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THE INFLUENCE OF A SMALL CARDINAL ON THE PRODUCT OF A LINDELÖF SPACE AND THE IRRATIONALS

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ABSTRACT. It is unknown whether there is in ZFC a Lindelöf space whose product with the irrationals is nonnormal. We give some necessary conditions based on the minimum cardinality of $a \leq^*$ unbounded family in ${}^{\omega}\omega$.

0. INTRODUCTION

History. In 1963 [9], E. A. Michael gave the first ZFC example of a normal space and a metric space with a nonnormal product, where the term "space" is an abbreviation for "Hausdorff regular topological space." (In 1955 [12], M. E. Rudin used a Souslin line to construct a Dowker space—a normal space whose product with the closed unit interval is nonnormal; the existence of a Souslin line was shown to be independent of ZFC in the late 1960s. Rudin constructed a real Dowker space in 1971 [13].) Let $\mathbb{P}$, $\mathbb{Q}$, and $\mathbb{R}$ denote respectively the irrationals, rationals, and reals with their usual topologies; and let $\mathbb{M}$ denote the Michael line which is the refinement of $\mathbb{R}$ obtained by isolating each irrational point. In Michael's example, $\mathbb{M}$ is the normal space and $\mathbb{P}$ is the metric space; $\mathbb{M}$ is moreover hereditarily paracompact.

A topological space X is concentrated on a subset $A \subseteq X$ if for every open set $U \supseteq A$, $X \setminus U$ is countable. The Continuum Hypothesis implies the existence of an uncountable subset $X \subseteq \mathbb{R}$ with X concentrated on $\mathbb{Q}$ (A.S. Besicovitch [2], 1934; in fact, $\underline{b} = \omega_1$ is all his proof requires—see Propositions 2 and 3 below). Michael obtained this result independently and pointed out that as a subspace of $\mathbb{M}$, X is Lindelöf whereas $X \times \mathbb{P}$ is nonnormal ([10], 1971; first mentioned in a footnote of [9]).

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Also in the 1963 paper, Michael gave a ZFC example of a Lindelöf space $\mathbb{M}'$ and a separable metric space S with $\mathbb{M}' \times S$ nonnormal. Let A be a Bernstein set in $\mathbb{R}$; that is, choose A so that each of A and $\mathbb{R} \setminus A$ intersects every uncountable closed set. (A set of this type was first constructed by F. Bernstein in 1908 using transfinite recursion and the Cantor-Bendixson theorem: each of *the* $\underline{c}$*-many closed sets in* $\mathbb{R}$ *is either countable or has cardinality* $\underline{c}$, *where* $\underline{c} = 2^{\omega}$; see [11, pp. 23–24.]) Note that $\mathbb{R}$ is concentrated on A. Let $\mathbb{M}'$ be the refinement of $\mathbb{R}$ obtained by isolating each point of $\mathbb{R} \setminus A$, and let S be $\mathbb{R} \setminus A$ with the subspace topology (note that S is not complete).

Is there a real Lindelöf space whose product with $\mathbb{P}$ is nonnormal? This is equivalent to the existence in ZFC of a Lindelöf space and a separable completely metrizable space with a nonnormal product (see Proposition 1 below).

Michael spaces. Define a Michael space to be a Lindelöf space whose product with $\mathbb{P}$ is nonnormal. Is there a Michael space in ZFC? In particular, can we build a real Michael space by the method of isolating points in a separable metric space? Define a concentrated Michael space to be a Michael space that is concentrated on a closed subset A where $A \times \mathbb{P}$ is normal. This concept captures the construction of isolating points mentioned above, in which case A is the derived set (i.e., the closed subset of nonisolated points) of the refinement; since A is a (separable) metrizable subspace of the refinement, its product with $\mathbb{P}$ is normal.

In his handbook article ("The Integers and Topology" in [7, pp. 111–167, and in particular, pp. 150–153]), E. K. van Douwen noted some necessary conditions for the existence of a real Michael space which are the point of departure for the current paper. We first establish that regardless of special set-theoretic axioms, a Michael space cannot be constructed by isolating points in an initial space that is completely metrizable (Theorem 1). We then show that the existence of a concentrated Michael space is independent of ZFC (Theorem 2). Finally, we give a lower bound for the cardinality and weight of an arbitrary Michael space (Theorem 3). (The weight of a space X, denoted by $w(X)$, is the minimum cardinality of a base for the topology of X.) As in the case of the results in the van Douwen article, the cardinal number $\underline{b}$ (defined below) is at the heart of the matter in Theorems 2 and 3.

