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SINGULAR INTEGRALS WITH POWER WEIGHTS

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ABSTRACT. This note contains proofs of weighted weak-type $(1, 1)$ and weighted L^p inequalities (with power weights $|x|^\alpha$) for singular integrals whose kernels satisfy Hörmander's condition, and also various size conditions. Some counter-examples are also given, yielding sharp results.

For singular integral operators in $\mathbf{R}^n$ defined by

$$Tf(x) \equiv \text{p.v.} \int K(x, y) f(y)\, dy,$$

the weakest smoothness condition on the kernel K known to yield a satisfactory L^1 theory is the so-called Hörmander condition:

$$\int_{|x-y|\geq 2|y-y'|} |K(x, y) - K(x, y')|\, dx \leq C. \tag{1}$$

(Recently this has been relaxed in low dimensions, especially $n = 2$. See [CR, H1, H2]). This smoothness condition is, however, too weak to prove weighted weak $(1, 1)$ inequalities by any known method in the context of arbitrary A_1 weights, although for power weights $|x|^\alpha$ one L^1 result had been known in $\mathbf{R}^n$. Consider the case where K is a homogeneous convolution kernel, i.e., $K(x) = \Omega(x)|x|^{-n}$ with Ω homogeneous of degree zero, integrable on the sphere and having mean value zero there. For such kernels, it is well known (see [CWZ, CZ]) that (1) is equivalent to the L^1-Dini Condition:

$$\int_0^1 \omega(t) t^{-1}\, dt < \infty, \quad \text{where } \omega(t) \equiv \sup \int_{|\sigma|=1} |\Omega(\rho\sigma) - \Omega(\sigma)|\, d\sigma, \tag{2}$$

the sup running over all rotations ρ with magnitude $|\rho| \leq t$. Kurtz and Wheeden [KW], have shown that, for Ω satisfying (2), the corresponding singular integral operator T satisfies the weighted weak $(1, 1)$ inequality

$$\int_{\{|Tf|>\lambda\}} |x|^\alpha\, dx \leq \frac{c}{\lambda} \int |f(x)||x|^\alpha\, dx \tag{3}$$

if $-1 < \alpha < 0$. Also, the same authors give a counterexample of an Ω for which (2) holds, but (3) fails if $\alpha < -1$ (or $\alpha > 0$.) It turns out that in fact more is

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true: by imposing a stronger size condition on Ω, the range of α for which (3) holds can be expanded, even without strengthening the smoothness condition (2). Although not stated in [KW], for $\Omega \in L^1 - \text{Dini} \cap L^q(|x| = 1)$, $1 < q \le \infty$, (3) holds if $-n+(n-1)/q < \alpha < 0$, and furthermore this result could actually be obtained by modifying the argument in that paper, using a more general version of Lemma 1 of [KW] (see [MW, Lemma 1]). It is also possible, however, to give a simpler argument, and at the same time generalize the conditions on the kernel. For example, consider kernels $K(x, y)$ satisfying

$$|K(x, y)| \le A/|x - y|^n . \tag{4}$$

For $p > 1$, Stein [S] has shown that if K satisfies (4), and if the corresponding operator T is bounded on unweighted L^p, then T is also bounded on $L^p(|x|^\alpha dx)$, $-n < \alpha < n(p-1)$. For $p = 1$, we have

Theorem 1. *Let* $Tf(x) \equiv \text{p.v.} \int K(x, y) f(y)\, dy$, *where* T *is bounded on* (*unweighted*) L^2, *and* K *satisfies* (1) *and* (4). *Then the weighted weak* $(1, 1)$ *bound* (3) *holds if* $-n < \alpha < 0$.

