NONEXISTENCE OF POSITIVELY EXPANSIVE MAPS
ON COMPACT CONNECTED MANIFOLDS WITH BOUNDARY

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(Communicated by Dennis Burke)

Dedicated to Professor Ryosuke Nakagawa on his sixtieth birthday

Abstract. In this note we prove that no compact connected manifold with boundary admits a positively expansive map.

Let $(X, d)$ be a compact metric space and $f: X \to X$ be a continuous map (not always surjective). We say that $f$ is positively expansive if there is a constant $c > 0$ such that if $x, y \in X$ and $x \neq y$ then $d(f^i(x), f^i(y)) > c$ for some $i \geq 0$ ($c$ is called an expansive constant for $f$). The notion of positive expansiveness is independent of the metrics compatible with original topology, and preserved under topological conjugacy. In D. W. Curtis and S. Miklos [3] they proved that no positively expansive map of $X$ onto $X$ can be a homeomorphism if $X$ is nontrivial and connected. In fact, whenever $X$ admits a positively expansive homeomorphism $f$, $X$ is finite (cf. [1, 5]). This is easily checked as follows. By a result of W. L. Reddy [10], $f$ is expanding with respect to some metric $\rho$ for $X$, i.e. there exist constants $\delta > 0$ and $0 < \lambda < 1$ such that $\rho(x, y) < \delta$ implies $\rho(f^{-1}(x), f^{-1}(y)) \leq \lambda \rho(x, y)$. Thus $\Phi^- = \{f^{-i}: i \geq 0\}$ is uniformly equicontinuous. By a metric $D$ defined by $D(f, g) = \max\{\rho(f(x), g(x)): x \in X\}$, $\Phi^-$ is totally bounded. Let $\Phi^+ = \{f^i: i \geq 0\}$ and define a map $G: \Phi^- \to \Phi^+$ by $G(f^{-i}) = f^i$ for $i \geq 0$. Then $G$ is $D$-isometric. Therefore $\Phi^+$ is totally bounded. Since $X$ is compact, $\Phi^+$ is uniformly equicontinuous and so there is $\epsilon > 0$ such that $\rho(x, y) < \epsilon$ implies $\rho(f^i(x), f^i(y)) < \epsilon$ for all $i \geq 0$ ( $\epsilon$ is an expansive constant for $f$). This shows $x = y$ and therefore $x$ is an isolated point.

The study of positively expansive maps is an interesting subject in topological dynamics. In [3] the following is posed: “Characterize all compact connected manifolds which admit positively expansive maps.” Our aim is to give an answer for this problem.
Theorem 1. No compact connected manifold with boundary admits a positively expansive map.

M. Shub [11, 12], J. Franks [4], M. W. Hirsch [9], and M. Gromov [6] studied the problem of characterization of expanding differentiable maps on closed smooth manifolds. In [6] M. Gromov proved finally that an expanding differentiable map of an arbitrary closed smooth manifold is topologically conjugate to an expanding infra-nil-endomorphism. On the other hand, E. M. Coven and W. L. Reddy [2] studied positively expansive maps of closed topological manifolds and showed that such maps are expanding with respect to certain metrics. Recently the author [8] generalized a result of M. Gromov [6] as follows: a positively expansive map of an arbitrary closed topological manifold is topologically conjugate to an expanding infra-nil-endomorphism. Combining this and Theorem 1, we can conclude the following

Theorem 2. Every compact connected manifold which admits a positively expansive map is homeomorphic to an infra-nil-manifold, and such a map of an infra-nil-manifold is topologically conjugate to an expanding infra-nil-endomorphism.

Theorem 1 will be obtained by using the following.

Lemma. Let \( X \) be a compact connected locally connected metric space and \( f: X \to X \) be a positively expansive map. If a closed proper subset \( K \) of \( X \) satisfies the following conditions:

1. \( f(X \setminus K) \subseteq X \setminus K \),
2. \( f|_{X \setminus K}: X \setminus K \to X \setminus K \) is an open map,

then \( K = \emptyset \) (compare with Theorem 3 of [3]).

