) x_i.$$

Since $\alpha_i + \beta_i$ is not necessarily in U' we are not finished. But $\alpha_i + \beta_i$ is in W and W is an algebraic ring in the sense we gave to this term in the previous section. Thus

$$\alpha_i + \beta_i = \sum_{j=0}^m \tau_{ij} p^j$$

and each $\tau_{ij} \in U'$ depends polynomially on α_i and β_i (but the polynomial is independent of α_i and β_i). Putting this expression in the calculations of $x + y$ we obtain

$$x + y = \sum_i \sum_j \tau_{ij} p^j x_i.$$

But with the convention of the beginning of this paragraph, $p^j x_i = 0$ if $j \neq 0$. Thus

$$x + y = \sum_i \tau_{i0} x_i,$$

and we are done with the first step of the induction.

Let us assume we know how to calculate the sum of elements of S_i in an algebraic way. Take x and y in S_{i+1} . Assume again, without loss of generality that if $x_j \notin S_{i+1}$ then $\alpha_j x_j, \beta_j x_j$ do not occur in the expressions of x and y . Then separate the basis x_1, \dots, x_n into two parts, the ones that belong to S_i , the ones that do not:

$$\begin{aligned} x &= \sum \alpha_k x_k + \sum \alpha_j x_j, \\ y &= \sum \beta_k x_k + \sum \beta_j x_j, \end{aligned}$$

where $x_k \in S_{i+1} \setminus S_i, x_j \in S_i$ with a lot of notational abuse. First add $\sum \alpha_k x_k$ and $\sum \beta_k x_k$ the way we did above, but this time $p x_k \in S_i$ for all k :

$$\sum \alpha_k x_k + \sum \beta_k x_k = \sum_k \tau_{k0} x_k + \sum_k \sum_{l>0} \tau_{kl} p^l x_k.$$

Now $x + y$ is equal to the sum of $\sum_k \tau_{k0} x_k$ and of three elements of S_i , namely of

$$\sum \alpha_j x_j, \quad \sum \beta_j x_j \quad \text{and} \quad \sum_k \sum_{l>0} \tau_{kl} p^l x_k = \sum_m \sum_k \sum_{l>0} \tau_{kl} \alpha_{klm} x_m.$$

By induction this second sum (of the above three elements) is algebraic; by the algebraicity of W, τ_{k0} are polynomials. This proves the lemma. \square

Lemma 2. *The action of R^* on G' is an algebraic action.*

Proof. $R^* = U + M$. We know that the action of U on G' is componentwise (see the last paragraph of §3) and G' is an algebraic group. Thus it remains to prove that the action of M on G' is algebraic. We will first choose a “ U -base” of M , the same way we did for G' . Remember that M^i/M^{i+1} is a finite-dimensional vector space over R/M . Choose a set of representatives of M^i/M^{i+1} for a K -base and do this for each i , to get a set of elements $y_1, \dots, y_r \in M$ for which

$$M = U'y_1 + \dots + U'y_r.$$

Then we have the usual “uniqueness” property: if $u_1, \dots, u_r, v_1, \dots, v_r \in U'$ and if $u_1 y_1 + \dots + u_r y_r = v_1 y_1 + \dots + v_r y_r$ then $u_i = v_i$ for each i . With this decomposition and with what we already know about the action of U' on G' (see the end of §3) we reduce the problem to the algebraicity of the action of each y_i on G' . But this is clear. Let x_1, \dots, x_n be a U' -basis of G' as in the proof of Lemma 1. Then

$$y_i x_j = \sum_k \beta_{ijk} x_k$$

for some fixed $\beta_{ijk} \in U' (\simeq K)$. The lemma is now proved. \square

Lemma 3. R^* is an algebraic group.

Proof. Let y_1, \dots, y_r be as in Lemma 2. Thus

$$R^* = U + U'y_1 + \dots + U'y_r$$

in a “unique” way. Identify R^* with the variety $K^* \times K^r$ via this decomposition. Let

$$y_i y_j = \sum_k \gamma_{ijk} y_k$$

for some unique γ_{ijk} . Now the proof of the lemma is as the previous lemmas and we omit it. \square