The unweighted L^2 bound is, of course, crucial. It is therefore worth noting that, for general nonconvolution kernels, it is an open problem to determine whether the Hörmander condition (1) (for both K and its adjoint $K^*(x, y) = K(y, x)$) is enough to imply a " $T1$ " type criterion for L^2 boundedness. The weakest such smoothness condition known to yield a " $T1$ " theorem is due to Meyer [M]. Meyer's condition is analogous to, but slightly stronger than (1), although it is still too weak to obtain A_1 weighted inequalities by any known method. Suppose that, for all $R > 0$, $|u| + |v| \le R$, and $k = 1, 2, 3, \ldots$, that

$$\int_{2^k R \le |x-y| \le 2^{k+1} R} |K(x, y) - K(x+u, y+v)|\, dx \le \varepsilon(k).$$

Meyer's condition is that both K and K^* satisfy the above, and that

$$\sum k\varepsilon(k) < \infty .$$

It is well known that, for certain specific nonconvolution kernels, the L^2 bound can be obtained without Meyer's condition. One example is the commutator type operator with kernel

$$K(x, y) \equiv \Omega(x-y)|x-y|^{-n-1}(a(x) - a(y)),$$

where Ω is homogeneous of degree 0, has first moment zero, and belongs to $L\log^+ L(|x| = 1)$, and $\|\nabla a\|_\infty < \infty$ (see [C] or [BC]). If Ω also satisfies (2), then K satisfies (1).

One can also prove (for a smaller range of α) weighted inequalities with weaker size conditions than (4), although it seems that some sort of homogeneity must be assumed to obtain optimal results. In the unweighted case, L^p and

$w(1, 1)$ bounds were proved by Benedek, Calderon, and Panzone [BCP] for convolution kernels satisfying

$$\left|\int_{0<\varepsilon\leq|x|<R<\infty} K(x)\,dx\right| \leq B \tag{5}$$

$$\int_{|x|\geq 2|y|} |K(x-y) - K(x)|\,dx \leq B \tag{6}$$

and

$$\int_{|x|\leq R} |x||K(x)|\,dx \leq BR. \tag{7}$$

For homogeneous kernels, (5) is equivalent to Ω having mean value zero, (6) is equivalent to (2), and (7) is equivalent to $\Omega \in L^1$. As mentioned above, the operator T induced by such a kernel satisfies the weighted $w(1, 1)$ bound (3) if $-1 < \alpha < 0$ [KW]. Also, since $\Omega \in L^1 \cap L^1$-Dini implies $\Omega \in L\log^+ L$ [CWZ], by [MW], T is bounded on $L^p(|x|^\alpha\,dx)$, $-1 < \alpha < p-1$. In [MW] it is also shown that, for $\Omega \in L^q$, $q > 1$, T is bounded on $L^p(|x|^\alpha\,dx)$ if $\max(-n, -1-(n-1)p/q') < \alpha < \min(n(p-1), p-1+(n-1)p/q')$. The condition analogous to (7) corresponding to $\Omega \in L^q$ is

$$\int_{|x|\leq R} |x|^{nq-n+1}|K(x)|^q\,dx \leq BR. \tag{8}$$

Unfortunately, in this more general setting, the results of [KW] and [MW] need not hold.

Theorem 2. *Let* $1 \leq q < \infty$. *For each* $p \geq 1$, *and each* $\alpha < -np/q'$ *or* $\alpha > np/q'$, *there exists a kernel* $K(x)$ *satisfying* (5), (6), *and* (8), *but for which the corresponding operator* T *fails to be bounded on* $L^p(|x|^\alpha\,dx)$, $p > 1$; *if* $p = 1$, T *fails to satisfy the weighted* $w(1, 1)$ *bound* (3). *In particular, for* $q = 1$ (*condition* (7)), *the weighted bounds do not hold in general for any* $\alpha \neq 0$.

If however, we assume a sort of average homogeneity, the standard results can be recovered. For $1 \leq q < \infty$, consider (not necessarily convolution) kernels for which both $K(x, y)$ and $K^*(x, y) \equiv K(y, x)$ satisfy

$$\int_{a\leq|x-y|\leq b} |x-y|^{nq-n+1}|K(x, y)|^q\,dx \leq B(b-a). \tag{9}$$

Theorem 3. *Suppose* T *is bounded on unweighted* L^p, $1 < p < \infty$, *and suppose* K *and* K^* *satisfy Equation* (9). *Then* T *is bounded on* $L^p(|x|^\alpha\,dx)$ *if* $\max(-n, -1-(n-1)p/q') < \alpha < \min(n(p-1), p-1+(n-1)p/q')$. *If* K *also satisfies* (1), *then* T *satisfies the* $w(1, 1)$ *bound* (3) *for* $-n+(n-1)/q < \alpha < 0$.