1. Propositions

Remark. The propositions below are now more or less standard. We sketch the proofs for the convenience of the reader.

Proposition 1. *Suppose X and Y are spaces where Y is separable, completely metrizable, and not σ-compact. Then $X \times Y$ is normal iff $X \times \mathbb{P}$ is normal.*

Proof. Sufficiency follows from three standard theorems: Normality is preserved under a closed continuous map (see [4, p. 69]); every separable completely metrizable space is the image under a perfect map of a closed subspace of $\mathbb{P}$ (a map is perfect if it is closed and continuous, and each point-inverse set

is compact); and, a Cartesian product of perfect mappings is perfect [4, p. 237]. (Since $\mathbb{P}$ is homeomorphic to ${}^{\omega}\mathbb{P}$ [4, p. 348], and every separable completely metrizable space can be embedded as a closed subspace in ${}^{\omega}\mathbb{R}$ [4, p. 342], we can prove the second theorem by using ω-many copies of the Cantor ternary function [7, pp. 203–204] to define a perfect map from a closed subspace of $\mathbb{P}$ onto $\mathbb{R}$, and in turn, from a closed subspace of ${}^{\omega}\mathbb{P}$ onto ${}^{\omega}\mathbb{R}$.)

Necessity. Y has a closed subspace homeomorphic to $\mathbb{P}$ (W. Hurewicz; see van Douwen's proof in [7, pp. 141–142]).

Notation. Recall that $\mathbb{P}$ can be identified with ${}^{\omega}\omega = \{f | f\colon \omega \to \omega\}$ with the product topology: for each $\xi \in {}^{<\omega}\omega = \{\eta | \eta\colon [0, n] \to \omega$ for some $n \in \omega\}$, $\{f \in {}^{\omega}\omega\colon \xi \subseteq f\}$ is a basic open set (see [7, p. 204]). For $f, g \in {}^{\omega}\omega$, define $f \leq g$ if for each $n \in \omega$, $f(n) \leq g(n)$; and define $f \leq^* g$ if there exists $m \in \omega$ such that for each $n \geq m$, $f(n) \leq g(n)$. For each $g \in {}^{\omega}\omega$, let $C(g) = \{f \in {}^{\omega}\omega\colon f \leq g\}$, and let $C^*(g) = \{f \in {}^{\omega}\omega\colon f \leq^* g\}$.

Compact sets. (1) For each $g \in {}^{\omega}\omega$, $C(g)$ is compact and $C^*(g)$ is σ-compact.

(2) For each $T \subseteq {}^{\omega}\omega$, if T is compact (resp., σ-compact), then there exists $g \in {}^{\omega}\omega$ such that $T \subseteq C(g)$ (resp., $T \subseteq C^*(g)$) (Note that the compactness relationship implies the analogue for σ-compactness, since for any sequence $g\colon \omega \to {}^{\omega}\omega$, $\bigcup_{k\in\omega} C(g_k) \subseteq C^*(h)$ where $h(n) = \sum_{k\leq n} g_k(n)$).

(3) For all $f, g \in {}^{\omega}\omega$, $C(f) \subseteq C(g)$ (resp., $C^*(f) \subseteq C^*(g)$) iff $f \leq g$ (resp. $f \leq^* g$).

Cardinals. With respect to $\leq^*$, let $\underline{b}$ be the minimum cardinality of an unbounded family and let $\underline{d}$ be the minimum cardinality of a dominant (cofinal) family. Then in ZFC we have:

(1) $\omega_1 \leq \underline{b} \leq \underline{d} \leq \underline{c}$;

(2) $\underline{b}$ is regular because there is a $\leq^*$ well-ordered unbounded family of order type $\underline{b}$. (To prove this, let $f\colon \underline{b} \to {}^{\omega}\omega$ with Range(f) unbounded, and then define $g\colon \underline{b} \to {}^{\omega}\omega$ recursively so that for all $\alpha, \beta \in \underline{b}$ with $\alpha < \beta$, $f(\alpha) \leq^* g(\alpha) \leq^* g(\beta)$);

(3) $\underline{b} = \underline{d}$ iff there is a $\leq^*$ well-ordered dominant family (often called a scale). (To prove sufficiency, note that if $g\colon \gamma \to {}^{\omega}\omega$ is an order-isomorphism onto a dominant subcollection, then $\underline{d} \leq$ cofinality of $\gamma \leq \underline{b}$; and to prove necessity, let $f\colon \underline{b} \to {}^{\omega}\omega$ with Range(f) dominant, and then define g as in the proof that $\underline{b}$ is regular.)

