ON HANKEL TRANSFORM

ANTONIO J. DURAN

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Abstract. In this paper, using some results of the author on Hankel transform in the Schwartz and Gel'fand-Shilov spaces, we characterize the integral operators of Hankel type which are isomorphisms between the spaces $H^\mu_\mu$ of Zemanian. As a particular case, we obtain the classical Zemanian results on Hankel transform, some results of Mendez, and improve some results of Lee. Finally, we use these results to characterize the functions $f$ of the Schwartz space which satisfy $\int_0^\infty t^{\alpha+n} f(t) \, dt = 0$ for all $n \geq 0$ and $\alpha > -1$.

Introduction

In [9], Zemanian defined the spaces $H^\mu_{\mu}$ ($\mu \geq -\frac{1}{2}$) as

1. $H^\mu_{\mu} = \{ f \in \mathcal{S}^\infty((0, +\infty)) : \forall k, n \geq 0 \, t^k (t^{-1} D)^n t^{-1/2-\mu} f(t) < C_k, n \}$

and he proved that the Hankel transform:

2. $H^\mu_{\mu}(f)(x) = \int_0^\infty f(t) \sqrt{x} i J_\mu(x t) \, dt$

is an automorphism of the space $H^\mu_{\mu}$.

In [5], Lee generalized these spaces in the following way (similar to the Gel'fand-Shilov spaces), given $\alpha, \beta \geq 0, A, B > 0$, and $\mu \geq -\frac{1}{2}$, the space $H^\beta_{\mu, \alpha, A}$ is defined by

3. $H^\beta_{\mu, \alpha, A} = \{ f \in \mathcal{S}^\infty((0, +\infty)) : \exists C, \delta, \rho > 0 \text{ such that } \forall k, n \geq 0 \, t^k (t^{-1} D)^n t^{-1/2-\mu} f(t) < C (A + \delta)^n (B + \rho)^k k^\alpha n^{\beta n} \}$

and

4. $H^\beta_{\mu, \alpha} = \bigcup_{A, B \geq 1} H^\beta_{\mu, \alpha, A}$.

He studied the nontriviality of these spaces and the Hankel operator (2).
In this paper, we obtain the spaces $H_\mu$ and $H_{\mu, \beta}^\alpha$ through the spaces $S_\alpha^+$ and $S_\alpha^{+\beta}$ (see [2, 3]):

(5) $S^+ = \{ f \in C^\infty((0, +\infty)) : \forall k, n \geq 0 |t^k f^{(n)}(t)| < C_k, n \}$

(6) $S_\alpha^{+\beta} = \{ f \in C^\infty(0, +\infty) : \exists C, A, B > 0 \forall k, n \geq 0 |t^k f^{(n)}(t)| < CA^kB^{\alpha n} \}$

Using some results of [2, 3], we characterize the operators defined in a similar form as (2) which are isomorphism between the spaces $H_\alpha$ and $H_\beta$. As particular cases, we obtain the classical results of Zemanian in [9], those of Méndez in [7] and we improve those of Lee in [5, 6].

Finally, we use our theorems to characterize the functions $f$ of the space $S^+$ which satisfy $\int_0^\infty t^{\alpha+n} f(t) dt = 0$ for all $n \geq 0$ and $\alpha > -1$ (this should be compared with Theorem 4.1 and 4.4 of [4]). We also solve the problem which we had proposed in [4].

Preliminaries

We consider the integral operators $\mathcal{K}_\alpha$:

(7) $\mathcal{K}_\alpha(f)(x) = \frac{1}{2} \int_0^\infty f(t)(tx)^{-\alpha/2} t^\alpha J_\alpha(\sqrt{tx}) dt.$

Now, we state a theorem for the space $S^+$, which was proved in [2, Theorem 6.2] for the space $S_1^{+0}$. The proof is based on [3, Theorem 2.5] and is analogous to that given in [2]. For $\alpha \geq 0$, another proof of this theorem can be found in [8].

**Theorem 1.** For $\alpha > -1$, the integral operator defined by (7) is an isomorphism of $S^+$ onto itself and $\mathcal{K}_\alpha^2 = \text{Id}$.

Now, we shall prove that this Theorem is the best possible in the following sense.

**Theorem 2.** Let $\alpha > -1$ and $H$ be the integral operator defined by $H(f)(x) = \int_0^\infty f(t)x^2 J_\alpha(\sqrt{tx}) dt$. Then $H$ is an isomorphism of $S^+$ onto itself if and only if $a = -\alpha/2$ and $b = \alpha/2$.