Remarks. For homogeneous convolution kernels, the L^p result is that of [MW], and (for $q = 1$) the $w(1, 1)$ inequality is that of [KW]. For $q > 1$, the weak $(1, 1)$ result is new even for homogeneous convolution kernels, although in that special case the arguments of [KW] could have been modified as mentioned in the introduction of the present paper.

By Theorem 2, the following result is sharp except for the end point values of α:

Theorem 4. *Let* $1 < q < \infty$, *and suppose* K *and* K^* *satisfy* (9), *but* only *for* $a = 0$ (*i.e., this is the nonconvolution version of* (8)). *Suppose also that* T *is bounded on unweighted* $L^p(1 < p < \infty)$. *Then* T *is bounded on* L^p $(|x|^\alpha\, dx)$ *if* $\max(-n, -np/q') < \alpha < \min(n(p-1), np/q')$. *Also, if* K *satisfies* (1), *then the weak* $(1, 1)$ *bound* (3) *holds for* $-n/q' = -n + n/q < \alpha < 0$.

We now proceed to the proof of Theorem 1. Although easy and direct, it illustrates the basic set-up for all the $w(1, 1)$ results in this paper.

The proof is a simple modification of Calderon–Zygmund arguments. For $\lambda > 0$ fixed, perform a Whitney decomposition of the set $\{Mf > \lambda\}$ (where M is the Hardy–Littlewood Maximal operator) into a union of nonoverlapping closed cubes Q_j. We can write $f = b+g$, where $\int |g|^2|x|^\alpha\, dx \leq c\lambda \int |f||x|^\alpha\, dx$ and $b = \sum b_j$, with b_j supported on Q_j, $\int b_j = 0$, and $\int |b_j| \leq c\lambda |Q_j|$. One handles g by the result of Stein [S] which implies that T is bounded on $L^2(|x|^\alpha\, dx)$, $-n < \alpha < n$.

Let Q_j^* have the same center as Q_j, but side length $5\sqrt{n}$ times as large (thus, for $x \in (Q_j^*)^c$ and $y \in Q_j$, one has $|x-y| \geq 2 \operatorname{diam} Q_j$), and let E be the union of the cubes Q_j^*. By properties of A_1 weights, $\int_E |x|^\alpha\, dx \leq c\lambda^{-1} \int |f||x|^\alpha\, dx$, if $-n < \alpha < 0$. Thus, as usual, it is enough to prove

$$\int_{E^c} |Tb(x)||x|^\alpha\, dx \leq c \int |b(y)||y|^\alpha\, dy.$$

In fact, if $y_j \in Q_j$ is chosen so that $|y_j|^\alpha = \min_{Q_j} |y|^\alpha$, it is then enough to show

$$\int_{(Q_j^*)^c} |Tb_j(x)||x|^\alpha\, dx \leq c \int |b_j(y)||y_j|^\alpha\, dy. \tag{10}$$

The left-hand side of this last inequality is equal to

$$\int_{\substack{(Q_j^*)^c \\ |x| \leq |y_j|/2}} + \int_{\substack{(Q_j^*)^c \\ |x| > |y_j|/2}} \equiv I_1 + I_2.$$

Since $\alpha < 0$, in I_2 we have $|x|^\alpha \leq c|y_j|^\alpha$, so I_2 is bounded by

$$c \int_{(Q_j^*)^c} \left| \int b_j(y) K(x, y)\, dy \right| |y_j|^\alpha\, dx,$$

which, as usual, by the mean value zero property of b_j and Fubini's Theorem, is in turn no larger than

$$c \int |b_j(y)||y_j|^\alpha \int_{(Q_j^*)^c} |K(x, y) - K(x, y_j)|\, dx\, dy.$$

Applying (1) and the definition of Q_j^*, we obtain the desired estimate.