To use the lemma for the proof of Theorem 1, let \( M \) be a compact connected manifold and \( \partial M \) denote the boundary of \( M \). Suppose that \( M \) admits a positively expansive map \( f \). From the definition it follows that \( f \) is locally injective. Combining this fact and Brouwer's theorem on invariance of domain, we have that \( f(M \setminus \partial M) \subseteq M \setminus \partial M \) and \( f|_{M \setminus \partial M}: M \setminus \partial M \to M \setminus \partial M \) is an open map. Use the lemma here, then we have \( \partial M = \emptyset \).

We must prove the lemma to obtain the conclusion. Let \( f: X \to X \) be as in the lemma. Then there exist a metric \( \rho \) for \( X \) and constants \( \delta > 0 \) and \( \lambda > 1 \) such that if \( \rho(x, y) < \delta \) then \( \rho(f(x), f(y)) \geq \lambda \rho(x, y) \). This is proved in the same way as in [10] (notice that \( f \) is not always surjective). For \( \varepsilon > 0 \) and \( x \in X \), let \( U_\varepsilon(x) = \{ y \in X : \rho(x, y) < \varepsilon \} \) and denote by \( C_\varepsilon(x) \) the connected component of \( x \) in \( U_\varepsilon(x) \). Obviously \( C_\varepsilon(x) \) is open in \( X \). Since \( X \) is locally arcwise connected (Theorem 6.29 of [7]), we have that \( C_\varepsilon(x) \) is arcwise connected.

We first check the following claim: Let \( 0 < \varepsilon < \delta/2 \) and \( x \in X \setminus K \). If \( C_\varepsilon(x) \subseteq X \setminus K \), then \( f(C_\varepsilon(x)) \supseteq C_{\lambda \varepsilon}(f(x)) \).

Assume \( y \in C_{\lambda \varepsilon}(f(x)) \setminus f(C_\varepsilon(x)) \neq \emptyset \). Since \( C_{\lambda \varepsilon}(f(x)) \) is arcwise connected, there exists an arc \( \omega: [0, 1] \to C_{\lambda \varepsilon}(f(x)) \) such that \( \omega(0) = f(x) \) and \( \omega(1) = y \).
Since $C_\varepsilon(x) \subset X \setminus K$, $f(C_\varepsilon(x))$ is open in $X$ by (2). Obviously $\omega(0) = f(x) \in f(C_\varepsilon(x))$. Hence we can take $0 < t_0 < 1$ such that $\omega([0, t_0)) \subset \overline{f(C_\varepsilon(x))}$ and $\omega(t_0) \notin f(C_\varepsilon(x))$. Then $\omega([0, t_0]) \subset \overline{f(C_\varepsilon(x))}$ where $\overline{f(C_\varepsilon(x))}$ denotes the closure of $f(C_\varepsilon(x))$ in $X$.

Obviously $f(\overline{C_\varepsilon(x)}) = \overline{f(C_\varepsilon(x))}$. Since $C_\varepsilon(x) \subset U_{\delta/2}(x)$, by the choice of $\delta$ we have that $f|_{\overline{C_\varepsilon(x)}}: C_\varepsilon(x) \to \overline{f(C_\varepsilon(x))}$ is injective, and so $f|_{\overline{C_\varepsilon(x)}}: C_\varepsilon(x) \to \overline{f(C_\varepsilon(x))}$ is a homeomorphism. Since $\omega([0, t_0]) \subset \overline{f(C_\varepsilon(x))}$, we can find an arc $\overline{w}: [0, t_0] \to \overline{C_\varepsilon(x)}$ such that $f \circ \overline{w} = \omega$. Note that $\omega([0, t_0]) \subset f(C_\varepsilon(x))$. Then we have that $\overline{w}([0, t_0]) \subset C_\varepsilon(x)$. Since $\omega(t_0) \notin f(C_\varepsilon(x))$, obviously $\omega(t_0) \notin C_\varepsilon(x)$.

On the other hand, since $\omega(t_0) \in C_{\lambda \varepsilon}(f(x))$, we have

$$\lambda \rho(\overline{w}(t_0), x) \leq \rho(\omega(t_0), f(x)) < \lambda \varepsilon,$$

and so $\overline{w}(t_0) \in U_{\varepsilon}(x)$. Combining this and the fact that $x \in \overline{w}([0, t_0]) \subset C_\varepsilon(x) \subset U_{\varepsilon}(x)$, we have that $\overline{w}(t_0) \in C_\varepsilon(x)$, thus a contradiction.