**Proof.** We consider the following subspace of $S^+$ denoted by $S^+ \cap S$: a function $f \in S^+ \cap S$ if and only if $f \in S^+$ and $f^{(n)}(0) = 0$ for all $n \geq 0$.

Let us assume that $H$ is an isomorphism of $S^+$ onto itself. Let $c \in \mathbb{R}$ such that $b + c = \alpha/2$. We suppose that $a + \alpha/2 \notin \mathbb{N}$. Let $f, g \in S^+ \cap S$ such that $f(t) = t^c g(t)$. Then

$$H(f)(x) = H(t^c g)(x) = x^a \int_0^\infty g(t) t^{b+c} J_\alpha(\sqrt{tx}) dt = 2x^{a+n/2} \mathcal{K}_\alpha(g)(x).$$

Since $a + \alpha/2 \notin \mathbb{N}$ and $H(f) \in S^+$, we deduce that $\mathcal{K}_\alpha(g)(0) = 0$ for all $g \in S^+ \cap S$. Now, let $g$ be a $C^\infty$-function with support contained in $[0,1]$ and
such that \( \int_0^\infty t^\alpha g(t) \, dt \neq 0 \). As
\[
J_\alpha(x) = c_\alpha x^\alpha \sum_k a_k x^{2k}
\]
we have
\[
(9) \quad \mathcal{H}_\alpha g(x) = \frac{1}{2} \int_0^\infty g(t)(tx)^{-\alpha/2} J_\alpha(\sqrt{xt}) \, dt = \frac{c_\alpha}{2} \int_0^\infty g(t)t^\alpha \sum_k a_k (xt)^k \, dt.
\]
Now, \( x^{-\alpha} J_\alpha(x) \) is bounded as \( x \) tends to 0 and from the asymptotic expansion of \( J_\alpha \), we see that \( x^{1/2} J_\alpha(x) \) is bounded when \( x \) tends to \( \infty \), so we get
\[
(10) \quad |(xt)^{-\alpha/2} J_\alpha(\sqrt{xt})| \leq C \quad \text{if } 0 < x, t.
\]
From (9) and (10), we deduce that \( \mathcal{H}_\alpha g(0) = c_\alpha \int_0^\infty a_0 g(t)t^\alpha \, dt \neq 0 \). As \( g \in \mathcal{S}^+ \cap \mathcal{S} \), this shows that \( a_\alpha/2 \in \mathbb{N} \).

Now, we suppose that \( c \notin \mathbb{N} \), so that \( t^\alpha e^{-\alpha/2} \notin \mathcal{S}^+ \). Since \( H \) is an isomorphism of \( \mathcal{S}^+ \) onto itself, \( H(t^\alpha e^{-\alpha/2}) \notin \mathcal{S}^+ \), but \( H(t^\alpha e^{-\alpha/2}) = 2x^{\alpha_0/\alpha} \mathcal{H}_\alpha(e^{-\alpha/2}) \), and as \( a_\alpha/2 \in \mathbb{N} \) it follows that \( x^{\alpha_0/\alpha} \mathcal{H}_\alpha(e^{-\alpha/2}) \in \mathcal{S}^+ \). Hence \( c \in \mathbb{N} \).

Let \( n, m \in \mathbb{N} \) such that \( b = \alpha_\alpha/2 - n \) and \( a = -\alpha_\alpha/2 + m \). By (8), we can write
\[
H(f)(x) = x^{-\alpha_\alpha/2 + n} \int_0^\infty f(t)t^{-\alpha_\alpha/2 + m} J_\alpha(\sqrt{xt}) \, dt = c_\alpha x^n \int_0^\infty f(t)t^{\alpha_\alpha - m} \sum_k a_k (xt)^k \, dt.
\]
If \( n > 0 \), from (10) it follows that \( H(f)(0) = 0 \) for all \( f \in \mathcal{S}^+ \), and so \( H \) would not be an isomorphism of \( \mathcal{S}^+ \) onto itself. Therefore \( n = 0 \).

Now, if \( m > 0 \), as \( t^m e^{-\alpha/2} \notin \mathcal{S}^+ \), it follows that \( \mathcal{H}_\alpha(t^m e^{-\alpha/2}) \notin \mathcal{S}^+ \) but \( \mathcal{H}_\alpha(t^m e^{-\alpha/2}) = \frac{1}{2} H(e^{-\alpha/2}) \in \mathcal{S}^+ \). So \( m = 0 \) and the theorem is proved.