I_1 is also easy to handle. It is dominated by

(11)
$$\int |b_j(y)| \int_{\substack{(Q_j^*)^c \\ |x|\le |y_j|/2}} |K(x,y)||x|^\alpha\,dx \le CA\int |b_j(y)| \int_{|x|\le |y_j|/2} |y_j|^{-n}|x|^\alpha\,dx\,dy,$$

where in the last step we have used (4) and also the fact that, for $x \in (Q_j^*)^c$, $y \in Q_j$ and $|x| \le |y_j|/2$, we have $|x-y| \approx |y_j|$. Since $\alpha > -n$, the inner integral in the right-hand side of (11) is no bigger than $c|y_j|^\alpha$, which concludes the proof. □

We remark that this result also holds for the maximal singular integral operator $\widetilde{T}f \equiv \sup_{\varepsilon>0}|T_\varepsilon f|$, where $T_\varepsilon f(x) \equiv \int_{|x-y|>\varepsilon} K(x,y)f(y)\,dy$. If $\widetilde{T}$ is bounded on unweighted L^2, then it is also bounded on $L^2(|x|^\alpha\,dx)$, $-n<\alpha<n$, by the results of [MW]. (That paper only discussed homogeneous kernels, but the same argument goes over unchanged for kernels which are merely bounded *pointwise* by a homogeneous kernel, i.e., $|K(x,y)| \le |\Omega(x-y)|/|x-y|^n$, with $\Omega \in L^q(S^{n-1})$. The case $q=\infty$ is the bound (4).) This takes care of the "good" function g. To handle the "bad" function b, we observe that, for each $x \in E^c$,

$$\begin{aligned}\widetilde{T}b(x) &\le \sup_{\varepsilon>0}\sum_j \chi_j(x)|T_\varepsilon b_j(x)| \\ &\quad + \sup_{\varepsilon>0}\sum_j (1-\chi_j(x))|T_\varepsilon b_j(x)|. \\ &\equiv A + B,\end{aligned}$$

where $\chi_j(x)$ is the characteristic function of $\{x \in E^c : |x| \le |y_j|/2\}$. Now

$$A \le \sum_j \chi_j(x)\int |K(x,y)||b_j(y)|\,dy,$$

which can be handled exactly like I_1 above by integrating over E^c with respect to the measure $|x|^\alpha\,dx$. By exactly the same reasoning as in Stein's book [S2, pp. 43–44],

$$B \le \sum_j (1-\chi_j(x))\int |K(x,y)-K(x,y_j)||b_j(y)|\,dy + cMb(x),$$

where M is the Hardy–Littlewood maximal operator. This last sum is handled exactly like I_2 above by taking the weighted integral over E^c, and M is well known to be $w(1,1)$ with respect to any A_1 weight.

We now proceed to the (simultaneous) proofs of Theorems 3 and 4, and defer the counter–examples of Theorem 2 until last. Let us assume for now the L^p (or at least the L^2) results, and prove the weak $(1,1)$ bounds. We begin with the same decomposition as in the proof of Theorem 1, except that for technical reasons we take Q_j^* to be large enough that, for $x \in (Q_j^*)^c$ and $y \in Q_j$, we have $|x-y| \ge 4\operatorname{diam} Q_j$. Then, as before, modulo the L^2 result for the "good" function g, it is enough to prove inequality (10). The same splitting of the left side of (10) into $I_1 + I_2$ is now used, with I_2 estimated exactly as in Theorem 1 (that estimate used only the Hörmander condition and the fact that $\alpha \le 0$). The estimate for I_1 imposes the lower bound on α, and is the only new difficulty here. Again, it is enough to show that

$$\int_{\substack{(Q_j^*)^c \\ |x|\le|y_j|/2}} |K(x,y)||x|^\alpha\,dx \le C|y|^\alpha, \tag{12}$$

for all $y \in Q_j$ and $\alpha > -n+(n-1)/q$ ($1 \le q < \infty$, if (9) holds as in Theorem 3) or $\alpha > -n + n/q$ ($q > 1$, if (9) only holds for $a = 0$ as in Theorem 4). Note that if $|y_j| \le 2\operatorname{diam} Q_j$, then the left side of (12) is zero, because in that case $|x| \ge |x - y_j| - |y_j| \ge 4\operatorname{diam} Q_j - 2\operatorname{diam} Q_j \ge |y_j|$. Thus we make take $|y_j| > 2\operatorname{diam} Q_j$, so, for all $y \in Q_j$, $|y| \approx |y_j| \approx |x-y|$. The left side of (12) is therefore bounded by