The Continuum Hypothesis implies $\omega_1 = \underline{b} = \underline{d} = \underline{c}$, while under Martin's axiom, $\underline{b} = \underline{d} = \underline{c}$ (see [6, p. 87, Example 8]). Since Martin's axiom is consistent with the negation of the Continuum Hypothesis [6, pp. 278–281], we have that the combinatorial statement $\underline{b} = \omega_1$ is independent of ZFC.

Consistency results. In [3], D. K. Burke and S. Davis constructed a Michael space using an ω_1-scale (equivalently, $\underline{d} = \omega_1$); while in [7], van Douwen

obtained a Michael space under $\underline{b} = \omega_1$. The hypotheses $\underline{c} = \omega_1$, $\underline{d} = \omega_1$, and $\underline{b} = \omega_1$ are successively weaker statements (see [5]).

Proposition 2. *Suppose X is an uncountable subset of $\mathbb{P}$. Then the following are equivalent*:

(1) *every uncountable subset of X is $\leq^*$ unbounded in ${}^\omega\omega$*;
(2) *X has countable intersection with every compact subset of $\mathbb{P}$*;
(3) *$X \cup \mathbb{Q}$ is concentrated on $\mathbb{Q}$*;
(4) *$X \cup \mathbb{Q}$ with the subspace topology of $\mathbb{M}$ is Lindelöf.*

Proposition 3. *There is an uncountable subset of $\mathbb{P}$ satisfying each condition of Proposition 2 iff $\underline{b} = \omega_1$.*

Proof of Proposition 2. Each of (1) and (2) is equivalent to X having countable intersection with $C^*(g)$ for every $g \in {}^\omega\omega$.

For (2) implies (3), note that a closed set in $\mathbb{R}$ which is contained in $\mathbb{R}\backslash\mathbb{Q}$ is a σ-compact subset of $\mathbb{P}$ and therefore is contained in $C^*(g)$ for some $g \in {}^\omega\omega$. The converse follows from the fact that the complement in $\mathbb{R}$ of a compact subset of $\mathbb{P}$ is an open set in $\mathbb{R}$ containing $\mathbb{Q}$.

The equivalence of (3) and (4) is immediate from the definitions.

Proof of Proposition 3. Suppose X is an uncountable subset of $\mathbb{P}$ satisfying the first condition of Proposition 2. Let $Y \subseteq X$ with $|Y| = \omega_1$. By hypothesis, Y is unbounded so $\underline{b} = \omega_1$. For the converse, let X be a $\leq^*$ well-ordered unbounded family in ${}^\omega\omega$ of type ω_1.

2. Theorems

Lemma 1. (Rudin and M. Starbird, [14], 1975). *The product of a Lindelöf space and a separable metric space is normal iff it is Lindelöf.*

Lemma 2 (Characterization). *Suppose S is a separable metric space and X is a Lindelöf space that is concentrated on a closed subset $A \subseteq X$ where $A \times S$ is normal. Then $X \times S$ is normal iff for every uncountable subset $B \subseteq X\backslash A$ and $1-1$ function $F: B \to S$, $\mathrm{cl}(\mathrm{Graph}(F)) \cap (A \times S) \neq \varnothing$.*

Proof. Sufficiency of the contrapositive. We will show that $\mathrm{cl}(\mathrm{Graph}(F))$ and $A \times S$ cannot be separated by disjoint open sets. Let U be an open set in the product with $A \times S \subseteq U$. Let D be a dense countable subset of S. For each $d \in D$, let $Y_d = \{x \in X: \langle x, d\rangle \notin U\}$. By the concentration of X on A, each Y_d is countable, and since B is uncountable, we can choose $x_0 \in B\backslash \bigcup_{d\in D} Y_d$. Then $\{x_0\} \times D \subseteq U$, so $\langle x_0, F(x_0)\rangle \in \mathrm{cl}(U)$.

Necessity of the contrapositive. By Lemma 1 in application to A and S, we have that $A \times S$ is Lindelöf. So if every open cover of $A \times S$ also covers each point of all but countably many vertical sections (i.e., the point-inverse sets under projection to X), then $X \times S$ is Lindelöf.