**The Hankel transform.** Now, we shall use the preliminary results to study the Hankel transform in the spaces \( H_\alpha \).

Let \( f, g \) be two functions such that \( f(t) = g(t^2) \). Formally, it is clear that if \( x = t^2 \), then \( \frac{1}{2} \frac{\partial}{\partial t} f(t) = 2^n \left( \frac{1}{2} \frac{\partial}{\partial x} \right)^n g(x) \). Using this formula, we obtain the following relation between the spaces defined in the introduction.

**Lemma 3.** Let \( \mu \geq -\frac{1}{2}, \alpha, \beta \geq 0 \). Then
\[
(11) \quad H_{-\frac{1}{2}} = \{ g(t^2): g \in \mathcal{S}^+ \},
\]
\[
(12) \quad H_\mu = t^{1/2+\mu} \{ g(t^2): g \in \mathcal{S}^+ \},
\]
\[
(13) \quad H^\beta_{\mu, \alpha} = t^{1/2+\mu} \{ g(t^2): g \in \mathcal{S}_\alpha^{+\beta} \}.
\]

As these spaces are Frechet ((11), (12)) or \( LF \) spaces ((13)), using the Closed Graph theorem it is clear that \( T_\mu: \mathcal{S}^+(\text{resp. } \mathcal{S}_\alpha^{+\beta}) \to H_\mu(\text{resp. } H^\beta_{\mu, \alpha}) \) defined by
$T_\mu(f)(t) = t^{1/2+\mu} f(t^2)$ is an isomorphism. Its inverse is given by $T_\mu^{-1}(f)(t) = t^{-1/4-\mu/2} f(\sqrt{t})$. From (13) and the Appendix in [2] Theorem 4 follows.

**Theorem 4.** Let $\mu \geq -\frac{1}{2}$, $\alpha$, $\beta \geq 0$. Then $H_{\mu,\alpha}^\beta = \{0\}$ if and only if

(a) $\alpha \neq 0$, $\beta \neq 0$, and $\beta + 2\alpha < 1$, or
(b) $\alpha = 0$ and $\beta \leq 1$, or
(c) $\beta = 0$ and $\alpha < \frac{1}{2}$.

This theorem improves the results of Lee in [5], since he only proved that $H_{\mu,0}^0 = \{0\}$ and $H_{\mu,1/2}^0 \neq \{0\}$.

Using Lemma 3 and Theorem 1, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
S^+ & \xrightarrow{\mathcal{F}_\mu} & S^+ \\
\downarrow T_\alpha & & \downarrow T_\beta \\
H_\alpha & \xrightarrow{L_{\beta,\mu,\alpha}} & H_\beta
\end{array}
\]

where

$L_{\beta,\mu,\alpha} = T_\beta \circ \mathcal{F}_\mu \circ T_\alpha^{-1}$

is an isomorphism of $H_\alpha$ onto $H_\beta$. Its inverse is $L_{\beta,\mu,\alpha}^{-1} = T_\alpha \circ \mathcal{F}_\mu \circ T_\beta^{-1}$. So, when $\alpha = \beta$, $L_{\beta,\mu,\alpha}^0 = \text{Id}$. It is easy to check that

$L_{\beta,\mu,\alpha}(f)(x) = x^{\alpha+1/2-\mu} \int_0^\infty f(u)u^{\mu-\alpha+1/2} J_\mu(ux) \, du$

and

$L_{\beta,\mu,\alpha}^{-1}(f)(x) = x^{\alpha+1/2-\mu} \int_0^\infty f(u)u^{\mu-\beta+1/2} J_\mu(ux) \, du$.

Now, from Theorem 2, we deduce the following theorem.

**Theorem 5.** Let $\alpha > -\frac{1}{2}$ and $H$ be the integral operator defined by $H(f)(x) = \int_0^\infty f(t)x^a t^b J_\mu(xt) \, dt$. Then $H$ is an isomorphism of $H_\alpha$ onto $H_\beta$ if and only if $a = \beta + \frac{1}{2} - \mu$ and $b = \mu - \alpha + \frac{1}{2}$. Moreover, $H^2 = \text{Id}$ if and only if $\alpha = \beta$.