$$\int_{a|y|\le|x-y|\le b|y|} |K(x,y)||x|^\alpha\,dx, \tag{13}$$

where $0 < a < b < \infty$ and a, b depend only on dimension. Theorem 4 ($p = 1$ case) now follows directly from Hölder's inequality. In fact, for $\alpha > -n+n/q$ (i.e., $aq' > -n$), (13) is less than or equal to

$$\left(\int_{a|y|\le|x-y|\le b|y|} |K(x,y)|^q\,dx\right)^{1/q} \left(\int_{|x|\le(b+1)|y|} |x|^{\alpha q'}\,dx\right)^{1/q'}.$$

By the assumption on α, the second factor equals $C|y|^{\alpha+n/q'}$. The first factor is no larger than

$$C\left(|y|^{-qn+n-1}\int_{a|y|\le|x-y|\le b|y|} |K(x,y)|^q|x-y|^{qn-n+1}\,dx\right)^{1/q} \le c|y|^{-n+n/q}.$$

Multiplying then gives the desired estimate.

The $w(1,1)$ part of Theorem 3 will be an easy consequence of

Lemma 1. *Let K satisfy* (9), *and let $0 < \gamma < 1$. Then*

$$\int_{a|y|\le|x-y|\le b|y|} |x|^{-\gamma}|K(x,y)|^q\,dx \le C|y|^{-nq+n-\gamma}. \tag{14}$$

Proof. Assume for now that Lemma 1 holds. If $q = 1$, set $-\gamma = \alpha$ and we are done. For $q > 1$, and $\alpha > -n + (n-1)/q$, choose $\varepsilon > 0$ so that $\alpha > -n+(n-1)/q+\varepsilon$, and set $\beta = 1/q - \varepsilon$ (thus $\beta q < 1$). Write $|x|^\alpha = |x|^{-\beta}|x|^{\alpha+\beta}$ and apply Hölder's inequality to (13), which is then dominated by

$$\left(\int_{a|y|\le|x-y|\le b|y|} |K(x,y)|^q|x|^{-\beta q}\,dx\right)^{1/q} \left(\int_{|x|\le(b+1)|y|} |x|^{(\alpha+\beta)q'}\,dx\right)^{1/q'}.$$

By (14), with $\gamma = \beta q$, the first factor is less than or equal to $c|y|^{-n+n/q-\beta}$. The second factor equals $c|y|^{n/q'+\alpha+\beta}$, since the definition of β and the assumptions on α and ε imply that $(\alpha+\beta)q' > -n$. The estimate (12) then follows, so it is enough to prove Lemma 1.

We remark that, for convolution kernels, Lemma 1 is very easy to prove: For $-\gamma < 0$, we can replace $|x|$ by $||x-y|-|y||$ in the left side of (14), write the integral in polar coordinates centered at y, and use the fact that, by Lebesque's differentation theorem, inequality (9) for convolution kernels is equivalent to $r^{nq}\int_{S^{n-1}} |K(r\sigma)|^q \, d\sigma \le B$ for a.e. $r \in (0, \infty)$.

To prove (14) in the general case, we split the integral into

$$\int_{a|y|\le|x-y|\le|y|} |x|^{-\gamma}|K(x,y)|^q\,dx + \int_{|y|\le|x-y|\le b|y|} |x|^{-\gamma}|K(x,y)|^q\,dx.$$