Suppose $X \times S$ is nonnormal. Then by the preceding paragraph, we can choose an open set $U \supseteq A \times S$ such that the complement of U intersects uncountably many vertical sections. For each $s \in S$, again let $Y_s = \{x \in X: \langle x, s\rangle \notin U\}$, and let $Z = \bigcup_{s\in S} Y_s$. Then each Y_s is countable while

Z is uncountable, so we can define $\sigma\colon \omega_1 \to Z$ and $\tau\colon \omega_1 \to S$ by transfinite recursion so that for each $\alpha \in \omega_1$, $\sigma(\alpha) \in Y_{\tau(\alpha)} \setminus \bigcup_{\beta<\alpha} Y_{\tau(\beta)}$. Let $B = \mathrm{Range}(\sigma)$ and let $F = \{\langle \sigma(\alpha), \tau(\alpha)\rangle : \alpha \in \omega_1\}$.

Notation. Suppose S is a separable metric space and $A \subseteq S$ where S is concentrated on A. Define $\mathbb{L}(S, A)$ to be the Lindelöf refinement of S obtained by isolating each point of $S \setminus A$.

Corollary 1. *Suppose S is a separable metric space and $A \subseteq S$, where S is concentrated on A and $S \setminus A$ is uncountable. Then $\mathbb{L}(S, A) \times (S \setminus A)$ is nonnormal (where $S \setminus A$ has the subspace topology).*

Proof. Use Lemma 2 with F the identity function on $S \setminus A$. (This choice of closed sets that cannot be separated is the prototypical construction discovered by Michael.)

Remark. In Lemma 3, and consequently in Theorem 1 and Corollary 2 also, $\mathbb{P}$ can be replaced by any separable completely metrizable space.

Lemma 3. *Suppose Y is a separable completely metrizable space, and $A \subseteq X \subseteq Y$, where X is concentrated on A and $\mathbb{L}(X, A) \times \mathbb{P}$ is nonnormal. Then there is a Cantor set $K \subseteq Y$ such that K is disjoint from A while contained in the Y closure of A.*

Proof. Choose F according to Lemma 2. Let G be the closure of $\mathrm{Graph}(F)$ in the topology of $Y \times \mathbb{P}$. Note that if U is an open set in $\mathbb{L}(X, A) \times \mathbb{P}$ containing $A \times \mathbb{P}$, then there is a set $V \supseteq A \times \mathbb{P}$ which is open in $Y \times \mathbb{P}$ and satisfies $U \supseteq V \cap (X \times \mathbb{P})$; so if $U \cap \mathrm{Graph}(F) = \varnothing$, then $V \cap \mathrm{Graph}(F) = \varnothing$. Thus G is disjoint from $A \times \mathbb{P}$.

For the remainder of the proof we will work entirely in the topology of $Y \times \mathbb{P}$. We need the following theorem: Every uncountable subset T of a separable metric space contains a condensation point (i.e., there exists $t \in T$ such that for each open set U containing t, $|U \cap T| \geq \omega_1$). (For a proof, assume otherwise and use the Lindelöf property on the subspace to obtain a countable open cover of T where each open set contains only countably many points of T.)

By the theorem just quoted, we can choose a sequence $\langle U_n : n \in \omega\rangle$ of open sets in the product $Y \times \mathbb{P}$ such that U_0 is a basic ("rectangular") open set, and for each $n > 0$:

(1) $\mathrm{cl}(U_n) \subseteq U_{n-1}$;
(2) each component of U_{n-1} is a basic open set with diameter $< \frac{1}{n}$, and contains at least two, but no more than a finite number of, components of U_n;
(3) the projections to Y of the components of U_n are pairwise disjoint; and
(4) each component of U_n has uncountable intersection with $\mathrm{Graph}(F)$.

Let $H = \bigcap_{n\in\omega} U_n$. Then by the completeness of $Y \times \mathbb{P}$, H is a Cantor set which is contained in G and intersects uncountably many vertical sections. Let K be the projection of H to Y. Then $K \cap A = \varnothing$, while K is contained in the Y closure of A since X is concentrated on A and each neighborhood of each point of K has uncountable intersection with X.

Theorem 1. *Suppose Y is a separable completely metrizable space concentrated on a subset A. Then $\mathbb{L}(Y, A)\times\mathbb{P}$ is normal (and therefore Lindelöf by Lemma 1) in ZFC.*

Proof. The concentration of Y on A implies that A intersects every Cantor set contained in Y; so by the contrapositive of Lemma 3, $\mathbb{L}(Y, A) \times \mathbb{P}$ is normal.