Choosing $\alpha$ and $\beta$ in a suitable way, we can obtain some classical results. For example:

(I) Taking $\alpha = \beta = \mu$ we obtain the classical result of Zemanian in [9], i.e.

$L_{\mu,\mu,\mu} = H_\mu$,

where $H_\mu$ is the Hankel transform defined by (2). So, $H_\mu$ is an isomorphism of $H_\mu$ onto $H_\mu$, and $H_\mu^2 = \text{Id}$.

(II) Taking $\alpha = \beta = -\frac{1}{2} + \mu$ and $\alpha = \beta = \frac{1}{2} + \mu$, we obtain Theorems 3 and 4 of Mendez in [7].

(III) Now, we'll improve the results of Lee in [6]. Indeed, we consider the spaces $F_\mu$ defined by Lee in [6]

$F_\mu = \{ f \in C^\infty((0, +\infty)) : \exists C_{k,n} > 0 \text{ such that } |x^n(D^2 + (2\mu + 1)(x^{-1} D))^k f(x)| < C_{k,n} \}$.
By Remark 2 in [6], it is clear that $F_{\mu} = H_{-1/2}$ for all $\mu \geq -\frac{1}{2}$. So, if we take $\alpha = \beta = -\frac{1}{2}$ in the previous diagram, we have:

$$L_{-1/2, -1/2} = h_{\mu}(f)(x) = \int_0^\infty f(t)t^{2\mu+1}(tx)^{-\mu}f_{\mu}(tx)\,dt$$

and so, $h_{\mu}$ is an isomorphism of $H_{-1/2}$ onto itself, i.e. $h_{\mu}$ is an isomorphism of $F_{\mu}$ onto itself and $h_{\mu}^2 = I_d$. Lee had proved that $h_{\mu}$ is a continuous linear mapping from the space $F_{\mu}$ into the spaces $G_{\mu}$, where $F_{\mu} \subset G_{\mu}$. Moreover, as the operator (7) is defined for $\alpha > -1$, the operator $h_{\mu}$ defined by (11), is an isomorphism for $\mu > -1$. This gives a partial answer to problem 1 in [6]. So, we have proved the following theorem:

**Theorem 6.** Let $\mu > -1$, then the operator $h_{\mu}$ defined by (14) is an isomorphism of $F_{\mu}$ onto itself and $h_{\mu}^2 = I_d$.

Dualizing, we can extend the operator $h_{\mu}$ to the distributional space $F'_{\mu}$ as follows: if $u \in F'_{\mu}$ then $h_{\mu}(u)$ is defined by

$$\langle h_{\mu}(u), f \rangle = \langle u, h_{\mu}(f) \rangle \quad \text{for all } f \in F_{\mu}'.$$

The following Theorem follows from Theorem 6.

**Theorem 7.** Let $\mu > -1$, then the operator $h_{\mu}$ defined by (15) is an isomorphism of $F'_{\mu}$ onto itself and $h_{\mu}^2 = I_d$.

In a similar way, since the operator (7) is an automorphism of the space $S_{1+0}^s$, from the diagram, (13) and taking $\alpha = \beta = \mu$, we obtain that the operator $H_{\mu}$ defined by (2) is an isomorphism of $H_{\mu, 1/2}^0$ onto itself. This improves the results of Lee in [6].

**Appendix**

In this Appendix, we shall use Theorem 1 to give a characterization similar to [4, Theorem 4.1] of the functions in $S^+$ which satisfy \( \int_0^\infty f(t)t^{\alpha+n}\,dt = 0 \) for all $n \geq 0$ and $\alpha > -1$. We need the following results:

Let $\alpha > -1$, and $L_n^\alpha(t) = (n!/(\Gamma(n + \alpha + 1)))^{1/2}L_n^\alpha(t)t^{\alpha/2}e^{-t/2}$, where $L_n^\alpha(t)$ are the generalized Laguerre polynomials. It is well known that $(L_n^\alpha(t))_n$ is an orthonormal system in $L^2([0, \infty))$. If $\psi \in S^+$ we define the $\alpha$ Fourier-Laguerre coefficients of $\psi$ as $a_n = \int_0^\infty t^{\alpha/2}\psi(t)L_n^\alpha(t)\,dt$. From [3, Theorem 2.5] it follows that