We estimate only the second term (the first is handled analogously), and dominate it by

$$(15) \qquad \sum_{k=1}^{\infty}\int_{A_k} |K(x,y)|^q \, ||x-y|-|y||^{-\gamma}\,dx,$$

where $A_k = \{x\colon |y|(1+(b-1)2^{-k}) \le |x-y| \le |y|(1+(b-1)2^{-k+1})\}$. Now $|x-y| \approx |y|$ (independently of k), and, on A_k, $|x-y|-|y| \approx |y|2^{-k}$. Thus, (15) is bounded by

$$\begin{aligned} &C\sum_{k=1}^{\infty} |y|^{-\gamma}2^{k\gamma}|y|^{-qn+n-1}\int_{A_k}|x-y|^{qn-n+1}|K(x,y)|^q\,dx \\ &\quad \le C|y|^{-qn+n-\gamma}\sum_{k=1}^{\infty}2^{k(\gamma-1)}, \end{aligned}$$

where in the last inequality we have applied (9) to the integral over A_k. For $\gamma < 1$, (14) now follows. □

Next we prove the L^p $(p > 1)$ bounds of Theorems 3 and 4. The proof essentially follows that of [MW], the only modification being to use the techniques already discussed in the $w(1,1)$ case. The argument will therefore be kept brief. As in [MW], it is enough to prove weighted L^p bounds for

$$Rf(x) \equiv \int_{|y|\le|x|/2} |K(x,y)||f(y)|\,dy$$

and

$$Sf(x) \equiv \int_{|y|\ge 2|x|} |K(x,y)f(y)|\,dy.$$

The upper limit for α comes from the estimate for Rf, the lower limit from Sf. (The remaining part of the operator, corresponding to (4.2) on page 255 of [MW], is handled exactly as in that paper. The L^p bound for this part depends only on the *unweighted* L^p bound for T, and imposes no restriction on α.) In fact, it is enough to study Rf, because the estimate for Sf then follows by duality as in Lemma 5 of [MW]. Let $m = \min(1, q/p)$. Set $\beta = \alpha/p$. Following the proof of Lemma 3 of [MW], we apply Hölder's inequality and bound $|x|^{\beta} Rf(x)$ by the product of

$$\left(\int_{|y|\leq|x|/2} (|f(y)||y|^{\beta+\varepsilon}|K(x, y)|^{m})^{p}\, dy\right)^{1/p} \equiv P(x)$$

and

$$|x|^{\beta}\left(\int_{|y|\leq|x|/2} (|y|^{-\beta-\varepsilon}|K(x, y)|^{1-m})^{p'}\, dy\right)^{1/p'} \equiv |x|^{\beta} Q(x),$$

where ε is a small positive number to be chosen. If $m = 1$ (i.e., $q \geq p$), then $|x|^{\beta} Q(x)$ is bounded by $C|x|^{n/p'-\varepsilon}$, if ε is chosen small enough so that $p'(\beta+\varepsilon) < n$, which of course can always be done if $\alpha < n(p-1)$. If $q < p$, then $m = q/p$ and $(1-m)p' = (p-q)/(p-1)$. Set $\delta = q(p-1)/(p-q)$ (which is bigger than 1 for $q > 1$). If $q = 1 = \delta$, and K^{*} satisfies (9) as in Theorem 3, then Lemma 1 with $\gamma = p'(\beta+\varepsilon)$ can be applied directly to obtain the bound $|x|^{\beta} Q(x) \leq C|x|^{-\varepsilon}$, for $\beta + \varepsilon < 1/p' = (p-1)/p$. Now, for $1 < q < p$, apply Hölder's inequality again, so that

$$\begin{aligned} Q(x)^{p'} \leq{}& \left(\int_{|x|/2\leq|x-y|\leq 3|x|/2} |K(x, y)|^{q}|y|^{-\eta\delta}\, dy\right)^{1/\delta} \\ &\times \left(\int_{|y|\leq|x|/2} |y|^{\delta'(\eta-p'(\beta+\varepsilon))}\, dy\right)^{1/\delta'}, \end{aligned} \tag{16}$$

where $\eta = 1/\delta - \varepsilon$, so $\eta\delta = 1 - \delta\varepsilon < 1$. Applying Lemma 1 with $\eta\delta = \gamma$, we see that the first factor in (16) is bounded by