5. PROOF OF THEOREM 2

Now we have to put back the subscripts. Notation is as in §§2 and 3. Let

$$U = \bigoplus_{i=1}^n U_i,$$

$$X = \prod_{i=1}^n (1 + M_i),$$

$$X_b = \prod \{1 + M_i : R_i \text{ has characteristic } p \neq 0\},$$

$$X_d = \prod \{1 + M_i : R_i \text{ has characteristic } 0\}.$$

Then $X = X_b \times X_d$, X_b has bounded order, X_d is divisible, X_b and X_d are definable and connected ($1 + M_i^k / 1 + M_i^{k+1} \simeq M_i^k / M_i^{k+1}$ and the latter is a vector space over K_i). Notice that X_d has no elements of finite order. It is clear that $G'_i \rtimes (1 + M_i)$ is the set of unipotent elements of \tilde{G}_i and it can be easily checked to be the Fitting subgroup of \tilde{G}_i . Also $R^* = X \times U$. Thus

$$\begin{aligned} F(\tilde{G}) &= G' \rtimes X, \\ \tilde{G} &= F(\tilde{G}) \rtimes U. \end{aligned}$$

Since $U_i \simeq K_i^*$, U is a divisible group. When we will speak about \tilde{G} as a group of finite Morley rank, we will mean it is so with the G -language. In other words we assume that G is a definable subgroup of \tilde{G} even though it may not be in the pure language of groups.

By Macintyre [Mac 1], $T = B \oplus D$ where T is a complement of G' , B has bounded order and D is divisible. If $B^k = 1$ then $D = T^k$ and so D is definable and being divisible it is connected. But B is not necessarily definable. We will replace B with a definable subgroup. Let

$$C = \{t \in T : t^k = 1\}.$$

Since T is abelian, C is a definable subgroup containing B . We will show that $T = C^\circ \oplus D$.

Lemma 1. $C^\circ \subseteq F(G)$.

Proof. By [Ne, Theorem 6], G/F is a divisible group of finite Morley rank. Since $C^\circ F/F$ has finite exponent, $C^\circ F/F$ is finite. But $C^\circ F$ is clearly connected, thus so is $C^\circ F/F$. Hence $C^\circ F = F$, $C^\circ \subseteq F$. \square

Lemma 2. $F(G) \subseteq F(\tilde{G}) \cap G$.

Proof. $F(G)$ is a nilpotent, definable, connected subgroup of G . Since \tilde{G} is interpretable in G and we consider \tilde{G} with the G -language, these properties are preserved in \tilde{G} . It remains to show that $F(G) \triangleleft \tilde{G}$. Since $G' \subseteq F(G)$, $\tilde{G} = G' \rtimes R^*$ we have

$$F(G) = G' \rtimes (F(G) \cap R^*).$$

G' clearly normalizes $F(G)$. Let us show that R^* also normalizes $F(G) = G' \rtimes (F(G) \cap R^*)$; this will show that $F(G) \triangleleft \tilde{G}$. R^* normalizes G' by its definition. Since $F(G) \cap R^* \subseteq R^*$ and R^* is commutative, R^* also normalizes $F(G) \cap R^*$. Thus R^* normalizes $F(G)$. \square

Lemma 3. $C^\circ \subseteq X_b$.

Proof. We have $C^\circ < T < R^* = X \times U = X_b \times X_d U$. Let π be the second projection map from R^* onto $X_d U$. $\pi(C^\circ)$ is a subgroup of finite exponent of the divisible definable group $X_d U$. Thus $\pi(C^\circ)$ is finite. On the other hand C° being connected, so is its projection $\pi(C^\circ)$. Therefore $\pi(C^\circ) = 1$ and $C^\circ \subseteq X_b$. \square

Lemma 4. $X_b \cap (X_d U D) = 1$.

Proof. We have

$$X_d U \subseteq X_d U D \subseteq R^* = X \times U = X_b \times X_d U,$$

so

$$X_d U D = (X_b \cap X_d U D) \times X_d U.$$

Clearly $X_b \cap X_d U D$ is finite (one is bounded, the other one is divisible); thus, $X_d U$ is a subgroup of finite index of the divisible group $X_d U D$, so $X_d U = X_d U D$ and hence $X_b \cap X_d U D = 1$. \square

Lemma 5. $T = C^\circ \oplus D$.

Proof. By Lemmas 3 and 4, $C^\circ \cap D = 1$. Let us show that $C^\circ D = T$; then the lemma will be proved. Since C/C° is finite, so is $B/C^\circ \cap B$. Since $T = BD$ this shows that $[T: (B \cap C^\circ)D]$ is finite. But

$$(B \cap C^\circ)D \subseteq C^\circ D \subseteq T,$$

so also $[T: C^\circ D]$ is finite. T being connected and $C^\circ D$ definable we get $T = C^\circ D$. \square

Lemma 6.