We proceed to the proof of the lemma. For $\epsilon > 0$ let

$$X(\epsilon) = \{x \in X \setminus K: C_\varepsilon(x) \subset X \setminus K\}.$$

Since $K$ is a closed proper subset of $X$, there exist $0 < \epsilon_0 < \delta/2$ such that $X(\epsilon_0) \neq \phi$. Assume that $K \neq \phi$. Obviously $X(\epsilon_0) \not\subset X$. Let $x \in X(\epsilon_0)$. Then $C_{\epsilon_0}(x) \subset X \setminus K$. From the above claim $f(C_{\epsilon_0}(x)) \supset C_{\lambda \epsilon_0}(f(x))$. Hence $C_{\mu \epsilon_0}(f(x)) \subset X \setminus K$ by (1) and so $f(x) \in X(\lambda \epsilon_0)$. Therefore $f(X(\epsilon_0)) \subset X(\lambda \epsilon_0)$.

It is easily checked that $X(\lambda \epsilon_0) \subset X(\mu \epsilon_0) \subset X(\epsilon_0)$ for $1 < \mu < \lambda$. We show that $X(\lambda \epsilon_0) \subset X(\mu \epsilon_0)$. To do this, let $\{x_i\}_{i \geq 0}$ be a sequence of $X(\lambda \epsilon_0)$ and let $x_i \to x \in X$ as $i \to \infty$. Obviously $U_{\mu \epsilon_0}(x) \subset U_{\lambda \epsilon_0}(x_i)$ for sufficiently large $i$. Since $C_{\mu \epsilon_0}(x)$ is open in $X$, we may assume that $x_i \in C_{\mu \epsilon_0}(x)$. Then $C_{\mu \epsilon_0}(x) \subset C_{\lambda \epsilon_0}(x)$, which implies $x \in X(\mu \epsilon_0)$. Therefore $X(\lambda \epsilon_0) \subset X(\mu \epsilon_0)$.

Consequently we have

$$f(X(\epsilon_0)) = f(X(\epsilon_0)) \subset X(\lambda \epsilon_0) \subset X(\mu \epsilon_0) \subset X(\epsilon_0).$$

Hence $Y = \bigcap_{i \geq 0} f^i(X(\epsilon_0))$ is a nonempty closed set and $f(Y) = Y$. Since $Y \subset X(\epsilon_0) \not\subset X$ and $X$ is connected, obviously $Y$ is not open in $X$.

For $A \subset X$ and $\alpha > 0$ let $N_\alpha(A) = \bigcup_{a \in A} C_\alpha(a)$. Then we have $N_{\mu - 1}\epsilon_0(Y) \subset X(\epsilon_0)$. This is checked as follows. Let $z \in Y$ and $x \in C_{\mu - 1}\epsilon_0(z)$. Then $C_{\epsilon_0}(x) \subset C_{\mu \epsilon_0}(z)$. Since $z \in Y \subset X(\mu \epsilon_0)$, $C_{\epsilon_0}(x) \subset C_{\mu \epsilon_0}(z) \subset X \setminus K$ and so $x \in X(\epsilon_0)$.

By the above result $Y \subset N_{\mu - 1}\epsilon_0(Y) \subset X(\epsilon_0)$. Since $f(Y) = Y$, we have

$$Y \subset \bigcap_{i \geq 0} f^i(N_{\mu - 1}\epsilon_0(Y)) \subset \bigcap_{i \geq 0} f^i(X(\epsilon_0)) = Y,$$

and hence $Y = \bigcap_{i \geq 0} f^i(N_{\mu - 1}\epsilon_0(Y))$. 

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On the other hand, let \( z \in Y \). Then \( C_{(\mu-1)\epsilon_0}(z) \subset C_{\mu\epsilon_0}(z) \subset X\setminus K \). From the claim we have \( f(C_{(\mu-1)\epsilon_0}(z)) \supset C_{\lambda(\mu-1)\epsilon_0}(f(z)) \). Hence \( f(N_{(\mu-1)\epsilon_0}(Y)) \supset N_{\lambda(\mu-1)\epsilon_0}(Y) \supset N_{(\mu-1)\epsilon_0}(Y) \) (since \( f(Y) = Y \)). Therefore \( Y = N_{(\mu-1)\epsilon_0}(Y) \). This implies that \( Y \) is open in \( X \). We arrived at a contradiction.

References


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