ON APPROXIMATELY INNER AUTOMORPHISMS
OF CERTAIN CROSSED PRODUCT C*-ALGEBRAS

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Abstract. Let $G$ be a compact connected topological group having a dense subgroup isomorphic to $\mathbb{Z}$. Let $C(G) \rtimes \mathbb{Z}$ be the crossed product $C^*$-algebra of $C(G)$ with $\mathbb{Z}$, where $\mathbb{Z}$ acts on $G$ by rotations. Automorphisms of $C(G) \rtimes \mathbb{Z}$ leaving invariant the canonical copy of $C(G)$ are shown to be approximately inner iff they act trivially on $K_1(C(G) \rtimes \mathbb{Z})$.

Let $G$ be a compact abelian topological group. An element $s \in G$ is called a generator if the group algebraically generated by $s$ is dense in $G$. $G$ is called monothetic if it has at least one generator. If in addition $G$ is connected, this is equivalent to saying that the topology of $G$ has a base of cardinality $\leq c$. Moreover if $G$ is second countable then the set of generators is measurable and its Haar measure equals 1. (See [4], Theorems 24.15, 24.27.)

From now on, $G$ is a monothetic compact connected infinite topological group and $s \in G$ is a fixed generator. Let $A = C(G)$ be the $C^*$-algebra of all complex-valued continuous functions on $G$. We consider the action $\alpha: \mathbb{Z} \to \text{Aut}(A)$ given by

$$(\alpha_k(a))(x) = a(s^{-k}x), \quad \text{for } a \in A, \ x \in G$$

and the corresponding crossed product $C^*$-algebra $A \rtimes \mathbb{Z}$ (see [5, 8]). Denote by $\text{Aut}_\alpha(A \rtimes \mathbb{Z})$ the closed subgroup

$$\{\beta \in \text{Aut}(A \rtimes \mathbb{Z}): \beta(A) = A\}$$

where $\text{Aut}(A \rtimes \mathbb{Z})$ has the topology of pointwise norm convergence. Note that $\text{Aut}_\alpha(A \rtimes \mathbb{Z}) = \{\beta \in \text{Aut}(A \rtimes \mathbb{Z}): \beta(A) \subset A\}$, since $A$ is a maximal abelian self-adjoint subalgebra in $A \rtimes \mathbb{Z}$ (see [8], Proposition 4.14). We prove the following.

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383
1. **Theorem.** An automorphism $\beta \in \text{Aut}_\infty(A \times \mathbb{Z})$ is approximately inner iff $\beta$ induces the identity automorphism of $K_1(A \times \mathbb{Z})$.

For $G$ isomorphic to the one-dimensional torus $T$, the corresponding result is due to Brenken [2]. The proof uses the description of $\text{Aut}_\infty(A \times \mathbb{Z})$ which follows from more general results [3, Theorem 2.8].

Let $u$ be the generator of $\mathbb{Z}$ in $A \times \mathbb{Z}$, i.e. $A \times \mathbb{Z} = C^*(A, u)$ with $uau^* = \alpha_1(a)$ for $a \in A$. Then each $\beta \in \text{Aut}_\infty(A \times \mathbb{Z})$ is given by a unique triplet $(b, x, q) \in U(A) \times G \times \{-1, 1\}$ such that $\beta(u) = bu^q$ and $\beta(a)(y) = a(xy^q)$ for $a \in A$, $y \in G$. Here $U(A)$ denotes the unitary group of $A$ (with the norm topology) and the correspondence $\beta \leftrightarrow (b, x, q)$ is a homeomorphism.

It follows by ([3], Lemma 2.4) that an automorphism is inner iff $q = 1$, $x = s^k$ for some $k \in \mathbb{Z}$ and $b$ has the form $w(\cdot)w^*(s^{-1})$ for some $w \in U(A)$. In this case $\beta(t) = wu^{-k}tu^kw^*$, $t \in A \times \mathbb{Z}$. Therefore if $\beta \in \text{Aut}_\infty(A \times \mathbb{Z})$ is given by $(b, x, q)$ then $\beta$ is approximately inner provided that $q = 1$ and that $b$ is in the closure of the set

$$\{w(\cdot)w^*(s^{-1}) : w \in U(A)\}.$$

Indeed, if $w_n(\cdot)w_n^*(s^{-1})$ converges to $b$ in $U(A)$ and $s^{k_n}$ converges to $x$ in $G$ then, $\text{ad}(w_nu^{-k_n})$ converges to $b$ in $\text{Aut}_\infty(A \times \mathbb{Z})$.