- (i) $F(G) = F(\tilde{G}) \cap G$.
- (ii) $F(G) = G' C^\circ \rtimes (D \cap F(\tilde{G}))$.
- (iii) $D \cap F(\tilde{G})$ is a divisible group.
- (iv) $D \cap F(\tilde{G}) \subseteq X_d$.

Proof. Clauses (ii), (iii) and (iv) will be proved during the proof of (i). By Lemma 2, to prove (i), we only need to show that $F(\tilde{G}) \cap G \subseteq F(G)$. Now, we have

$$G' \subseteq F(\tilde{G}) \cap G \subseteq G = G' \rtimes T;$$

therefore

$$(1) \quad F(\tilde{G}) \cap G = G' \rtimes (T \cap F(\tilde{G}) \cap G) = G' \rtimes (T \cap F(\tilde{G})).$$

Thus we just need to prove that $T \cap F(\tilde{G}) \subseteq F(G)$. But $T = C^\circ \oplus D$ and $C^\circ \subseteq F(G) \subseteq F(\tilde{G})$; thus,

$$(2) \quad T \cap F(\tilde{G}) = C^\circ \oplus (D \cap F(\tilde{G})).$$

Since $C^\circ \subseteq F(G)$, we just need to prove that $D \cap F(\tilde{G}) \subseteq F(G)$. Also $D < T < R^*$, $R^* \cap G' = 1$ and $X \subseteq R^*$, so

$$(3) \quad D \cap F(\tilde{G}) \subseteq R^* \cap (G' \rtimes X) = R^* \cap X = X.$$

Now it follows from Lemma 4 that $D \cap F(\tilde{G}) \subseteq X_d$ (which is iv)). We distinguish two cases:

Case 1. $D \cap F(\tilde{G})$ is infinite. Then, being an infinite subgroup of the divisible group D , $D \cap F(\tilde{G})$ is divisible and hence connected. (2) shows that then $T \cap F(\tilde{G})$ is connected and by (1), $F(\tilde{G}) \cap G$ is connected. Since $F(\tilde{G}) \cap G$ is clearly definable, nilpotent and normal, with its connectedness we obtain $F(\tilde{G}) \cap G \subseteq F(G)$, proving the Lemma in Case 1.

Case 2. $D \cap F(\tilde{G})$ is finite. By (3) $D \cap F(\tilde{G}) \subseteq X = X_b X_d$. Since $D \cap F(\tilde{G})$ has finite exponent and X_d is torsion-free we have $D \cap F(\tilde{G}) \subseteq X_b$. Thus

$$D \cap F(\tilde{G}) \subseteq X_b \cap D,$$

but $X_b \cap D = 1$ by Lemma 4, so $D \cap F(\tilde{G}) = 1$. By (2) $T \cap F(\tilde{G}) = C^\circ \subseteq F(G)$ and the Lemma is now proved. \square

Lemma 7. Elements of $F(G)$ are the unipotent elements of G .

Proof. Since this is true for $F(\tilde{G})$, the lemma follows from the previous one. \square

Lemma 8. $G = F(G) \rtimes V$ for some divisible abelian group V of finite Morley rank.

Proof. Since $G = G' \rtimes T = G' C^\circ \rtimes D$, by Lemma 6 (ii) we need to prove that $X_d \cap D$ splits in D . But by Lemma 6 (iii) and (iv) $X_d \cap D$ is divisible. Thus $D = (X_d \cap D) \oplus V$ for some subgroup V (see, e.g., [Kap]). Now we have

$G = F(G) \rtimes V$. Since $G/F(G)$ is divisible and $V \simeq G/F(G)$ the rest of the Lemma is trivial. \square

It remains to prove (iv) of Theorem 2. If $\text{char } K_i \neq 0$ for all i , then $X_d = 1$ and thus, from Lemma 6 (iv) and (ii) we get

$$F(G) = G' C^\circ.$$

Since we know that $G = (G' C^\circ) \rtimes D$, we may take V to be D which is definable. Let us show that elements of D are semi-simple, i.e., that $D \subseteq U$. We have

$$U \subseteq DU \subseteq R^* = X \times U;$$

thus $DU = (DU \cap X) \times U$. But X has finite exponent this time. Thus $DU \cap X$ is finite (because DU is divisible). Therefore U is a subgroup of finite index of the divisible group DU , so $U = DU$, $D \subset U$. This proves Theorem 2 completely.

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