Corollary 2. *Suppose A is a Bernstein set in $\mathbb{R}$. Then $\mathbb{L}(\mathbb{R}, A) \times \mathbb{P}$ is Lindelöf in ZFC.*

Remark. Suppose $A \subseteq \mathbb{R}$ is Bernstein, and give both A and $\mathbb{R}\backslash A$ the subspace topology. By Corollaries 1 and 2, $\mathbb{L}(R, A) \times \mathbb{P}$ is Lindelöf whereas $\mathbb{L}(\mathbb{R}, A) \times \mathbb{R}\backslash A$ is nonnormal. However, completeness in $\mathbb{P}$, or its failure in $\mathbb{R}\backslash A$, is not the only relevant property. The status of $\mathbb{L}(\mathbb{R}, A)\times A$ depends upon the choice of A. For each n with $n \in \omega$ or $n = \omega$, there is a Bernstein set A such that for each $m < n$, $\mathbb{L}(\mathbb{R}, A) \times {}^m A$ is Lindelöf, whereas $\mathbb{L}(\mathbb{R}, A) \times {}^n A$ is nonnormal (see [8] for the proof).

Theorem 2. *There is a concentrated Michael space iff $\underline{b} = \omega_1$.*

Proof.

Sufficiency. Suppose $\underline{b} = \omega_1$. Use Propositions 2 and 3 to choose an uncountable Lindelöf subspace X of $\mathbb{M}$ with $\mathbb{Q} \subseteq X$. Then $X \times \mathbb{P}$ is nonnormal by Lemma 2 using the Michael construction: $S = \mathbb{P}$, $A = \mathbb{Q}$, $B = X\backslash A$, and F is the identity function on B.

Necessity. Suppose X is a Lindelöf space that is concentrated on a closed subset $A \subseteq X$ where $A\times\mathbb{P}$ is normal. Suppose further that $X\times\mathbb{P}$ is nonnormal. Choose F according to Lemma 2. Claim. Range(F) has countable intersection with every compact subset of $\mathbb{P}$. If the claim holds, then $\underline{b} = \omega_1$ by the propositions.

Assume the claim is false and let T be a compact subset of $\mathbb{P}$ and let G be a restriction of F such that $|\mathrm{Graph}(G)| = \omega_1$ and $\mathrm{Range}(G) \subseteq T$. For each $a \in A$, let $W(a)$ be a finite basic open cover of $\{a\} \times T$, where each element is disjoint from $\mathrm{cl}(\mathrm{Graph}(G))$. Let $V = \{\bigcap_{U\in W(a)} \mathrm{Proj}_X(U) \colon a \in A\}$ (where Proj_X is the projection function to X). Then V is an open cover of A with $\mathrm{Domain}(G) \cap \bigcup V = \varnothing$, which contradicts the concentration of X on A.

Corollary 3. *A real Michael space cannot be concentrated on a subset whose product with $\mathbb{P}$ is Lindelöf whether or not the subset is closed.*

Proof. Suppose X is a Lindelöf space concentrated on a subset $A \subseteq X$ where $A \times \mathbb{P}$ is Lindelöf. Let Y be the refinement of X obtained by isolating each

point of $X \backslash A$. By Theorem 2 and Lemma 1, if $\underline{b} > \omega_1$, then $Y \times \mathbb{P}$ is Lindelöf; and since $X \times \mathbb{P}$ has a coarser topology, it too must be Lindelöf.

Corollary 4. *The derived set of a real Michael space is itself a Michael space.*

Proof. A Lindelöf space is concentrated on its derived set, so Corollary 3 applies.

Lemma 4. *Suppose X is a Lindelöf space. Then for each $g \in {}^{\omega}\omega$, $X \times C^*(g)$ is a Lindelöf subspace of $X \times \mathbb{P}$.*

Proof. The result follows from two facts: (1) the product of a Lindelöf space and a compact space is Lindelöf and (2) $C^*(g)$ is σ-compact.

Notation. Let $\underline{\ell}$ be the minimum of $\underline{b}$ and $\operatorname{lub}\{\omega_n : n \in \omega\}$. (We are not using the customary symbol for the above least upper bound to avoid confusion with our use of ${}^{\omega}\omega$ for the function space.)