2. **Lemma.** Let $\beta \in \text{Aut}_\infty(A \times \mathbb{Z})$ be given by $(b, x, q)$. If $\beta$ induces the identity automorphism of $K_1(A \times \mathbb{Z})$ then $q = 1$ and $b \in U_0(A)$ (the connected component of the identity in $U(A)$).

**Proof.** Since $G$ is connected it follows that $\alpha_1$ induces the identity automorphism of $K_1(A)$. Using the Pimsner-Voiculescu exact sequence [6] one sees that the canonical map $K_1(A) \rightarrow K_1(A \times \mathbb{Z})$ is injective. The obvious map $\pi^1(G) := [G, T] \rightarrow K_1(A)$ is also injective (use for instance the determinant map). Consequently, if $a \in U(A)$ then $a \in U_0(A)$ iff $[a] = 0$ in $K_1(A \times \mathbb{Z})$.

For $\gamma \in \hat{G}$ (the Pontrjagin dual of $G$) we have $\beta(\gamma) = \gamma(x)\gamma^q$. Therefore $[\gamma] = [\gamma^q]$ in $K_1(A \times \mathbb{Z})$ and by the above remarks $\gamma$ is homotopic to $\gamma^q$ as maps $G \rightarrow T$. By a result of Scheffer [7] this is possible only if $q = 1$. The equation $\beta(u) = bu$ implies that $[\beta(u)] = [b] + [u]$ in $K_1(A \times \mathbb{Z})$ hence using the hypothesis on $\beta$ and the above remarks we find that $b \in U_0(A)$.

3. **Lemma.** The map $w \rightarrow w(\cdot)w^*(s^{-1})$ from $U(A)$ to $U_0(A)$ has dense range (compare with Theorem 4 in [2]).

**Proof.** Let $A_s = \{a(\cdot) - a(s^{-1}) : a \in A\}$. Our first aim is to prove that $A_s + C.1$ is a dense (linear, self-adjoint) subspace of $A$. This is accomplished by showing...
that it contains the \(*\)-subalgebra of \(C(G)\) generated by the characters of \(G\) (which is dense in \(C(G)\) by the Stone–Weierstrass Theorem). We use the fact that

\[
S = \{\chi(s) : \chi \in \hat{G} \setminus \{1\}\}
\]

is a dense subset of \(T\) and \(1 \notin S\) (see [4], Theorem 25.11). Thus if \(\gamma \in \hat{G} \setminus \{1\}\) then \(a = (1 - \gamma(s^{-1}))^{-1} \gamma\) is such that \(\gamma = a(\cdot) - a(s^{-1} \cdot) \in A_s\).

Any \(v \in U_0(A)\) has the form \(v = \exp(ih)\) for some \(h \in C(G, R)\). By the above discussion we can find \(a \in C(G, R)\) and \(\lambda \in R\) such that \(a(\cdot) - a(s^{-1} \cdot) + \lambda\) is arbitrarily close to \(h\) in norm. Also there is \(\gamma \in \hat{G} \setminus \{1\}\) such that \(|e^{i\lambda} - \gamma(s)|\) is arbitrarily small. Then for \(w = \gamma \exp(ia)\),

\[
w(\cdot)w^*(s^{-1} \cdot) = \gamma(s) \cdot \exp i(a(\cdot) - a(s^{-1} \cdot))
\]

will approximate \(v\) as well as we want.

**Proof of the theorem.** If \(\beta \in \text{Aut}_\infty(A \times \mathbb{Z})\) given by \((b, x, q)\) induces the identity automorphism of \(K_1(A \times \mathbb{Z})\) then by Lemma 2, \(b \in U_0(A)\) and \(q = 1\).

Using Lemma 3 we can find a sequence \(w_n \in U(A)\) such that \(w_n(\cdot)w_n^*(s^{-1} \cdot)\) converges to \(b\) in \(U_0(A)\). The discussion before Lemma 2 shows that \(\beta\) is approximately inner. The reverse implication is a general fact.

**References**


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