$$C|x|^{-\eta-n(q-1)/\delta}.$$

The second factor is no larger than

$$C|x|^{\eta-p'(\beta+\varepsilon)+n/\delta'}$$

if $\delta'(\eta - p'(\beta+\varepsilon)) > -n$. But a grubby computation shows that this is true if $\beta < (n-1)/q' + 1/p' - \varepsilon(1 + 1/p')$, which holds under the assumptions of Theorem 3, for ε small enough. For Theorem 4, we obtain the same estimates that $\eta = 0$, since $\beta < n/q'$ implies $\delta' p'(\beta+\varepsilon) < n$ for small ε. In any case multiplying these estimates, taking the power $1/p'$, and multiplying by $|x|^{\beta}$ shows that

$$|x|^{\beta} Q(x) \leq |x|^{n/p' - nq/\delta p' - \varepsilon} = |x|^{-\varepsilon + n(q-1)/p}.$$

Now we multiply $P(x)$ and the estimate for $|x|^{\beta} Q(x)$, so that, in the case

$q < p$ $(m = q/p)$, we have

$$\begin{aligned}
&\int (|x|^{\beta} Rf(x))^{p}\,dx \\
&\quad \le C \iint_{|y|\le|x|/2} |f(y)|^{p}|y|^{\beta p+\varepsilon p}|K(x\,,\,y)|^{q}\,dy|x|^{-\varepsilon p+nq-n}\,dx \\
&\quad = \int |f(y)|^{p}|y|^{\alpha}\int_{|y|\le|x|/2}(|y|/|x|)^{\varepsilon p}|K(x\,,\,y)|^{q}|x|^{qn-n}\,dx\,dy\,.
\end{aligned}$$

The inner integral equals

$$\begin{aligned}
&|y|^{p\varepsilon}\int_{|x|\ge 2|y|}|x|^{-\varepsilon p-1}|K(x\,,\,y)|^{q}|x|^{qn-n+1}\,dx \\
&\quad \le C|y|^{p\varepsilon}\sum_{k=1}^{\infty}|y|^{-p\varepsilon-1}2^{k(-p\varepsilon-1)}\int_{2^{k}|y|\le|x|\le 2^{k+1}|y|}|K(x\,,\,y)|^{q}|x-y|^{qn-n+1}\,dx \\
&\quad \le C\sum_{k=1}^{\infty}2^{-kp\varepsilon}(2^{k}|y|)^{-1}\int_{|x-y|\le 2^{k+2}|y|}|K(x\,,\,y)|^{q}|x-y|^{qn-n+1}\,dx \le CB\,,
\end{aligned}$$

by (9) applied with $a = 0$.

To conclude the proofs of Theorems 3 and 4, we consider the case $q \ge p$ $(m = 1)$. In this case we had the bound $|x|^{\beta}Q(x) \le c|x|^{n/p'-\varepsilon}$, so

$$\begin{aligned}
&\int (|x|^{\beta} Rf(x))^{p}\,dx \\
&\quad \le \int |f(y)|^{p}|y|^{\beta p}\int_{|x|\ge 2|y|}|K(x\,,\,y)|^{p}(|y|/|x|)^{\varepsilon p}|x|^{np-n}\,dx\,dy\,.
\end{aligned}$$

But the inner integral is bounded by a constant just as in the previous case, because if (9) is satisfied for a given $q > 1$, it also is satisfied for $p < q$ by Hölder's inequality.

We remark that these L^{p} results also carry over for the maximal singular integral $\widetilde{T}$, *if* $\widetilde{T}$ is known to be bounded on unweighted L^{p}. The operators R and S are clearly bounded independently of any truncation of the kernel, and the rest of the operator $\widetilde{T}$ is exactly like (4.2) of [MW, p. 255]. □

It remains only to discuss the counterexamples of Theorem 2. For simplicity, we give an example for the case $q = 1 = p$, followed by a brief sketch of the modifications necessary for the other cases.