**Theorem A.** The mapping $\mathcal{L} : S^+ \rightarrow s$ defined by

$$\mathcal{L}(\phi) = (a_n)_n = \left(\int_0^\infty t^{\alpha/2}\phi(t)L_n^\alpha(t)\,dt\right)_n$$

is an isomorphism of the space $S^+$ onto $s$. (We write $s$ for the space of rapidly decreasing sequences).
Now, we can complete Theorem 1. Indeed, as (see [2, 6.1])
\[
\int_0^\infty t^{\alpha/2} L_n^\alpha(t) e^{-t/2} J_\alpha(\sqrt{t}x) \, dt = 2(-1)^n x^{\alpha/2} L_n^\alpha(x) e^{-x/2},
\]
it is easy to prove (see [2, proof of Theorem 6.2]) that the \( \alpha \) Fourier-Laguerre coefficients of \( \mathcal{H}_\alpha(f) \) are \( ((-1)^n a_n^{(\alpha)})_n \) where \( (a_n^{(\alpha)})_n \) are the \( \alpha \) Fourier-Laguerre coefficients of \( f' \).

Let \( \phi \in L^1([0, +\infty)) \), and consider the following analytic function \( \tilde{\phi} \) in the lower half plane:
\[
\tilde{\phi}(x) = \int_0^{+\infty} \phi(t)e^{-2\pi i xt} \, dt.
\]

Using the bilinear transformation \( W(z) = (-\frac{1}{2} + 2\pi iz)/(\frac{1}{2} + 2\pi iz) \), which transforms the lower half plane in the unit disc, we obtain the following analytic function \( \phi \) in the unit disc:
\[
\phi(w) = \tilde{\phi}(Z(w)) = \int_0^\infty \phi(t)e^{-\frac{t}{2} \left( \frac{1+i}{1-w} \right)} \, dt.
\]

In the following Theorem [4, Theorem 4.2], we relate the function \( \tilde{\phi} \) with the Fourier-Laguerre coefficients of \( \phi \).

**Theorem B.** Let \( \phi \in S^+ \) and \( a_n^{\alpha} = \int_0^\infty t^{\alpha/2} \phi(t) L_n^\alpha(t) \, dt \). Then \( (t^\alpha \phi)^\sim(w) = (1-w)^{\alpha+1} \sum_{n=0}^{\infty} a_n^{\alpha} (\Gamma(n+\alpha+1)/n!)^{1/2} w^n \).

Now, we can prove the following theorem (to compare with [4, Theorems 4.1 and 4.4])

**Theorem 8.** Let \( \phi \in S^+ \), \( \alpha > -1 \), and \( a_n^{\alpha} = \int_0^\infty t^{\alpha/2} f(t) L_n^\alpha(t) \, dt \). The following conditions are equivalent:

(a) \( \int_0^\infty t^{\alpha+k} f(t) \, dt = 0 \) for all \( k \geq 0 \).

(b) \( \sum_n (-1)^n n^k (n+\alpha)^{1/2} a_n^{\alpha} = 0 \) for all \( k \geq 0 \).

(c) There exists a function \( \psi \in S^+ \cap S \) such that \( \phi = \mathcal{H}_\alpha(\psi) \).