Theorem 3. *Suppose X is a Michael space. Then $|X| \geq \underline{\ell}$ and $w(X) \geq \underline{\ell}$.*

Proof. Claim. $L(X \times \mathbb{P}) \geq \underline{\ell}$ (where $L(X \times \mathbb{P})$ is the Lindelöf degree of $X \times \mathbb{P}$ which is defined to be the minimum of $\{\kappa :$ every open cover of $X \times \mathbb{P}$ has a subcover of cardinality $\leq \kappa\}$). Before proving the claim, first note that the theorem is an immediate consequence since $L(X \times \mathbb{P}) \leq \min(|X|, w(X))$, which in turn follows from $w(X \times \mathbb{P}) = w(X)$ and the fact that each vertical section of $X \times \mathbb{P}$ is a Lindelöf subspace.

Proof of the claim. Let Λ be an open cover of $X \times \mathbb{P}$ of minimum cardinality which does not have a countable subcover, and let $\underline{\lambda} = |\Lambda|$. Note that $L(X \times \mathbb{P}) \geq \underline{\lambda}$. We will show that either $\underline{\lambda}$ has countable cofinality or $\underline{\lambda} \geq \underline{b}$. Thus $\underline{\lambda} \geq \underline{\ell}$.

Assume $\underline{\lambda}$ does not have countable cofinality, and let $\theta : \underline{\lambda} \to \Lambda$ be a $1-1$ correspondence. By transfinite recursion we now define strictly increasing sequences $\sigma : \underline{b} \to {}^{\omega}\omega$ and $\tau : \underline{b} \to \underline{\lambda}$ (where ${}^{\omega}\omega$ is partially ordered by $\leq^*$) such that for each $\alpha \in \underline{b}$, $X \times C^*(\sigma_\alpha)$ is not entirely covered by $\{\theta_\beta : \beta < \operatorname{lub}\operatorname{Range}(\tau|\alpha)\}$, but is covered by $\{\theta_\beta : \beta < \tau_\alpha\}$.

Suppose $\gamma \in \underline{b}$ and $\sigma|\gamma$ and $\tau|\gamma$ have been defined so as to satisfy the above condition. For the extension of σ we consider two cases. Let $\varepsilon = \operatorname{lub}\operatorname{Range}(\tau|\gamma)$. If γ has countable cofinality, then $\varepsilon < \underline{\lambda}$ by our starting assumption on $\underline{\lambda}$, so the existence of an appropriate choice for σ_γ follows from $\underline{b} \geq \omega_1$ and the minimality in the choice of Λ. Assume that γ does not have countable cofinality. Let σ_γ be a $\leq^*$ upper bound for $\sigma|\gamma$. If $X \times C^*(\sigma_\gamma)$ is covered by $\{\theta_\beta : \beta < \varepsilon\}$, then by Lemma 4 there is a countable subcover, which by our assumption on γ implies the existence of $\delta < \gamma$ such that $X \times C^*(\sigma_\gamma)$ is covered by $\{\theta_\beta : \beta < \tau_\delta\}$; this contradicts our hypothesis since $C^*(\sigma_{\delta+1}) \subseteq C^*(\sigma_\gamma)$. For the extension of τ, again use Lemma 4 and the uncountable cofinality of $\underline{\lambda}$.

Corollary 5. *Suppose X is a first countable Michael space. Then under Martin's axiom together with $\underline{c} = \omega_n$ for some $n \in \omega$, $|X| = w(X) = \underline{c}$.*

Proof. By Archangel'skiĭ's theorem (see [7, p. 19]), $\underline{c}$ is an upper bound for the cardinality and weight of a first countable Lindelöf space; and since Martin's axiom implies $\underline{b} = \underline{c}$, we have by Theorem 3 that $\underline{c}$ is also a lower bound.

Remark. Recently K. Alster showed in [1] that Martin's axiom implies the existence of a Michael space. Since $\underline{b} = \underline{c}$ under Martin's axiom and $\underline{c} \neq \omega_1$ is consistent with Martin's axiom, Alster's example cannot be concentrated. Thus the existence of a Michael space that is not concentrated is consistent with ZFC.

Problems. (1) Extend the independence result of Theorem 2 to a larger class of spaces. In particular, is the existence of a Michael space obtained as a refinement of a separable metric space independent of ZFC?

(2) Can Theorem 3 be improved by replacing $\underline{\ell}$ with $\underline{b}$?

(3) In both the Michael and Alster examples, one type of point and open set yields the Lindelöf property, while a second type guarantees nonnormality in the product with $\mathbb{P}$. Is there a homogeneous Michael space in ZFC?

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