Set $\overline{1} = (1\,,\,0\,,\,0\,,\,\dots\,,\,0)$, and, for $0 < \gamma < 1$, define

$$K(x) \equiv \begin{cases} |x+\overline{1}|^{-n+\gamma}, & \text{if } |x+\overline{1}| \le \frac{1}{2} \\ -|x-\overline{1}|^{-n+\gamma}, & \text{if } |x-\overline{1}| \le \frac{1}{2} \\ 0, & \text{otherwise}, \end{cases}$$

and, for $N = 4\,,\,5\,,\,6\,,\,\dots$,

$$f_{N}(y) \equiv \text{characteristic function of } \{|y-\overline{1}| \le 1/N\}\,.$$

Then $\int f_N(y)|y|^\alpha\,dy \approx CN^{-n}$. Note that K trivially satisfies (6) and (7), because it is integrable, and (5) because it is odd. Now for $|x| \le 1/N$ and $|y-\overline{1}| \le 1/N$, we have $|x-y+\overline{1}| = |y-x-\overline{1}| \le 2/N$. Thus, for such x,

$$\begin{aligned} K^* f_N(x) &= \int_{|y-\overline{1}|\le 1/N} |x-y+\overline{1}|^{-n+\gamma}\,dy \\ &\ge C\left(\frac{1}{N}\right)^{-n+\gamma} \int_{|y-\overline{1}|\le 1/N} dy \\ &= CN^{n-\gamma}\left(\tfrac{1}{N}\right)^n = CN^{-\gamma}. \end{aligned}$$

Observe that

$$C\int_{|x|\le 1/N} |x|^\alpha\,dx = \left(\frac{1}{N}\right)^{n+\alpha} = N^{-n-\alpha}, \qquad \alpha > -n.$$

If $Tf \equiv K^*f$ were of weak-type $(1, 1)$ with respect to $|x|^\alpha$, we would have

$$\begin{aligned} N^{-n-\alpha} &\le \int \chi\{|Tf_N| > cN^{-\gamma}\}|x|^\alpha\,dx \\ &\le cN^\gamma \int f_N(y)|y|^\alpha\,dy \\ &\le cN^{-n+\gamma}. \end{aligned}$$

But letting $N \to \infty$, we obtain a contradiction if $\alpha < -\gamma$. Since γ could be chosen arbitrarily close to zero, we are done.

The remaining cases of Theorem 2 are easy variants of the preceeding. For $q > 1$, use the kernel $K_q \equiv K^{1/q}$, where K is as above. The $w(1, 1)$ and also the L^p, $p > 1$ arguments for $\alpha < 0$ are then handled like the above. The upper limit for α with $p > 1$ is obtained by duality.

References

[BC] B. Bajsanski and R. Coifman, *On singular integrals*, Proc. Sympos. Pure Math. **10** (1967), 1–17.

[BCP] A. Benedek, A. P. Calderon, and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. **48** (1962), 356–365.

[C] A. P. Calderon, *Commutators of singular integral operators*, Proc. Nat. Acad. Sci. **53** (1965), 1092–1099.

[CWZ] A. P. Calderon, M. Weiss, and A. Zygmund, *On the existence of singular integrals*, Proc. Sympos. Pure Math. **10** (1967), 56–73.

[CZ] A. P. Calderon and A. Zygmund, *A note on singular integrals*, Studia Math. **65** (1979), 77–87.

[CR] M. Christ and J. L. Rudio de Francia, *Weak-type* $(1, 1)$ *bounds for rough operators* II, Invent. Math. **93** (1988), 225–237.

[H1] S. Hofmann, *Weak* $(1, 1)$ *boundedness of singular integrals with nonsmooth kernel*, Proc. Amer. Math. Soc. **103** (1988), 260–264.

[H2] ——, *Weighted weak-type* $(1, 1)$ *bounds for rough operators*, preprint.

[KW] D. S. Kurtz and R. L. Wheeden, *A note on singular integrals with weights*, Proc. Amer. Math. Soc. **81** (1981), 391–397.

[M] Y. Meyer, *La continuite des operateurs definis par des integrales singulieres*, preprint.

[MW] B. Muckenhoupt and R. L. Wheeden, *Weighted norm inequalities for singular and fractional integrals*, Trans. Amer. Math. Soc. **161** (1971), 249–258.

[S] E. M. Stein, *Note on singular integrals*, Proc. Amer. Math. Soc. **8** (1957), 250–254.

[S2] ——, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, 1970.

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