**Proof.** (a) \( \Leftrightarrow \) (b) It is clear that \( \int_0^\infty \phi(t)t^{\alpha+k} \, dt = 0 \) for all \( k \geq 0 \) if and only if \( \lim_{z \to 0} t^\alpha \phi^{(k)}(z) = 0 \) for all \( k \geq 0 \) and so if and only if \( \lim_{w \to -1} (t^\alpha \phi)^{(k)}(w) = 0 \) for all \( k \geq 0 \) (because \( \tilde{\phi}(w) = \tilde{\phi}(Z(w)) \) and \( d^nZ(w)/dw^n \) is bounded near \( w = -1 \)). It is clear that \( \lim_{w \to -1} (t^\alpha \phi)^{(k)}(w) = 0 \) for all \( k \geq 0 \) if and only if \( \lim_{w \to -1} ((t^\alpha \phi)^\sim(w)/(1-w)^{\alpha+1})^{(k)} = 0 \). We have \( (t^\alpha \phi)^\sim(w) = (1-w)^{\alpha+1} \sum_n a_n^{\alpha} (\Gamma(n+\alpha+1)/n!)^{1/2} w^n \) and \( (a_n^{\alpha})_n \in S \), so
\[
\frac{(t^\alpha \phi)^\sim(w)}{(1-w)^{\alpha+1}} = \sum_n a_n^{\alpha} \left( \frac{\Gamma(n+\alpha+1)}{n!} \right)^{1/2} w^n
\]
and
\[
\frac{(t^\alpha \varphi)(w)}{(1 - w)^{\alpha+1}}
\]
\[
= \sum_n n^k a_n^\alpha \left( \frac{\Gamma(n + \alpha + 1)}{n!} \right)^{1/2} w^{-k}
\]
\[
+ b_{k-1}^{(k)} \sum_n n^{k-1} a_n^\alpha \left( \frac{\Gamma(n + \alpha + 1)}{n!} \right)^{1/2} w^{-k}
\]
\[
+ \cdots + b_1^{(k)} \sum_n n a_n^\alpha \left( \frac{\Gamma(n + \alpha + 1)}{n!} \right)^{1/2} w^{-k}
\]
for certain \( b_{k-1}^{(k)}, \ldots, b_1^{(k)} \in \mathbb{N} \) and \( k \geq 1 \). Applying successively theorems of Tauber and Abel, we get \( \int_0^\infty \varphi(t) t^{n+k} \, dt = 0 \) if and only if
\[
\sum_n (-1)^n n^k a_n^\alpha (\Gamma(n + \alpha + 1)/n!)^{1/2} = 0
\]
for all \( k \geq 0 \). But \( \binom{\alpha+n}{n} = \Gamma(n + \alpha + 1)/n!\Gamma(\alpha) \).

(b) \( \Leftrightarrow \) (c) By the remark following Theorem A the Fourier-Laguerre coefficients of \( \mathcal{H}_\alpha(\varphi) \) are \((-1)^n a_n^\alpha \)_n. As by [2, Theorem 3.2]
\[
\sum_n (-1)^n n^k a_n^\alpha \left(\frac{n + \alpha}{n}\right)^{1/2} = 0
\]
if and only if \( \mathcal{H}_\alpha(\varphi) \in S^+ \cap S \). Since \( \mathcal{H}_\alpha^2 = \text{id} \), setting \( \psi = \mathcal{H}_\alpha(\varphi) \) we finish the proof.

Finally, we solve the problem which we had proposed in [4]. Indeed, we prove the following theorem.

**Theorem 9.** Let \( \varphi \in S^+ \), and \( a_n = \int_0^\infty \varphi(t) L_n(t) e^{-t/2} \, dt \). The following conditions are equivalent:

(a) \( \varphi \in S^+ \cap S \) and \( \int_0^\infty t^k \varphi(t) \, dt = 0 \) for all \( k \geq 0 \).

(b) \( \sum_n (-1)^n n^k a_n = 0 \) and \( \sum_n n^k a_n = 0 \) for all \( k \geq 0 \).

(c) There exist two functions \( \psi, \varphi \in S^+ \cap S \) such that \( \varphi = \varphi \ast \mathcal{H}_0 \psi \).

**Proof.** By [4], we only need to prove that (a) implies (c). Let
\[
P_1^\infty = \left\{ \text{the functions } f, \text{ analytics in the open unit disc such that } f \text{ and all its derivatives are bounded in the unit disc and } \lim_{w \to 1} f^{(k)}(w) = 0 \text{ for all } k \geq 0 \right\}
\]
For (a) implies (c) it is equivalent to prove that if \( f \in P_1^\infty \) and \( \lim_{w \to 1} f^{(k)}(w) = 0 \) for all \( k \geq 0 \), then there exist two functions \( g, h \) such that \( f = g \cdot h \).
$g \in P_1^\infty$ and $h$ is a bounded and analytic function in the open unit disc, with all its derivatives bounded in the unit disc and satisfying that $\lim_{w \to -1} h^{(k)}(w) = 0$. Now, given a function $f \in P_1^\infty$ such that $\lim_{w \to -1} f^{(k)}(w) = 0$ for all $k \geq 0$ by [1, Theorem 3, p. 22], there exist two functions $g, h \in P_1^\infty$ such that $f = g \cdot h$ and $h$ belongs to the closed ideal generated by $f$. So $\lim_{w \to -1} g^{(k)}(w) = 0$ and $\lim_{w \to -1} h^{(k)}(w) = 0$ for all $k \geq 0$. And the theorem is proved.

References


2. A. J. Duran, *The analytic functionals in the lower half plane as a Gel’fand-Shilov space*, preprint.


